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Robust stability of control systems: extreme point results for the stability of edges

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For the investigation of the robust stability of control systems with structured uncertainties many results have been presented recently that lead to stability tests of the edges of a polytope. In this paper some results are discussed where the stability of an edge is guaranteed by the stability of the vertices of the edge. Given the characteristic polynomial of a closed loop system with structured uncertainties in the parameters; for special types of uncertainties the family of polynomials is defined by a polytope in the space of parameters. In the frequency domain the corresponding value set of the family, for $s = j\omega$ fixed, is a polygon whose edges are formed by the so-called exposed edges of the polytope. For the investigation of the stability of the polynomial family these exposed edges must be tested (Kraus and Truöl 1991 a). This test can be simplified if an edge has the vertex property, i.e. if the stability of the edge is guaranteed by the stability of the two vertices.

1. Introduction

We shall first sketch the main idea for proving the vertex property of an edge. The following definition and stability theorem will be used frequently in the following.

Definition 1

A polynomial is called D-stable if all its roots lie in the domain D where the stability domain D is an open subset of the complex plane (Kraus and Truöl 1991 a).

Theorem 1

Let $f_0(s)$ and $f_1(s)$ be two D-stable polynomials of the same degree n. The edge between $f_0(s)$ and $f_1(s)$

$$f(s, \lambda) = f_0(s) + \lambda w(s) \quad 0 \leqslant \lambda \leqslant 1 \tag{1}$$

with $w(s) = f_1(s) - f_0(s)$, is D-stable if

(i) $f(s, \lambda)$ has degree n for all $\lambda \in [0, 1]$

(ii) there exists a one-to-one map $\mathscr{R}$

$$\mathscr{R}(w(\varphi(\delta))) \to w^*(\delta) \tag{2}$$

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(iv) The value set has at most

$$(\nu \times 4) \times \frac{1}{2} \times \prod_{i=1}^{\nu} 2^{i} = \nu \times 2^{\nu(n+1)/2+1} \tag{12}$$

exposed edges (Kraus and Truöl 1991 a, Bhattacharyya and Chapellat 1987). This number of edges can be reduced by considering the relative position of the phases of $p_i(s)$. For example with $\nu = 2$ the number of exposed edges of the value set is $8 \times \mu$ where μ is the number of quadrants crossed by the phase of $p_1(s)/p_2(s)$ (Kraus and Truöl 1991 a).

With these properties we present the main theorem.

Theorem 2

With notations as introduced above the polynomial family (11) is Hurwitz table iff all 4^{ν} critical vertex polynomials of $F(s, \underline{\gamma})$ are Hurwitz stable. The critical vertex polynomials of $F(s, \underline{\gamma})$ are formed using all combinations of the four Kharitonov polynomials (Kharitonov 1979) of each $g_i(s, \underline{a}^i)$, $i = 1, 2, ..., \nu$, i.e.

$$f_j(s) = f_0(s) + \sum_{i=1}^{\nu} p_i(s)\tilde{G}_i(s) \quad \tilde{G}_i(s) \in \{G_i^k(s), k = 1, ..., 4\} \tag{13}$$

where $G_i^k(s)$ are the four Kharitonov polynomials of $g_i(s, \underline{a}^i)$. □

Proof

The proof follows immediately with properties (i)–(iv). □

Remarks

(i) The number of critical vertex polynomials of $F(s, \underline{\gamma})$ can be reduced if the relative phase of the polynomials $p_i(s)$ is considered (Kraus and Truöl 1991 a). A further reduction is obtained for low order polynomials $g_i(s, \underline{a}^i)$ (Kraus *et al.* 1988).

(ii) The family of polynomials (11) can be associated with the closed loop characteristic polynomial of a SIMO or MISO system with an anti-Hurwitz compensator and an interval plant.

(iii) A significant simplification is obtained if all $p_i(s)$ are even or odd. Then the resulting value set is simply a rectangle with critical vertex polynomials determined by the sign of $p_i^0(j\omega)$. There results, at most, $4 \times 2^{\nu-1}$ critical vertex polynomials for $\omega \in [0, \infty)$.

