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Connections between real Schur polynomials and half order complex Schur polynomials

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Abstract: Results are available which connect the stability of continuous time real polynomials with the stability of continuous time complex polynomials of half the order. This work establishes parallel results for discrete-time polynomials.

Keywords: Linear system; stability; discrete-time system.

1. Introduction

There is a result connecting the stability properties of real coefficient polynomials with the stability properties of complex polynomials of approximately half the degree. One version of the result is as follows:

Theorem 1.1. [1]. *For real a_i, $i = 0, 1, \dots, 2n$, consider the three polynomials*

$$f(s) = \sum_{i=0}^{2n} a_i s^i, \tag{1.1}$$

$$f_1(s) = \sum_{i=0}^{n} \left[a_{2i}(\mathrm{j})^i + a_{2i+1}(\mathrm{j})^{i+1} \right] s^i, \tag{1.2}$$

$$f_2(s) = \sum_{i=0}^{n} \left[a_{2i}(-\mathrm{j})^i - a_{2i-1}(-\mathrm{j})^{i+1} \right] s^i \tag{1.3}$$

(*where* $a_{2n+1} = a_{-1} = 0$). *Suppose that* [*possibly after replacement of* $f(s)$ *by* $-f(s)$] *either* $a_{2i} > 0$ *for* $i = 0, 1, \dots, n$ *or* $a_0 > 0$, $a_{2n} > 0$ *and* $a_{2i+1} > 0$ *for* $i = 0, 1, \dots, n-1$. *Then if any of* $f(s)$, $f_1(s)$ *and* $f_2(s)$ *has all zeros in* $\mathrm{Re}[s] < 0$, *the other two polynomials have this property.*

In this work, we shall indicate how a parallel result can be obtained for discrete-time polynomials.

The above theorem can be modified to cope with odd degree $f(s)$ [5]. So can the discrete-time theorem. Stability of $f(s)$, $f_1(s)$ and $f_2(s)$ can be examined by checking the positive definiteness of the

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associated Hermite matrices [5]. It turns out that the so called reduced order (half size) Hermite matrices, definable from the Hermite matrix of $f(s)$, and by which the stability of $f(s)$ can be verified given satisfaction of the coefficient sign constraints in the theorem [1,2], are virtually the same as the Hermite matrices associated with $f_1(s)$ and $f_2(s)$. (See [4], Remark 7.)

The above idea also carries over to the discrete case. More precisely, we can associate with a $2n$ degree real polynomial $p(z)$ a Schur–Cohn matrix and two reduced order or half size Schur–Cohn matrices. Positive definiteness of the full size matrix, or positive definiteness of either half size matrix together with coefficient inequalities is necessary and sufficient for stability. We show in this paper that the reduced order Schur–Cohn matrices for $p(z)$ are very closely related to the Schur–Cohn matrices associated with two degree n complex polynomials $p_1(z)$ and $p_2(z)$, definable from $p(z)$. Modifications of these statements of course hold in case $p(z)$ has degree $2n+1$.

2. Discrete polynomial construction

Before stating the main result of this section, we aim to motivate it. Suppose n is even. Let $g(s^2)$ and $sh(s^2)$ be respectively the even and odd parts of $f(s)$ in (1.1), so that

$$f(s) = g(s^2) + sh(s^2) \tag{2.1}$$

Then it is not hard to verify, see [5, pp 62,63], that

$$f_1(s) = g(\mathrm{j}s) + \mathrm{j}h(\mathrm{j}s) \tag{2.2}$$

and

$$f_2(s) = g(-\mathrm{j}s) + sh(-\mathrm{j}s). \tag{2.3}$$

Now associated with every polynomial $\bar{f}(s)$, real or complex, of arbitrary degree r and with all zeros in $\mathrm{Re}[s] < 0$ is a second discrete-time polynomial $\bar{p}(z)$, also of degree r, with all zeros in $|z| < 1$, and the relation between $\bar{f}(s)$ and $\bar{p}(z)$ is

$$\bar{f}(s) = (s-1)^r \bar{p}\left(\frac{s+1}{s-1}\right), \qquad \bar{p}(z) = \frac{(z-1)^r}{2^r} \bar{f}\left(\frac{z+1}{z-1}\right). \tag{2.4}$$

This means that associated with f, f_1 and f_2 there exist discrete-time polynomials $p(z)$, $p_1(z)$ and $p_2(z)$ such that if certain linear inequalities on the coefficients are fulfilled, stability of any one of the polynomials implies stability of the other two. Note that (discrete-time) stability of $p(z)$ or equivalently (continuous-time) stability of $f(s)$ carries with it satisfaction of the coefficient inequalities (at least after introduction of an inessential -1 multiplier). Hence stability of $p(z)$ will imply stability of $p_1(z)$ and $p_2(z)$ (without having to separately verify linear coefficient inequalities).

