INT. J. CONTROL, 1992, VOL. 56, NO. 3, 531–545

Performance study of multi-rate output controllers under noise disturbances

MENG J. ER† and BRIAN D. O. ANDERSON†

This paper presents a comparative study of the performance of a multi-rate output controller (MROC) with a linear quadratic gaussian (LQG) controller in the presence of noise disturbances and anti-aliasing filters. The basis of comparison is to apply an LQG law with a one-step-ahead prediction-type Kalman filter (hereafter called LQG law I) and an LQG law with a current estimation-type Kalman filter (hereafter called LQG law II) to a linear time-invariant continuous-time plant model with a white gaussian process and measurement noise and compute a linear quadratic performance index for the discretized plant. Equivalent noise matrices for using the MROC law are derived and the same quadratic performance index computed. In order to have a fair comparison, the cut-off frequency of the anti-aliasing filter used to remove high frequency noise components prior to sampling is also kept the same when applying both laws. Application of both laws in typical situations shows that the performance of the MROC law is worse than either that of LQG law I or LQG law II.

1. Introduction

Owing to the rapid development in the technology of digital computers and micro-processors, considerable attention has been focused on the study of digital and sampled-data control (Franklin and Powell 1980, Jury 1958, Ogata 1987 and Ragazzini and Franklin 1958) and in particular, control schemes with multirate sampling (Araki and Yamamoto 1986, Berg *et al.* 1988, Boyklin and Frazier 1975, Glasson 1983, Kalman and Bertram 1959, Kranc 1957 and Meyer and Burrus 1975).

Multirate (MR) sampling got its start in the 1950s as an analytical tool. The idea was to attach a phantom sampler to the input or output of a single-input, single-output (SISO) system and operate it at some integer multiple of the basic sampling rate to detect intersample ripple. The first substantial treatment of MR systems from the state-space perspective was published by Kalman and Bertram (1959). That paper presents a method for forming a discrete-time state model of an MR sampled-data plant. For a linear time-invariant (LTI) plant and the general case of MR sampling, the discrete-time model is time varying. For the special case where all sampling rate ratios are rational numbers, the discrete-time model is periodically time varying with the same period as the sampling schedule.

In the last decade and particularly the last several years, a number of results on periodically time-varying (PTV) digital controllers have been reported (Francis and Georgiou 1988, Hagiwara and Araki 1989, Khargonekar *et al.* 1985, Anderson and Moore 1981, Araki and Hagiwara 1985, 1986, Chammas and Leondes 1978, 1979 a, b, Greschak and Verghese 1982 and Mita *et al.* 1987). PTV controllers used in conjunction with LTI plants offer a new dimension of flexibility in the design process.

Received 18 February 1991.

† Department of Systems Engineering, Research School of Physical Sciences and Engineering, Australian National University, GPO Box 4, Canberra ACT 2601, Australia.

In particular, they have been used to achieve equivalent state feedback without observers, pole assignment, zero assignment, gain margin improvement, strong and simultaneous stabilization and the removal of decentralised fixed modes in decentralised control. Evidently, PTV controllers can offer substantially more design freedom than conventional LTI controllers; also a PTV digital controller can be implemented in practice without any significant difficulty since it does not violate the constraint of finite memory in a computer. A survey of PTV digital controllers is reported by Hagiwara and Araki (1988).

MROCs, a special class of PTV controllers, are a new type of controller which detect the ith plant output at N_i uniformly spaced times and changes the plant input once during one frame period T_0. MROCs have the interesting features of allowing implementation of arbitrary linear state feedback and strong stabilization of unstable plant. Furthermore, the computational efforts required in the design procedure are almost the same as those required for ordinary time-invariant controllers and they do not change the plant inputs as rapidly as multirate-input controllers and other types of controller that use frequent changes of gains for regulation.