(iv) An important special case is obtained for $g_i(s, \underline{a}^i) = \lambda_i$, a constant. Then the polynomial family becomes

$$f(s, \underline{\lambda}) = f_0(s) + \sum_{i=1}^{\nu} \lambda_i p_i(s) \quad \underline{\lambda}_i \leq \lambda_i \leq \bar{\lambda}_i \tag{14}$$

which represents the known case of parpolytopic stability. In Kraus and Truöl (1991 a) the construction of the minimal sets of exposed edges is described. If all $p_i(s)$, $i = 1, ..., \nu$ are anti-Hurwitz only the corresponding critical vertex polynomials need to be tested for stability.

(v) Consider the family

$$f(s, \underline{\gamma}) = f_0(s) + \sum_{j=1}^{\mu} p_j(s) g_j(s, \underline{\alpha}^j) + \sum_{j=\mu+1}^{\nu} q_j(s) g_j(s, \underline{\alpha}^j) \tag{15}$$

where the polynomials $q_j(s)$ do not possess the anti-Hurwitz property. Then only those edges out of the list of exposed edges must be tested that are associated with $q_j(s)$, where q_i varies with $j \geqslant \mu + 1$. The stability of the other edges follows from the stability of the vertices (see Fig. 1). □

In the following we present some extensions to Theorem 2. Instead of the direct parametrization of the interval polynomials $g_i(s, \underline{\alpha}^i)$ we allow a separate parpolytopic parametrization of the even and odd parts of the polynomials g_j.

Let $F(s, \underline{\gamma})$ be a family of polynomials

$$F(s, \underline{\gamma}) = f_0(s) + \sum_{i=1}^{\nu} p_i(s) g_i(s, \underline{\alpha}^i, \underline{\beta}^i) \tag{16}$$

with

$$\underline{\gamma} = \operatorname{col}\begin{pmatrix} \underline{\alpha}^i \\ \underline{\beta}^i \end{pmatrix}, \quad i = 1, 2, \ldots, \nu \quad \gamma_j \in [\underline{\gamma}_j, \bar{\gamma}_j]$$

$$p_i(s) = p_i^+(s) p_i^0(s) \quad i = 1, \ldots, \nu$$

where $p_i^+(s)$ are anti-Hurwitz, $p_i^0(s)$ even or odd polynomials and

$$g_i(s, \underline{\alpha}^i, \underline{\beta}^i) = u_i(s^2, \underline{\alpha}^i) + s v_i(s^2, \underline{\beta}^i)$$

Thereby, $u_i(\cdot)$ and $v_i(\cdot)$ are the even and odd parts of $g_i(\cdot)$ and there are no common parameters between the different polynomials g_i and the even/odd parts respectively. In particular the polynomials $u_i(\cdot)$ and $v_i(\cdot)$ are affine functions of the interval parameters $\underline{\alpha}^i$ respectively $\underline{\beta}^i$, i.e.

$$u_i(\lambda, \bar{\alpha}^i) = u_0^i(\lambda) + \underline{\lambda}' T_0^i \underline{\alpha}^i \quad v_i(\lambda, \underline{\beta}^i) = v_0^i(\lambda) + \underline{\lambda}' T_v^i \underline{\beta}^i$$

where

$$\lambda := s^2 \quad \text{and} \quad \underline{\lambda}' = [1 \quad \lambda \quad \lambda^2 \quad \ldots]$$

$$T_u^i, T_v^i: \text{real transformation matrices}$$

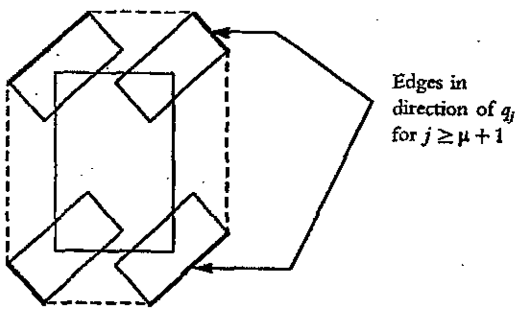

Figure 1. Value set: critical edges and vertices.

As the two polynomials $u_i(\cdot)$ and $v_i(\cdot)$ are real for $s = j\omega$ and independent from each other, the value set of $g_i(s, \underline{\alpha}^i, \underline{\beta}^i)$ is an axis parallel box. The four corners are determined by different sets of parameter choices $\underline{\alpha}^i$, $\underline{\beta}^i$ for a given $s = j\omega$. Indeed the $j\omega$-axis can be divided into a finite number of intervals δ, such that in each of these intervals the four corners of the value set of g^i are determined by four polynomials

$$G^k_{i\delta}(s), \quad k = 1, ..., 4$$

whereby each of them is characterized by a choice of extreme values of the parameters $\underline{\alpha}^i$, $\underline{\beta}^i$.