Evidently $p(z)$ is real, of degree $2n$, while $p_1(z)$ and $p_2(z)$ are complex, of degree n. The question arises as to how, given $p(z)$, the polynomials $p_1(z)$ and $p_2(z)$ might be constructed. This question is answered in Theorem 2.1 below. The construction rests on the composition of the maps

$$p(z) \underset{\substack{\text{via} \\ (2.4)}}{\longrightarrow} f(s) \underset{\substack{\text{via} \\ (2.1)}}{\longrightarrow} [g(s), h(s)] \underset{\substack{\text{via} \\ (2.2),(2.3)}}{\longrightarrow} f_1(s), f_2(s) \underset{\substack{\text{via} \\ (2.4)}}{\longrightarrow} p_1(z), p_2(z)$$

The main result, which is proved in Appendix A, is the following:

Theorem 2.1. *For real a_i, $i = 0, 1, \ldots, 2n$, consider the three polynomials*

$$p(z) = \sum_{i=0}^{2n} a_i z^i, \tag{2.5}$$

$$p_1(z) = \sum_{i=0}^{2n} a_i \Bigg[\sum_{\substack{k=0 \\ k+l \text{ even}}}^{2n-i} \sum_{l=0}^{i} \binom{2n-i}{k}\binom{i}{l} (z+1)^{(k+l)/2} (z-1)^{n-(k+l)/2} (-1)^{2n-i-k} \mathrm{j}^{(k+l/2)}$$
$$+ \sum_{\substack{k=0 \\ k+l \text{ odd}}}^{2n-i} \sum_{l=0}^{i} \binom{2n-i}{k}\binom{i}{l} (z+1)^{(k+l-1)/2} (z-1)^{n-(k+l-1)/2} (-1)^{2n-i-k} \mathrm{j}^{(k+l+1)/2} \Bigg], \tag{2.6}$$

$$p_2(z) = \sum_{i=0}^{2n} a_i \Bigg[\sum_{\substack{k=0 \\ k+l \text{ even}}}^{2n-i} \sum_{l=0}^{i} \binom{2n-i}{k}\binom{i}{l} (z+1)^{(k+l)/2} (z-1)^{n-(k+l)/2} (-1)^{2n-i-(k-l)/2} \mathrm{j}^{(k+l)/2}$$
$$+ \sum_{\substack{k=0 \\ k+l \text{ odd}}}^{2n-i} \sum_{l=0}^{i} \binom{2n-i}{k}\binom{i}{l} (z+1)^{(k+l+1)/2} (z-1)^{n-(k+l+1)/2} (-1)^{2n-i-(k-l+1)/2} \mathrm{j}^{(k-l-1)/2} \Bigg]. \tag{2.7}$$

Suppose that $f(s) = \sum_{i=0}^{2n} a_i(s-1)^{2n-i}(s+1)^i$ [possibly after replacement of $f(s)$ by $-f(s)$] has positive constant term and coefficient of s^{2n}, and either all odd power coefficients or even power coefficients positive (a property guaranteed to hold if $p(z)$ has all zeros in $|z| < 1$). Then if any of $p(z)$, $p_1(z)$ and $p_2(z)$ has all zeros in $|z| < 1$, the other two polynomials have this property.

Remarks. 1. When $z = \mathrm{e}^{\mathrm{j}\phi}$, the first and second triple sums in the expression (2.6) for $p_1(z)$ are, respectively, $\mathrm{j}^n \exp(\mathrm{j}n\phi/2) \times$ real quantity, and $\mathrm{j}^{n+1} \exp(\mathrm{j}n\phi/2) \times$ real quantity. By contrast, if we rewrite $p(z)$ as

$$p(z) = \frac{p(z) + z^{2n}p(z^{-1})}{2} + \frac{p(z) - z^{2n}p(z^{-1})}{2}$$
$$= \sum_{i=0}^{2n} \tfrac{1}{2}(a_i + a_{2n-i}) z^i + \sum_{i=0}^{2n} \tfrac{1}{2}(a_i - a_{2n-i}) z^i \tag{2.8}$$

and set $z = \mathrm{e}^{\mathrm{j}\phi}$; the first sum is $\mathrm{e}^{\mathrm{j}n\phi} \times$ real and the second sum is $\mathrm{j}\mathrm{e}^{\mathrm{j}n\phi} \times$ real.