However, the operational aspects of MROCs under disturbances such as process and/or measurement noise have not yet been reported in the literature. We showed (Er and Anderson 1991) that the frame periods and output sampling periods must fulfil certain inequality constaints to avoid the gains in the controller becoming very large; in this paper, we seek to identify yet another drawback of MROCs under noise disturbances. We show that the MROC law performs worse than two LQG laws, termed LQG law I and LQG law II, in the presence of noise disturbances. Note that the LQG law here is discrete-time and is associated with the equivalent discrete-time model of the underlying continuous-time plant; LQG law I uses a one-step prediction estimate of the state (so that the present control depends on measurements prior to the present time); LQG law II uses a true filtered estimate of the state (so that the present control depends on measurements prior to and at the present time). The basis of comparison is to apply to two types of LQG law to a LTI continuous-time plant with white, gaussian measurement and process noise and compute the optimal linear quadratic performance index for the discretized plant. Next, the MROC law used by Hagiwara and Araki (1988), seeking to implement the same state feedback law as the two LQG laws, is applied to the same plant. The equivalent noise matrices and performance index for the discretized plant with MROC law are then calculated. Simulation results show that the two types of LQG law perform better than the MROC law for a typical plant.

The paper is organized as follows. Section 2 describes the LQG and MROC problem. Specifically, we formulate the underlying continuous-time model based upon which the equivalent discrete-time models for using LQG laws and MROC laws are derived. In § 3, the performance indices for using the MROC law and LQG law I are computed. The performance index associated with the MROC law is formulated in a form similar to that associated with the LQG law. Section 4 presents a typical industrial plant to demonstrate that the MROC law has poorer performance than either LQG law I or LQG law II. Section 5 contains concluding remarks.

2. The LQG and MROC problem

One prime concern here will be to compare the two types of LQG law and the MROC law. At time $(k+1)T_0$, the two LQG laws feed back a linear feedback gain multiplied by an estimate of $x(k+1)T_0$. In the case of LQG law I, this estimate is obtained from measurements at times $\ldots, (k-2)T_0, (k-1)T_0$ and kT_0. For LQG law II, the estimate is obtained from measurements at times $\ldots, (k-2)T_0, (k-1)T_0, kT_0$ and $(k+1)T_0$. By contrast, the MROC law uses a different set of measurements; for each i, the ith plant output values at times $kT_0, kT_0+T_i, kT_0+2T_i, \ldots, (k+1)T_0-T_i$ are used where $T_i = T_0/N_i$, with T_0 the frame period and N_i the ith output multiplicity. Further, as it turns out, the MROC can be regarded as using these measurements to generate a (non-optimal) estimate of $x(k+1)T_0$ which is then multiplied by the same feedback gain as used in the two LQG laws. In comparison to LQG law I, the MROC law uses more recent measurements, but uses a lesser number of measurements. In comparison to LQG law II, it uses mostly more recent measurements. It is therefore not *a priori* obvious on the basis of the measurement strategies whether an MROC law will be inferior or superior to the LQG laws. The MROC law also uses its measurements non-optimally in comparison with the LQG laws. Of course, the LQG law II always performs better than the LQG law I, since a current estimation type Kalman filter always performs better than a one-step ahead prediction-type Kalman filter.

2.1. *Continuous-time plant model*

To fix ideas, we consider the following plant model which is an LTI continuous-time system given by

$$\dot{x}(t) = Ax(t) + Bu(t) + \omega_u(t) \tag{2.1 a}$$

$$y(t) = Cx(t) + \omega_y(t) \tag{2.1 b}$$

where the state vector $x \in \mathbb{R}^{n_x}$, the process noise $\omega_u \in \mathbb{R}^{n_x}$, the control $u \in \mathbb{R}^{n_u}$, the measurement vector $y \in \mathbb{R}^{n_y}$ and the measurement noise $\omega_y \in \mathbb{R}^{n_y}$. The initial condition $x(0)$ is a zero mean random vector with variance $E\{x(0)x'(0)\} = P_0$. The process disturbances $\omega_u(t)$ and $\omega_y(t)$ are assumed to be zero mean independent white noise processes with intensities $\Omega_u \geqslant 0$ and $\Omega_y \geqslant 0$.

As the output of the plant is contaminated by process noise (via the state) and measurement noise, the output needs to be filtered prior to sampling. This prefiltering is necessary to remove high frequency components which can confuse the interpretation of the sampled signal due to aliasing. The type of prefilter used for this purpose is called an analogue anti-aliasing filter (AAF). There are many types of AAFs and a good discussion of them can be found in Åström and Wittenmark (1990), Franklin *et al.* (1990), and Middleton and Goodwin (1990). For simplicity, we consider passing each entry of the plant output through an AAF of the low-pass filter type, i.e. one with transfer function given by

$$H_{aa}(s) = \frac{\alpha}{s+\alpha} \tag{2.2}$$

where α is the bandwidth of the filter. Here the nominal α is chosen to be $\omega_s/2$ as a rule of thumb where $\omega_s = 2\pi/T_0$ is the sampling rate. Note that the AAF is required even if there is only process noise and no measurement noise.