Theorem 3

With the notations as before, the family of polynomials $F(s, \underline{\gamma})$ in (16) is Hurwitz stable if all its vertex polynomials of the form

$$f_j(s) = f_0(s) + \sum_{i=1}^{\nu} p_i(s) G^k_{i\delta}(s) \tag{17}$$

are Hurwitz stable. □

Proof

The proof follows immediately. □

Example 1

With $\nu = 2$ there results in each interval $4^\nu = 16$ vertices of the value set (see Fig. 1). The vertex polynomials (17) in each interval δ are

$$f_j(s) = f_0(s) + p_1(s) G^k_{1\delta}(s) + p_2(s) G^\kappa_{2\delta}(s) \quad \kappa, k \in \{1, 2, 3, 4\}$$

With the number of intervals $\#\delta$ there result totally $4^\nu \cdot \#\delta$ vertices. From Fig. 1 it is obvious that for every interval only 8 or 16 vertices are critical as they form the convex hull of the value set of (16). □

Now we determine the boundaries of the δ-intervals. Associated with these δ-intervals on the imaginary axis, there exist λ-intervals on the negative real axis. Therefore the extreme polynomials of $u_i(\lambda, \underline{\alpha}^i)$ respectively $v_i(\lambda, \underline{\beta}^i)$ stay the same inside of fixed λ-intervals. The boundaries of the finite number of λ-intervals are determined by

$$\delta_\lambda = \bigcup_i \bigcup_k \{\arg(t^i_{uk}(\lambda) = 0) \cup \arg(t^i_{vk}(\lambda) = 0); \lambda < 0\} \tag{18}$$

where $t_{uk}(\lambda)$, $t_{vk}(\lambda)$ are polynomials associated to the kth column of the matrices T_u respectively T_v.

From the boundaries λ_i of the λ-intervals the corresponding boundaries ω_i of the intervals on the $j\omega$-axis followed by

$$\delta_\lambda \to \delta_\omega: \quad \omega_i = \pm\sqrt{-\lambda_i} \tag{19}$$

A more detailed analysis of the vertex polynomials of $F(s, \underline{\gamma})$ shows that they are invariant in intervals $\delta_\omega \cup \delta_c$. These additional intervals δ_c follow from a collinearity condition of $p_j(j\omega)$ and $p_k(j\omega)$ $\forall j, k$ where essential changes in the structure of the value set occur (the relative distortion of the two different boxes in Example 1, Fig. 1, is caused by the phase of the two polynomials $p_1(j\omega)$ and $p_2(j\omega)$—Kraus and Truöl 1991 a). Let

$$p_i(s) = \tilde{u}_i(s^2) + s\tilde{v}_i(s^2) \tag{20}$$

where $\tilde{u}_i(\cdot)$, $\tilde{v}_i(\cdot)$ are the even and odd parts of $p_i(s)$. With $\lambda = s^2$ we get the boundaries of the additional λ-intervals for the changes of the value set of $F(s, \underline{\gamma})$ caused by the polynomials $p_i(s)$, $i = 1, 2, ..., \nu$ as:

$$\delta'_\lambda = \bigcup_{i,k} \left\{ \arg\left(\begin{vmatrix} \tilde{u}_i(\lambda) & \tilde{u}_k(\lambda) \\ \tilde{v}_i(\lambda) & \tilde{v}_k(\lambda) \end{vmatrix} = 0 \right) \cup \arg\left(\begin{vmatrix} \tilde{u}_i(\lambda) & \tilde{v}_k(\lambda) \\ \tilde{v}_i(\lambda) & \tilde{u}_k(\lambda) \end{vmatrix} = 0 \right); \lambda < 0 \right\} \tag{21}$$

and with

$$\delta'_\lambda \to \delta_c: \quad \omega_i = \pm\sqrt{-\lambda_i}$$

we obtain finally the boundaries δ_c of the additional δ-intervals.