2. A result is available for odd degree $p(z)$ which is proved in the same way. The continuous time result concerns the simultaneous stability of

$$f(s) = \sum_{i=0}^{2n+1} a_i s^i = g(s^2) + sh(s^2), \tag{2.9}$$

$$f_1(s) = \sum_{i=0}^{n} \left[a_{2i}(\mathrm{j})^i + a_{2i+1}(\mathrm{j})^{i+1} \right] s^i = g(\mathrm{j}s) + \mathrm{j}h(\mathrm{j}s), \tag{2.10}$$

$$f_2(s) = \sum_{i=0}^{n+1} \left[a_{2i}(-\mathrm{j})^i - a_{2i-1}(-\mathrm{j})^{i+1} \right] s^i = g(-\mathrm{j}s) + sh(-\mathrm{j}s), \tag{2.11}$$

given sign constraints on some of the a_i ($a_{2n+1} > 0$ and $a_{2i} > 0$ for all i, or $a_0 > 0$ and $a_{2i+1} > 0$ for all i, conditions automatically fulfilled by $f(s)$ or $-f(s)$ if $f(s)$ is stable). The discrete time result concerns the stability of

$$p(z) = \sum_{i=0}^{2n+1} a_i z^i, \tag{2.12}$$

$$\begin{aligned} p_1(z) = \sum_i a_i \Bigg[& \sum_{k=0}^{2n+1-i} \sum_{\substack{l=0 \\ k+l \text{ even}}}^{i} \binom{2n+1-i}{k}\binom{i}{l}(z+1)^{(k+l)/2}(z-1)^{n-(k+l)/2}\mathrm{j}^{(k+l)/2}(-1)^{2n+1-i-k} \\ & + \sum_{k=0}^{2n+1-i} \sum_{\substack{l=0 \\ k+l \text{ odd}}}^{i} \binom{2n+1-i}{k}\binom{i}{l}(z+1)^{(k+l-1)/2}(z-1)^{n-(k+l-1)/2}\mathrm{j}^{(k+l+1)/2}(-1)^{2n+1-i-k} \Bigg] \end{aligned} \tag{2.13}$$

$$\begin{aligned} p_2(z) & \\ = \sum_i a_i \Bigg[& \sum_{k=0}^{2n+1-i} \sum_{\substack{l=0 \\ k+l \text{ even}}}^{i} \binom{2n+1-i}{k}\binom{i}{l}(z+1)^{(k+l)/2}(z-1)^{n+1-(k+l)/2}\mathrm{j}^{(k-l)/2}(-1)^{2n+1-i-(k+l)/2} \\ & + \sum_{k=0}^{2n+1-i} \sum_{\substack{l=0 \\ k+l \text{ odd}}}^{i} \binom{2n+1-i}{k}\binom{i}{l}(z+1)^{(k+l-1)/2}(z-1)^{n+1-(k+l-1)/2}\mathrm{j}^{(k+l-1)/2}(-1)^{2n-i-(k-l+1)/2} \Bigg]. \end{aligned} \tag{2.14}$$

Provided linear inequality constraints on the a_i are satisfied (and if $p(z)$ is stable they are), then stability of any one of $p(z)$, $p_1(z)$ and $p_2(z)$ implies stability of the other two. Note that $p_1(z)$ and $p_2(z)$ have degrees n and $n+1$, respectively.

3. If a complex polynomial $f_1(s)$ of degree n is given, it is easy to construct a real polynomial $f(s)$ of degree $2n$ so that f and f_1 are connected as per (2.9) and (2.10), or (2.1) and (2.2), i.e. not only can one easily construct f_1 (or f_2) from f, but one can easily construct f from f_1 or f_2. The same is true in respect of the construction of p from p_1 or p_2, although the formulas are a little more complicated. Of course given p_1, the stability of p so constructed is subject to the satisfaction of the linear inequalities mentioned above. That these inequalities are critical is confirmed by the following example. With $f(s)$ equalling $f(s) = s^2 + s$ the corresponding $f_1(s)$ becomes $f_1(s) = f(s+1)$. Thus even though $f_1(s)$ is Hurwitz, $f(s)$ is not.