It is not difficult to see that the cascade of the plant with an AAF can be defined by the following augmented system:

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}u(t) + \tilde{B}_1\tilde{\omega}_u(t) \tag{2.3 a}$$

$$\tilde{y}(t) = \tilde{C}\tilde{x}(t) \tag{2.3 b}$$

where

$$\tilde{A} = \begin{bmatrix} A & 0 \\ \alpha C & -\alpha I \end{bmatrix} \tag{2.4 a}$$

$$\tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} \tag{2.4 b}$$

$$\tilde{B}_1 = \begin{bmatrix} I & 0 \\ 0 & \alpha I \end{bmatrix} \tag{2.4 c}$$

$$\tilde{C} = [0 \quad I] \tag{2.4 d}$$

$$\tilde{x}(t) = \begin{bmatrix} x(t) \\ y_f(t) \end{bmatrix} \tag{2.4 e}$$

$$\tilde{\omega}_u(t) = \begin{bmatrix} \omega_u(t) \\ \omega_y(t) \end{bmatrix} \tag{2.4 f}$$

and $\tilde{\omega}_u(t)$ has covariance

$$E[\tilde{\omega}_u(t)\tilde{\omega}'_u(s)] = \tilde{\Omega}_u \delta(t-s) = \begin{bmatrix} \Omega_u & 0 \\ 0 & \Omega_y \end{bmatrix} \delta(t-s) \tag{2.5}$$

The conventional discrete-time system will be obtained by sampling $\tilde{y}(t)$ every T_0 seconds and applying an input $u(\cdot)$, which is the following pulse amplitude modulated signal, to (2.1):

$$u(t) = \sum_k u_d(k) p(t - kT_0) \quad k = 1, 2, \ldots \tag{2.6}$$

where $p(t) = 1$ for $0 \leqslant t < T_0$ but is otherwise zero. The MROC system is obtained by using the same style of input, but by sampling the output a number of times in an interval of length T_0.

2.2. *Equivalent discrete-time model of augmented system*

Conventional sampling of the model (2.3) yields the discrete time model

$$\tilde{x}_d(k+1) = F\tilde{x}_d(k) + Gu_d(k) + \omega_{ud}(k) \tag{2.7 a}$$

$$\tilde{y}_d(k) = \tilde{C}\tilde{x}_d(k) \tag{2.7 b}$$

where

$$F = \exp(\tilde{A}T_0) \tag{2.8 a}$$

$$G = \int_0^{T_0} \exp(\tilde{A}\sigma)\, \tilde{B}\, d\sigma \tag{2.8 b}$$

$$\omega_{ud}(k) = \int_0^{T_0} \exp((\tilde{A}(T_0 - \sigma))\, \tilde{B}_1 \tilde{\omega}_u(\sigma + kT_0)\, d\sigma \tag{2.8 c}$$

The process $\omega_{ud}(k)$ is zero mean and white with

$$E\{\omega_{ud}(k)\omega'_{ud}(l)\} = \int_0^{T_0} \exp(\tilde{A}\tau)\, \tilde{B}_1 \tilde{\Omega}_u \tilde{B}'_1 \exp(\tilde{A}'\tau)\, \delta_{kl}\, d\tau \geqslant 0 \tag{2.9 a}$$

$$= \hat{Q}\delta_{kl} \tag{2.9 b}$$

There is no output noise here. This is because we are performing ideal sampling of a continuous-time model with no input noise.

2.3. *Equivalent MROC model of augmented system*

Before describing the model when noise is present, we recall the construction applicable in the noiseless case. This construction is applied to the system (2.3) and it is crucial (and therefore assumed) that this system is observable. Let us therefore note the following simple result.