Remark

If the intervals of δ_ω are used (without consideration of δ_c), all 4^ν potential vertex polynomials in each interval must be considered. If the relative phase of the polynomials $p_j(s)$ is additionally used, then the number of intervals increases, but at the same time the number of vertex polynomials to be considered reduces to 4ν in each interval. □

3. Vertex property for Schur stability

Following the same lines of development, we can also treat the Schur stability. The role of the even/odd and anti-Hurwitz polynomials will be played now by symmetric/antisymmetric and anti-Schur polynomials respectively. Consider the edge

$$f(z, \lambda) = f_0(z) + \lambda q^+(z) q^s(z)(\alpha z + \beta) \quad \lambda \in [0, 1] \tag{22}$$

where $q^+(z)$ is an anti-Schur polynomial, $q^s(z)$ has symmetrical roots w.r.t. the unit circle, i.e.

$$q^s(z^*) = 0 \Rightarrow q^s(1/z^*) = 0 \tag{23}$$

and $(\alpha z + \beta)$ is a degree one polynomial. Then the multiplication of $f(z, \lambda)$ with a Schur polynomial

$$q^-(z) = z^n q^+(1/z) \tag{24}$$

does not change the stability properties. The resulting extended polynomial

$$\Delta(z) = q^+(z) q^-(z) q^s(z) \tag{25}$$

is a symmetrical/antisymmetrical polynomial of order ν, i.e.

$$\Delta(z) = z^\nu \Delta(1/z) \quad \dots \quad \Delta(z) \text{ symmetrical} \tag{26}$$

$$\Delta(z) = -z^\nu \Delta(1/z) \quad \dots \quad \Delta(z) \text{ antisymmetrical} \tag{27}$$

It is obvious that the polynomial $q^s(z)$ itself is a symmetric or antisymmetric polynomial. For $z = e^{j\theta}$ the polynomial $\Delta(z)$ can be decomposed in

$$\Delta(e^{j\theta}) = e^{jv\theta/2}\,\tilde{\Delta}(\theta) \tag{28}$$

where $\tilde{\Delta}(\theta) \in \mathbb{R}$ for $\Delta(z)$ symmetrical and $\tilde{\Delta}(\theta) \in j\mathbb{R}$ for $\Delta(z)$ antisymmetrical (Mansour *et al.* 1988). To apply Theorem 1 we used a map $\mathscr{R}$ that eliminates the rotation $e^{jv\theta/2}$ and at the same time preserves the monotonicity of argument of $\mathscr{R}(f(e^{j\theta}))$ for all Schur stable $f(z)$. Such a mapping is derived in Mansour and Kraus (1990) and in Kraus and Mansour (1991). For $\Delta(z)$ symmetric we used the mapping

$$\mathscr{R}(f_0(e^{j\theta})) \rightarrow \frac{\mathrm{Re}\,[e^{-j\theta v/2} f_0(e^{j\theta})]}{\cos(\theta/2)} + j\,\frac{\mathrm{Im}\,[e^{-j\theta v/2} f_0(e^{j\theta})]}{\sin(\theta/2)} \tag{29}$$

For $\Delta(z)$ antisymmetric the mapping is

$$\mathscr{R}(f_0(e^{j\theta})) \rightarrow \frac{\mathrm{Re}\,[e^{-j\theta v/2} f_0(e^{j\theta})]}{\sin(\theta/2)} + j\,\frac{\mathrm{Im}\,[e^{-j\theta v/2} f_0(e^{j\theta})]}{\cos(\theta/2)} \tag{30}$$

It can be shown (Mansour and Kraus 1990, Kraus and Mansour 1991) that, as well as preserving the argument monotonicity for Schur stable $f(z)$, $\mathscr{R}$ maps the extended edge to an edge with a constant slope. Therefore, the conditions of Theorem 1 are fulfilled.

Theorem 4

Consider the polynomial set $f(z, \lambda)$ of (22) then the set $f(z, \lambda), \lambda \in [0, 1]$ is robust Schur stable if the two vertex polynomials $f(z, 0)$ and $f(z, 1)$ are Schur stable and the degree of $f(z, \lambda)$ is λ-invariant. □

Proof

For the proof see Kraus *et al.* (1991). □

Using this theorem, different families of uncertain polynomials can be defined which possess the total or partial vertex property of edges. Consider first the family of polynomials

$$F(z, \underline{\gamma}) = f_0(z) + \sum_{i=1}^{v} q_i^+(z) q_i^0(z) w_i(z, \underline{a}^i) \tag{31}$$

where $q_i(\cdot)$ are anti-Schur polynomials, $q_i^0(\cdot)$ contains only roots on the unit circle and $w_i(\cdot)$ is a symmetric or antisymmetric polynomial with parametrization $\underline{a}^i$. Evidently $F(z, \underline{\gamma})$ may be the characteristic polynomial of a closed loop system where $w_i(\cdot)$ originates from an uncertain plant and $q_i^+(z)$, $q_i^0(z)$ are determined by the controller.