4. It is worth noting that the coefficients of $f(s)$ defined in the theorem statement are related via a simple matrix equality to those of $p(z)$ (see [5, Section 3.2] for a precise definition of the relation). Thus the inequalities that the coefficients of $f(s)$ must obey translate in a simple way to inequalities on the coefficients of $p(z)$.

3. Relation to reduced order stability criteria

Let $\bar{p}(z)$ be an r-th degree polynomial, with possibly complex coefficients. The associated $r \times r$ Schur–Cohn matrix $\bar{S} = (\bar{s}_{ij})$ is defined by

$$\frac{\bar{p}(z) w^n \bar{p}^*(w^{-1}) - \bar{p}(w) z^n \bar{p}^*(z^{-1})}{z - w} = \sum_{i,j=1}^{r} z^{i-1} \bar{s}_{ij} w^{n-j}. \tag{3.1}$$

[Here $\bar{p}^*(z)$ denotes the polynomial obtained from $\bar{p}$ by complex conjugation of coefficients.] The Schur–Cohn matrix is necessarily hermitian, and real symmetric if $\bar{p}(z)$ is a real polynomial. It is positive definite if and only if $\bar{p}$ has all roots in $|z|<1$ [5]. When $\bar{p}(z)$ is a real polynomial $p(z)$, say of degree $2n$, two half-size Schur–Cohn matrices can be formed, as follows. Define the $n\times n$ matrix

$$R=\begin{bmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & & & \vdots \\ 1 & \dots & 0 & 0 \end{bmatrix}. \tag{3.2}$$

Then

$$\begin{bmatrix} I & I \\ -I & I \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & R \end{bmatrix} S \begin{bmatrix} I & 0 \\ 0 & R \end{bmatrix}\begin{bmatrix} I & -I \\ I & I \end{bmatrix}=\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \tag{3.3}$$

for some $n\times n$ matrices A and B, the reduced-order Schur–Cohn matrices. It is established in [2] that $\bar{p}(z)$ has all roots in $|z|<1$ if and only if one of A and B is positive definite and certain linear inequalities on the coefficients of $\bar{p}(z)$ are fulfilled. (These inequalities are identical with those in the statement of Theorem 2.1.)

Because of the connection between reduced Hermite matrices and complex polynomials explained in Section 1, it is not surprising that there should be a connection between the reduced order Schur–Cohn matrices A and B associated with $p(z)$, (both of size $n\times n$), and the Schur–Cohn matrices S_1 and S_2 associated with $p_1(z)$ and $p_2(z)$ (again of size $n\times n$). In this section, we record the connection.

Again, our approach will invoke the continuous time polynomials (and now their Hermite matrices). Call H, H_1, and H_2 the Hermite matrices of f, f_1 and f_2. For each r, there exists an $r\times r$ matrix $\Gamma^{(r)}=(\Gamma_{ij}^{(n)})$ derivable from binomial coefficients through

$$2^{-(r-1)/2}(w+1)^{i-1}(w-1)^{r-i}=\sum_{j=1}^{r}\Gamma_{ij}^{(r)}w^{j-1},\quad i=1,2,\dots,r, \tag{3.4}$$

with the property that

$$H=\Gamma^{(2n)\prime}S\Gamma^{(2n)}, \tag{3.5a}$$

$$H_1=\Gamma^{(n)\prime}S_1\Gamma^{(n)}, \tag{3.5b}$$

$$H_2=\Gamma^{(n)\prime}S_2\Gamma^{(n)}. \tag{3.5c}$$

Moreover, $\Gamma^{(n)}$ is idempotent:

$$\Gamma^{(n)2}=I. \tag{3.6}$$

Let M' be the $2n\times 2n$ row rearranging matrix

$$M'=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \end{bmatrix}. \tag{3.7}$$

Via M_1 the reduced order Hermite matrices, call them C and D, are defined, see [3]:

$$M'HM=\begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}. \tag{3.8}$$

The matrices C and D are connected to both H_1 and H_2, and to A and B, as set out in [3]:

$$H_1 = \operatorname{diag}\begin{bmatrix} \mathrm{j}^{n-1} & \mathrm{j}^{n-2} & \cdots & 1 \end{bmatrix}^* C \operatorname{diag}\begin{bmatrix} \mathrm{j}^{n-1} & \mathrm{j}^{n-2} & \cdots & 1 \end{bmatrix}, \tag{3.9a}$$