Lemma 2.1: *Consider the systems* (2.1) *and* (2.7) *with the pairs* (A, C) *and* $(\tilde{A}, \tilde{C})$ *related as in* (2.4). *Then* (A, C) *is observable if and only if* $(\tilde{A}, \tilde{C})$ *is observable.*

Proof: Define

$$O(C, A) \triangleq [C' \quad A'C' \quad \dots \quad A'^{(n-1)}C']$$

$$O_a(\tilde{C}, \tilde{A}) \triangleq [\tilde{C}' \quad \tilde{A}'\tilde{C}' \quad \dots \quad \tilde{A}'^{(n-1)}\tilde{C}']$$

then substituting $\tilde{C}$ and $\tilde{A}$ into $O_a(\tilde{C}, \tilde{A})$ gives

$$O_a(\tilde{C}, \tilde{A}) = \begin{bmatrix} 0 & \alpha C' & \alpha A'C' - \alpha^2 C' & \dots & \alpha A'^{(n+p-2)}C' - \alpha^2 A'^{(n+p-3)}C' + \dots + (-1)^{(n+p-2)}\alpha^{n+p-1}C' \\ I - \alpha I & \alpha^2 I & & \dots & (-1)^{(n+p-1)}\alpha^{(n+p-1)}I \end{bmatrix}$$

Next, elementary column operations give

$$O_a(\tilde{C}, \tilde{A}) = \begin{bmatrix} 0 & \alpha C' & \alpha A'C' & \dots & \alpha A'^{(n+p-2)}C' \\ I & 0 & 0 & \dots & 0 \end{bmatrix}$$

It is then easy to see that

$$\operatorname{rank}\{O_a(\tilde{C}, \tilde{A})\} = n + p \Leftrightarrow \operatorname{rank}\{O(C, A)\} = n$$

i.e.

$$(C, A) \text{ is observable} \Leftrightarrow (\tilde{C}, \tilde{A}) \text{ is observable.} \qquad \square$$

Sampling rates for the different entries of the output $\tilde{y}(t)$ of (2.3 *b*) need to be defined. Let us suppose that the ith entry of $y(t)$ is sampled N_i times in each interval T_0, i.e. its sampling interval is $T_i = T_0/N_i$. Every T_0 seconds, the input changes and all outputs are synchronously sampled; additional output samples are taken before another T_0 seconds elapses. Further $\sum N_i \geqslant n$.

It follows that, in the noiseless case,

$$\tilde{y}_i(kT_0 + jT_i) = \tilde{c}_i \exp(j\tilde{A}T_i)\,\tilde{x}(kT_0) + \tilde{c}_i \int_0^{jT_i} \exp(\tilde{A}\sigma)\,\tilde{B}\,\mathrm{d}\sigma u_\mathrm{d}(k) \tag{2.10}$$

where $\tilde{c}_i$ is the ith row of $\tilde{C}$, $i = 1, 2, \dots, p$ and $j = 1, 2, \dots, (N_i - 1)$.

Since in the noiseless case

$$\tilde{x}_\mathrm{d}(k+1) = F\tilde{x}_\mathrm{d}(k) + Gu_\mathrm{d}(k) \tag{2.11}$$

with F invertible, one can express all the output samples collected in $[kT_0, (k+1)T_0)$ as a linear combination not of $\tilde{x}(kT_0)$ and $u_\mathrm{d}(k)$, but of $\tilde{x}[(k+1)T_0] = \tilde{x}_\mathrm{d}(k+1)$ and $u_\mathrm{d}(k)$ as in (2.10), so that

$$\tilde{y}_\mathrm{d}(k) = \hat{C}\tilde{x}_\mathrm{d}(k+1) + \hat{G}u_\mathrm{d}(k) \tag{2.12}$$

$$\tilde{y}_d(k) = \begin{bmatrix} \tilde{y}_1(kT_0) \\ \vdots \\ \tilde{y}_1(kT_0 + \overline{N_1 - 1}T_1) \\ \vdots \\ \tilde{y}_p(kT_0) \\ \vdots \\ \tilde{y}_p(kT_0 + \overline{N_p - 1}T_p) \end{bmatrix} \tag{2.13 a}$$

$$\hat{C} = \begin{bmatrix} \tilde{c}_1 \exp(-\tilde{A}N_1 T_1) \\ \vdots \\ \tilde{c}_1 \exp(-\tilde{A}T_1) \\ \vdots \\ \tilde{c}_p \exp(-\tilde{A}N_p T_p) \\ \vdots \\ \tilde{c}_p \exp(-\tilde{A}T_p) \end{bmatrix} \tag{2.13 b}$$

$$\hat{G} = \begin{bmatrix} \tilde{c}_1 \int_0^{-N_1 T_1} \exp(\tilde{A}t)\, \tilde{B}\, dt \\ \vdots \\ \tilde{c}_1 \int_0^{-T_1} \exp(\tilde{A}t)\, \tilde{B}\, dt \\ \vdots \\ \tilde{c}_p \int_0^{-N_p T_p} \exp(\tilde{A}t)\, \tilde{B}\, dt \\ \vdots \\ \tilde{c}_p \int_0^{-T_p} \exp(\tilde{A}t)\, \tilde{B}\, dt \end{bmatrix} \tag{2.13 c}$$

The reason for obtaining (2.12) is that it serves as the basis for defining a feedback law; this point is reviewed subsequently. Note that the observability assumption implies that $\hat{C}$ has full column rank, i.e. has a left inverse (see Hagiwara and Araki 1988).