Note that for every variation of a single parameter a_k^i the product of the polynomial $w_i(z, \underline{a}^i)$ with $q_i^0(z)$ can be rewritten as

$$q_i^0(z) w_i(z, \underline{a}^i) = \lambda q^s(z) + w_{i0}(z) \quad \lambda \in [0, 1] \tag{32}$$

and with Theorem 4 we obtain the vertex property for every edge. To prove the robust stability of the whole family only the vertex polynomials of the exposed edges must be checked.

Similarly to the Hurwitz case, the family of polynomials $F(z, \underline{\gamma})$ can be extended to cover the first order component of a Lead/Lag controller, more restrictive parametrizations of the plant or violation of the anti-Schur property by some of the $q_i^+(z)$.

Another possible extension of the allowed edge polynomials uses the idea of Hollot and Bartlett (1986). Consider the edge of polynomials

$$f(z, \lambda) = f_0(z) + \lambda\, \Delta(z) z^{\kappa} \quad \lambda \in [0, 1] \tag{33}$$

where $\Delta(z)$ is a symmetric/antisymmetric polynomial of order ν, the order of $f_0(z)$ is n_0 and the following inequality holds

$$\nu + 2\kappa \leqslant n_0 \tag{34}$$

First the extended polynomial $\tilde{f}(z, \lambda)$ is obtained by multiplication with $z^{n_0 - \nu - 2\kappa}$

$$\tilde{f}(z, \lambda) = \tilde{f}_0(z) + \lambda\, \Delta(z) z^{n_0 - \nu - \kappa} \tag{35}$$

The stability of $\tilde{f}(z, \lambda)$ is preserved by this extension. But for $\Delta(z)$ symmetrical, the antisymmetric part of $\tilde{f}(z, \lambda)$ is λ-invariant. Similarly, for $\Delta(z)$ antisymmetrical, the symmetric part of $\tilde{f}(z, \lambda)$ is not a function of λ. Therefore, by variation of λ only the symmetric/antisymmetric part varies. The vertex property follows by the same argument as in Hollot and Bartlett (1986). Otherwise the symmetric/antisymmetric property of $\Delta(z)$ can be obtained from $q^+(z)q^0(z)q^s(z)$ by extension with the anti-Schur polynomial $q^-(z)$. The combination of both ideas for construction of edges with vertex property and the associated polynomial families $F(z, \underline{\gamma})$ is straightforward.

4. Vertex property for D-stability domains

Consider the family of complex interval polynomials of degree n

$$P(s) = \prod_{i=1}^{n} (s - p_i) = s^n + (t_1 + jv_1)s^{n-1} + \cdots + (t_n + jv_n) \quad \underline{t}_k \leqslant t_k \leqslant \bar{t}_k \quad \underline{v}_k \leqslant v_k \leqslant \bar{v}_k \tag{36}$$

This family is isomorphic to an axis-parallel parpolytope in the parameter space. Let D be a stability domain in the $[s]$-plane associated with a rational function $f(s) = g(s)/h(s)$. Precisely, the region D is that s-plane partition with the maximal Nyquist index μ (Kraus and Truöl 1991 b) (see Fig. 2). As shown in Kraus and Truöl (1991 b), Sondergeld (1983) and Petersen (1989) the problem of investigating the D-stability of the polynomial family $P(z)$ is very easy if $\mu = \max\{\text{degree } g(s), \text{degree } h(s)\}$ because, in that case, the stability problem is equivalent to the Hurwitz stability of the polynomial family $P_f(s)$, where $P_f(s)$ is the numerator of $P(g(s)/h(s))$, i.e. the family of polynomials $P(z)$ is D-stable if $P_f(s)$ is Hurwitz stable. So we have to investigate the robust Hurwitz stability of the family of polynomials

$$P_f(s) = g^n(s) + (t_1 + jv_1)g^{n-1}(s)h(s) + \cdots + (t_n + jv_n)h^n(s) \tag{37}$$

As the family $P_f(s)$ is a parpolytope, the edge theorem of Bartlett *et al.* (1988) can be directly applied. Consider the edge polynomial that results by variation of the coefficient t_k

$$P_f(s) = \tilde{P}_f(s) + t_k\, \Delta P(s) \quad \underline{t}_k \leqslant t_k \leqslant \bar{t}_k \tag{38}$$

with $\Delta P(s) = g^{n-k}(s)h^k(s)$ and $\tilde{P}_f(s)$ the constant part of the edge. For $s = j\omega$ the difference polynomial $\Delta P(s)$ defines the slope of the edge.