$$H_2 = \operatorname{diag}\begin{bmatrix} \mathrm{j}^{n-1} & \mathrm{j}^{n-2} & \cdots & 1 \end{bmatrix}^* D \operatorname{diag}\begin{bmatrix} \mathrm{j}^{n-1} & \mathrm{j}^{n-2} & \cdots & 1 \end{bmatrix}, \tag{3.9b}$$

and

$$C = 2XBX', \tag{3.10a}$$

$$D = 2YAY', \tag{3.10b}$$

where X and Y are obtained from M and $\Gamma^{(2n)}$:

$$2^{(2n-1)/2} M' \Gamma^{(2n)'} \begin{bmatrix} I_n \\ 0 \end{bmatrix} = \begin{bmatrix} -X \\ Y \end{bmatrix}. \tag{3.11}$$

So evidently, the following matrices are congruent:

$$\begin{aligned} B &\leftrightarrow C \quad [\text{by (3.10a)}] \\ &\leftrightarrow H_1 \quad [\text{by (3.9a)}] \\ &\leftrightarrow S_1 \quad [\text{by (3.5b)}] \end{aligned}$$

and

$$\begin{aligned} A &\leftrightarrow D \quad [\text{by (3.10b)}] \\ &\leftrightarrow H_2 \quad [\text{by (3.9b)}] \\ &\leftrightarrow S_2 \quad [\text{by (3.5c)}]. \end{aligned}$$

These congruences show that positive definiteness of one of the reduced order Schur–Cohn matrices associated with $p(z)$ is equivalent to positive definiteness of the Schur–Cohn matrix of $p_1(z)$, $p_2(z)$. The actual congruency transformation linking B to S_1 and A to S_2 can be easily written down, and is evidently independent of the polynomial coefficients, merely involving binomial coefficients, powers of 2, and j, etc:

$$\begin{aligned} S_1 &= \Gamma^{(n)'} H_1 \Gamma^{(n)} \\ &= \Gamma^{(n)'} \operatorname{diag}\begin{bmatrix} \mathrm{j}^{n-1} & \mathrm{j}^{n-2} & \cdots & 1 \end{bmatrix}^* C \operatorname{diag}\begin{bmatrix} \mathrm{j}^{n-1} & \mathrm{j}^{n-2} & \cdots & 1 \end{bmatrix} \Gamma^{(n)} \\ &= 2\Gamma^{(n)'} \operatorname{diag}\begin{bmatrix} \mathrm{j}^{n-1} & \mathrm{j}^{n-2} & \cdots & 1 \end{bmatrix}^* XBX' \operatorname{diag}\begin{bmatrix} \mathrm{j}^{n-1} & \mathrm{j}^{n-2} & \cdots & 1 \end{bmatrix} \Gamma^{(n)} \end{aligned} \tag{3.12}$$

Similarly,

$$S_2 = 2\Gamma^{(n)'} \operatorname{diag}\begin{bmatrix} \mathrm{j}^{n-1} & \mathrm{j}^{n-2} & \cdots & 1 \end{bmatrix} YAY' \operatorname{diag}\begin{bmatrix} \mathrm{j}^{n-1} & \mathrm{j}^{n-2} & \cdots & 1 \end{bmatrix} \Gamma^{(n)}. \tag{3.13}$$

A minor variant on these results applies for odd degree polynomials.

4. Conclusions

Formulas connecting real Schur polynomials with complex Schur polynomials of half the degree have been presented. Connection between the reduced order Schur–Cohn matrices of the real polynomial and the Schur–Cohn matrices of the complex polynomials have also been established.

Appendix A. Proof of Theorem 2.1

With $f(s)=(s-1)^{2n}p((s+1)/(s-1))=g(s^2)+sh(s^2)$, it is evident that

$$g(\mathrm{j}s)=\frac{\left(\sqrt{\mathrm{j}s}-1\right)^{2n}p\left(\dfrac{\sqrt{\mathrm{j}s}+1}{\sqrt{\mathrm{j}s}-1}\right)+\left(\sqrt{\mathrm{j}s}+1\right)^{2n}p\left(\dfrac{\sqrt{\mathrm{j}s}-1}{\sqrt{\mathrm{j}s}+1}\right)}{2}. \tag{A.1}$$