Now, we aim to indicate the changes to (2.11) and (2.12) when noise is included. The change to (2.11) has already been recorded in (2.7 *a*). To obtain the change to (2.12), consider first the variation to (2.10). It easily follows from (2.3) that

$$\begin{aligned} \tilde{y}_i(kT_0 + jT_i) &= \tilde{c}_i \exp(j\tilde{A}T_i)\, \tilde{x}(kT_0) + \tilde{c}_i \int_0^{jT_i} \exp(\tilde{A}\sigma)\, \tilde{B}\, d\sigma u_d(k) \\ &\quad + \tilde{c}_i \int_{kT_0}^{kT_0 + jT_i} \exp\{\tilde{A}(kT_0 + jT_i - \sigma)\}\, \tilde{B}_1 \tilde{\omega}_u(\sigma)\, d\sigma \end{aligned} \tag{2.14}$$

The third summand reflects the noise. However, an additional noise term enters when we seek to replace $\tilde{x}(kT_0)$ in (2.14) using (2.7 *a*):

$$\tilde{x}(kT_0) = F^{-1}\tilde{x}_d(k+1) - F^{-1}Gu_d(k) - F^{-1}\int_0^{T_0} \exp\{\tilde{A}(T_0-\sigma)\}\,\tilde{B}_1\tilde{\omega}_u(\sigma + kT_0)\,d\sigma \tag{2.15}$$

The overall result is

$$\tilde{y}_d(k) = \hat{C}\tilde{x}_d(k+1) + \hat{G}u_d(k) + \omega_{\tilde{y}}(k) \tag{2.16}$$

where

$$\omega_{\tilde{y}}(k) = \begin{bmatrix} \tilde{c}_1 \displaystyle\int_0^{-N_1T_1} \exp(\tilde{A}t)\,\tilde{B}_1\tilde{\omega}_u(kT_0 + T_1 - t)\,dt \\ \vdots \\ \tilde{c}_1 \displaystyle\int_0^{-T_1} \exp(\tilde{A}t)\,\tilde{B}_1\tilde{\omega}_u(kT_0 + (N_1-1)T_1 - t)\,dt \\ \vdots \\ \tilde{c}_p \displaystyle\int_0^{-N_pT_p} \exp(\tilde{A}t)\,\tilde{B}_1\tilde{\omega}_u(kT_0 + T_p - t)\,dt \\ \vdots \\ \tilde{c}_p \displaystyle\int_0^{-T_p} \exp(\tilde{A}t)\,\tilde{B}_1\tilde{\omega}_u(kT_0 + (N_p-1)T_p - t)\,dt \end{bmatrix} \tag{2.17}$$

2.4. *Comparison of feedback law implementation*

Suppose that our desire is to implement, to the best extent possible, a discrete-time feedback law

$$u_d(k) = -L\tilde{x}_d(k) \tag{2.18}$$

(which may be derived by minimizing a linear-quadratic law). With the conventional discrete time system, (2.18) is replaced by either

$$u_d(k) = -L\hat{\tilde{x}}_d(k/k) \tag{2.19}$$

or

$$u_d(k) = -L\hat{\tilde{x}}_d(k/k-1) \tag{2.20}$$

where $\hat{\tilde{x}}_d(k/k)$ and $\hat{\tilde{x}}_d(k/k-1)$ are the true filtered estimates, and one-step ahead predicted estimate of $\tilde{x}_d(k)$, generated using a Kalman filter.