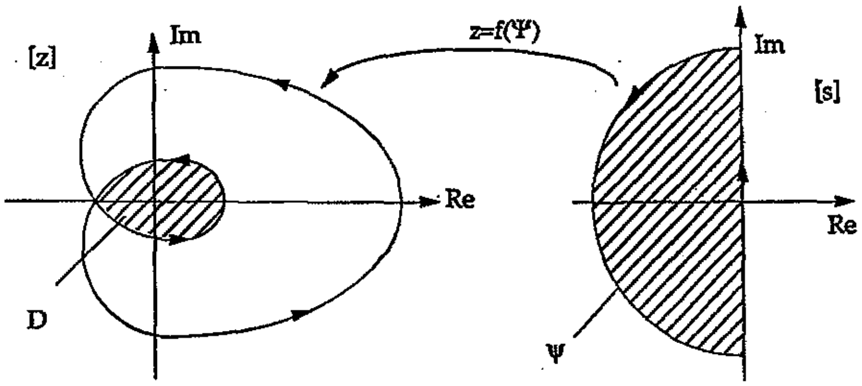

Figure 2. Mapping of the s-plane.

In the following we give the conditions for the edge to have the vertex property. To use the result of Minnichelli *et al.* (1989) for the vertex property of an edge we have to check that the slope of the edge is constant for $s = j\omega$.

Suppose there exists the following decomposition of $\Delta P(s)$

$$\Delta P(s) = \Delta P_s(s)\, \Delta P_+(s) \quad \text{or} \quad \Delta P(s) = s\, \Delta P_s(s)\, \Delta P_+(s) \tag{39}$$

where $\Delta P_s(s)$ denotes that part of $\Delta P(s)$ with all roots symmetrically located relative to the $j\omega$-axis, i.e. $\Delta P_s(s^*) = 0 \Rightarrow \Delta P_s(-s^*) = 0$ and $\Delta P_+(s)$ is anti-Hurwitz. Then there exists a Hurwitz polynomial $\Delta P_-(s) = \Delta P_+(s)$. After extension with $\Delta P_-(s)$ the slope of the edge $\Delta P(s)\, \Delta P_-(s)$ for $s = j\omega$ is constant. With this decomposition the t_k-edge has the following vertex property.

Theorem 5

If the difference polynomial $\Delta P(s)$ of the edge

$$P_f(s) = \tilde{P}_f(s) + t_k\, \Delta P(s)$$

can be decomposed as $\Delta P(s) = \Delta P_s(s)\, \Delta P_+(s)$ and the two vertex polynomials of the edge are Hurwitz stable, then the edge $P_f(s)$ for $\underline{t}_k \leqslant t_k \leqslant \bar{t}_k$ is Hurwitz stable. □

Proof

For the proof see Mansour and Kraus (1990). □

In consequence the vertex property for every edge t_k, v_k, $k = 0, 1, \ldots, n$ is obtained if the polynomials $g(s)$ and $h(s)$ can be decomposed as

$$g(s) = g_s(s) g_+(s) \quad \text{and} \quad h(s) = h_s(s) h_+(s) \tag{40}$$

and thereafter

$$\begin{aligned} \Delta P(s) &= g_s^{n-k}(s) h_s^k(s) g_+^{n-k}(s) h_+^k(s) \\ &= \Delta P_s(s)\, \Delta P_+(s) \quad \forall\, k = 0, \ldots, n \end{aligned} \tag{41}$$

5. Conclusion

In this paper the vertex property of an edge is investigated. For three types of stability—the Hurwitz, Schur and a rational D-stability—the conditions for this property are obtained from the decomposition of the edge polynomials. The associated polynomial families, which can result by control of uncertain plants are discussed.

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