Hence

$$\begin{aligned}
(z-1)^n g\left(\mathrm{j}\frac{z+1}{z-1}\right) &= \tfrac{1}{2}\left\{\left(\sqrt{\mathrm{j}}\sqrt{z+1}-\sqrt{z-1}\right)^{2n}p\left(\frac{\sqrt{\mathrm{j}}\sqrt{z+1}+\sqrt{z-1}}{\sqrt{\mathrm{j}}\sqrt{z+1}-\sqrt{z-1}}\right)\right.\\
&\qquad\left.+\left(\sqrt{\mathrm{j}}\sqrt{z+1}+\sqrt{z-1}\right)^{2n}p\left(\frac{\sqrt{\mathrm{j}}\sqrt{z+1}-\sqrt{z-1}}{\sqrt{\mathrm{j}}\sqrt{z+1}+\sqrt{z-1}}\right)\right\}\\
&= \tfrac{1}{2}\left\{\sum_i a_i\left(\sqrt{\mathrm{j}}\sqrt{z+1}-\sqrt{z-1}\right)^{2n-i}\left(\sqrt{\mathrm{j}}\sqrt{z+1}+\sqrt{z-1}\right)^{i}\right.\\
&\qquad\left.+\sum_i a_i\left(\sqrt{\mathrm{j}}\sqrt{z+1}+\sqrt{z-1}\right)^{2n-i}\left(\sqrt{\mathrm{j}}\sqrt{z+1}-\sqrt{z-1}\right)^{i}\right\}\\
&= \tfrac{1}{2}\sum_{i=0}^{2n}a_i\left\{\sum_{k=0}^{2n-i}\binom{2n-i}{k}\mathrm{j}^{k/2}(z+1)^{k/2}(-1)^{2n-i-k}(z-1)^{(2n-i-k)/2}\right.\\
&\qquad\times\sum_{l=0}^{i}\binom{i}{l}\mathrm{j}^{l/2}(z+1)^{l/2}(z-1)^{(i-l)/2}\\
&\qquad+\sum_{k=0}^{2n-i}\binom{2n-i}{k}\mathrm{j}^{k/2}(z+1)^{k/2}(z-1)^{(2n-i-k)/2}\\
&\qquad\left.\times\sum_{l=0}^{i}\binom{i}{l}\mathrm{j}^{l/2}(z+1)^{l/2}(-1)^{i-l}(z-1)^{1/2(i-l)}\right\}\\
&= \sum_{i=0}^{2n}a_i\sum_{\substack{k=0\\k+l\ \text{even}}}^{2n-i}\sum_{l=0}^{i}\binom{2n-i}{k}\binom{i}{l}(z+1)^{(k+l)/2}(z-1)^{n-(k+l)/2}(-1)^{2n-i-k}\mathrm{j}^{(k+l)/2}.
\end{aligned} \tag{A.2}$$

Next, it is evident that

$$h(\mathrm{j}s)=\frac{\left(\sqrt{\mathrm{j}s}-1\right)^{2n}p\left(\dfrac{\sqrt{\mathrm{j}s}+1}{\sqrt{\mathrm{j}s}-1}\right)-\left(\sqrt{\mathrm{j}s}+1\right)^{2n}p\left(\dfrac{\sqrt{\mathrm{j}s}-1}{\sqrt{\mathrm{j}s}+1}\right)}{2\sqrt{\mathrm{j}s}}.$$

and so, using similar calculation to those for obtaining (A.2),

$$\begin{aligned}
&\mathrm{j}(z-1)^n h\left(\mathrm{j}\frac{z+1}{z-1}\right)\\
&\quad= \sum_{i=0}^{2n}a_i\sum_{\substack{k=0\\k+l\ \text{odd}}}^{2n-i}\sum_{l=0}^{i}\binom{2n-i}{k}\binom{i}{l}(z+1)^{(k+l-1)/2}(z-1)^{n-(k+l-1)/2}(-1)^{2n-i-k}\mathrm{j}^{(k+l+1)/2}
\end{aligned}$$

Inspection of the formula for $p_1(z)$ in (2.6) shows that

$$p_1(z) = (z-1)^n\left[g\left(\mathrm{j}\frac{z+1}{z-1}\right) + \mathrm{j}h\left(\mathrm{j}\frac{z+1}{z-1}\right)\right] = (z-1)^n f_1\left(\frac{z+1}{z-1}\right).$$

Since also

$$p(z) = \frac{(z-1)^{2n}}{2^{2n}} f\left(\frac{z+1}{z-1}\right)$$

(by construction of $f(s)$), Theorem 1.1 establishes the claim of Theorem 2.1 as regards $p(z)$ and $p_i(z)$. The claim regarding $p_2(z)$ is proved the same way.

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