To understand the arrangement for the MROC case with noise, we recall first the noiseless case of Hagiwara and Araki (1988). A controller of the form

$$u_d(k+1) = Mu_d(k) - H\tilde{y}_d(k) \tag{2.21}$$

is adopted. Note that it is causal (in fact strictly causal). In the light of (2.12), this means that

$$u_d(k+1) = Mu_d(k) - H\hat{C}\tilde{x}_d(k+1) - H\hat{G}u_d(k) \tag{2.22}$$

Accordingly, the choice of H so that

$$H\hat{C} = L \tag{2.23}$$

(i.e. $H = L\hat{C}^{-L}$, with $\hat{C}^{-L}$ a left inverse for $\hat{C}$) and then M so that

$$M = H\hat{G} \tag{2.24}$$

yields (2.18) (with time index adjusted by 1).

In the noisy case, since obviously we need to work with measurable quantities, we adopt the same controller (2.21), with (2.23) and (2.24) still determining H and M. Now, however, (2.22) does not hold. Rather, from (2.16), what is actually implemented, is

$$\begin{aligned} u_d(k+1) &= Mu_d(k) - H\hat{C}\tilde{x}_d(k+1) - H\hat{G}u_d(k) - H\omega_{\bar{y}}(k) \\ &= -L\tilde{x}_d(k+1) - H\omega_{\bar{y}}(k) \end{aligned} \tag{2.25}$$

Evidently, noise is perturbing the correct feedback signal, so that, as with the conventional approach of (2.18), the feedback is inexact. The actual error in the feedback signal is however different in (2.21) from that in (2.18).

Note that because $H = L\hat{C}^{-L}$, we can rewrite (2.25) as

$$u_d(k+1) = -L[\tilde{x}_d(k+1) + \hat{C}^{-L}\omega_{\bar{y}}(k)] \tag{2.26}$$

This has the interpretation that we are using an estimate of $\tilde{x}_d(k+1)$ with error $\hat{C}^{-L}\omega_{\bar{y}}(k)$, as opposed to $\tilde{x}_d(k+1)$ itself, in the linear feedback law.

For the conventional system, the state estimate used in $u_d(k)$ depends on samples of $\bar{y}(t)$ at times T_0 apart, i.e. on either $\dots, \bar{y}[(k-j)T_0], \bar{y}[(k-j+1)T_0], \dots, \bar{y}[(k-1)T_0]$ or on $\dots, \bar{y}[(k-j)T_0], \bar{y}[(k-j+1)T_0], \dots, \bar{y}[(k-1)T_0], \bar{y}(kT_0)$.

For the MROC system, in effect we are also using a state estimate, computed using a different set of measurements. If $N_1 = N_2 = \dots = N$, these are $\bar{y}[(k-1)T_0], \bar{y}[(k-1+(1/N))T_0], \dots, \bar{y}[(k-1+(N-1/N))T_0]$. This latter estimate is not necessarily optimal.

There is no obvious comparison of the quality of the estimates, since it is not the case that one set of measurements includes the other.

3. Performance comparison

In order to compare the performance of the non-MROC and MROC schemes, we shall use as an underlying criterion an LQG index. Thus we shall suppose that a performance index is prescribed.

$$J = \lim_{N\to\infty} \frac{1}{N} E\left\{\sum_{k=0}^{N-1} [\tilde{x}_d'(k)Q\tilde{x}_d(k) + u_d'(k)Ru_d(k)]\right\} \tag{3.1}$$

This index may arise by discretizing a quadratic performance index associated with the underlying continuous time system (2.1) (in which case, there might arise an additional cross product term $2\tilde{x}_d'(k)M_d u_d(k)$ which can be treated with very minor variations).

The following result is reasonably well known, and is relevant to both the non-MROC and MROC cases, since it depends purely on the state update equation (2.7 *a*), which is common to both. Nevertheless, for completeness, a proof is indicated in the Appendix.

Lemma 3.1: *Consider the discrete-time model* (2.7 *a*), (2.8) *and* (2.9) *and the index*

$$J_N = \frac{1}{N} E \left\{ \sum_{k=0}^{N-1} [\tilde{x}'_d(k) Q \tilde{x}_d(k) + u'_d(k) R u_d(k)] \right\} \tag{3.2}$$

Suppose that the sequences $P(k)$ *and* $L(k)$ *are defined for* $k = N-1, N-2, \ldots, 0$ *by*

$$P(k) = Q + F'P(k+1)F - P^*(k+1) \tag{3.3 a}$$

$$P^*(k+1) = F'P(k+1)G[R + G'P(k+1)G]^{-1}G'P(k+1)F \tag{3.3 b}$$

$$P(N) = 0 \tag{3.3 c}$$

$$L(k) = [R + G'P(k+1)G]^{-1}G'P(k+1)F \tag{3.3 d}$$

Assume further (as is reasonable) that $\tilde{x}_d(0)$ *and, no matter how it is obtained,* $u_d(k)$ *[which is the value assumed by* $u(t)$ *in* (2.1) *at time* $t = kT_0$ *and persisting over* $[kT_0, (k+1)T_0)$*] are independent of* $\omega_{ud}(k)$ *(which depends on the noise input* $\omega_u(t)$ *to* (2.1) *over* $[jT_0, (j+1)T_0]$ *for all* $j \geq k$*). Then,*

$$\begin{aligned} J_N = \frac{1}{N} E \Bigg\{ & \tilde{x}'_d(0) P(0) \tilde{x}_d(0) \\ & + \sum_{k=0}^{N-1} [u_d(k) + L(k)\tilde{x}_d(k)]'[R + G'P(k+1)G][u_d(k) + L(k)\tilde{x}_d(k)] \Bigg\} \\ & + \sum_{k=0}^{N-1} \operatorname{tr}[\hat{Q}P(k+1)] \end{aligned} \tag{3.4}$$

Proof: See the Appendix. □

Now, for the purpose of comparing the different schemes, it is more straightforward to consider time-invariant problems. Accordingly, we shall adopt henceforth the following assumption.

Assumption 1: *The pair* $[F, G]$ *is stabilizable and the pair* $[F, Q^{1/2}]$ *is detectable.*

As is well known, this ensures that when $N \to \infty$ in (3.3), the matrices $P(k)$ and $L(k)$ become independent of k, and $F - GL$ has all eigenvalues inside the unit circle.

Now, let Σ_f and Σ_p denote the optimal error covariance associated with Kalman filter estimates $\hat{x}_d(k/k)$ and $\hat{x}_d(k/k-1)$ of $x_d(k)$. When such estimates are used in place of $x_d(k)$ in the optimal feedback law, $u_d(k)$ fulfils the independence requirement cited in the lemma statement.

From (3.4), it follows that the optimal performance index associated with LQG law *II* is

$$\begin{aligned} J_f &= E\{[\tilde{x}_d(k) - \hat{\tilde{x}}_d(k/k)]'L'[R + G'PG]L[\tilde{x}_d(k) - \hat{\tilde{x}}_d(k/k)]\} + \operatorname{tr}[\hat{Q}P] \\ &= \operatorname{tr}[\Sigma_f L'(R + G'PG)L] + \operatorname{tr}[\hat{Q}P] \\ &= \operatorname{tr}[\Sigma_f P^*] + \operatorname{tr}[\hat{Q}P] \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} P^* &= F'PG[R + G'PG]^{-1}G'PF \\ &= L'[R + G'PG]L \end{aligned} \tag{3.6}$$

Similarly, the optimal performance index associated with LQG law I is given by

$$J_p = \operatorname{tr}[\Sigma_p P^*] + \operatorname{tr}[\hat{Q}P] \tag{3.7}$$

Next, consider the cost of using MROC control. Observe first that $\omega_{\tilde{y}}(k)$ depends on $\tilde{\omega}_u(t)$ for values of t in the interval $[kT_0, (k+1)T_0)$; since by (2.25), $u_d(k)$ depends on $\omega_{\tilde{y}}(k-1)$, this ensures that $u_d(k)$ possesses the independence property of the lemma statement. Suppose that

$$E[\omega_{\tilde{y}}(k)\omega'_{\tilde{y}}(k)] = \hat{R} \tag{3.8}$$

Then, by (2.25) and (3.4), we have

$$\begin{aligned} J_{\text{MROC}} &= E\{\omega'_{\tilde{y}}(k-1)(\hat{C}^{-L})'L'[R+G'PG]L\hat{C}^{-L}\omega_{\tilde{y}}(k-1)\} + \operatorname{tr}[\hat{Q}P] \\ &= \operatorname{tr}[\hat{C}^{-L}\hat{R}(\hat{C}^{-L})'P^*] + \operatorname{tr}[\hat{Q}P] \end{aligned} \tag{3.9}$$

Consequently, despite the very different pattern in MROC control, *the performance is ultimately still dependent on the quality of the estimate of the state.* This is measured, for the three controllers considered, by Σ_f, Σ_p and $\hat{C}^{-L}\hat{R}(\hat{C}^{-L})'$, respectively.

It therefore remains to obtain a feel for how these quantities are likely to compare in typical situations.

4. An example

To illustrate the ideas presented, we provide a performance comparison between the three controllers for control of the altitude of a single-axis satellite altitude given by Franklin *et al.* (1990). The equation of motion of the system can be represented by the following state-space equation:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = [1 \quad 0]$$

which is basically a double-integrator plant.

The output of the system is the angle of the satellite axis with respect to an 'inertial' reference. Appropriate process noise intensity Ω_u, measurement noise intensity Ω_y and the weighting matrices Q_c and R_c include the following values:

$$\Omega_u = \begin{bmatrix} 0{\cdot}1 & 0 \\ 0 & 0{\cdot}1 \end{bmatrix}, \quad \Omega_y = 0{\cdot}1, \quad Q_c = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad R_c = 0{\cdot}2$$

respectively. The frame period, T_0 of 0·5 is selected according to the guidelines of T_0 for proper operation of MROCs given by the authors previously (Er and Anderson 1990) and the recommended T_0 given by Åström and Wittenmark (1990), Franklin *et al.* (1990) and Middleton and Goodwin (1990).

It is easy to see that the augmented system is given by

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \alpha & 0 & -\alpha \end{bmatrix}; \quad \tilde{B} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \tilde{C} = [0 \quad 1 \quad 0]$$

where $\alpha = \omega_s/2$. Since the augmented system has an observability index of three, we let the output multiplicity be $N_1 = 3$. Further, since $\operatorname{tr}[\hat{Q}P]$ is common to the performance indices associated with the three controllers, it suffices to compare $\operatorname{tr}[\Sigma_f P^*]$, $\operatorname{tr}[\Sigma_p P^*]$ and $\operatorname{tr}[\hat{C}^{-L}\hat{R}(\hat{C}^{-L})'P^*]$ provided the value of $\operatorname{tr}[\hat{Q}P]$ does *not* swamp the values of $\operatorname{tr}[\Sigma_f P^*]$, etc.

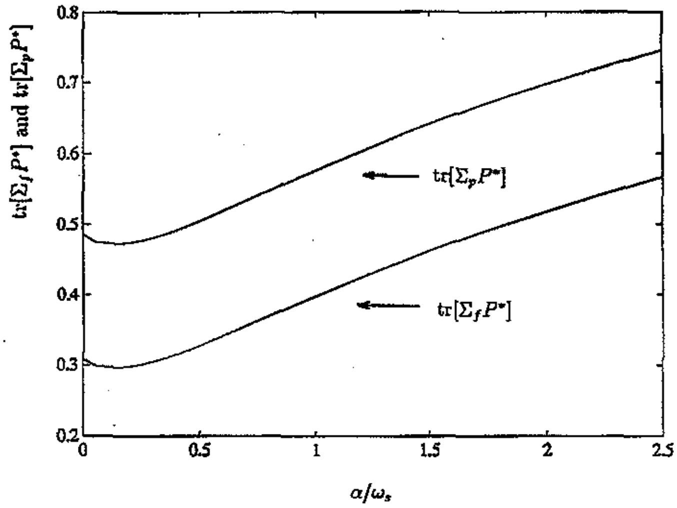

Figure 4.1. tr [$\Sigma_f P^*$] and tr [$\Sigma_p P^*$] versus α/ω_s.

Figure 4.1 shows the variation of tr [$\Sigma_f P^*$] and tr [$\Sigma_p P^*$] as a function of α/ω_s when LQG law I and LQG law II are applied to the plant respectively. Observe that the trace values reach a minimum value at $\alpha/\omega_s \simeq 0{\cdot}2$. The reason why the trace values are increasing when α/ω_s becomes smaller than 0·2 is that measurement becomes poorer when α/ω_s gets smaller since the AAF bandwidth becomes sufficiently narrow that significant distortion is introduced. The implication of this is that the entries of the error covariance matrix Σ becomes much bigger as α/ω_s becomes smaller. On the other hand, the reason that the trace values are increasing when α/ω_s gets bigger than 0·2 is that more noise passes through the AAF as α/ω_s increases. This also leads to an increase in the entries of Σ. That tr [$\Sigma_f P^*$] is smaller than tr [$\Sigma_p P^*$] is expected. In this example, the value of tr [$\hat{Q}P$] does *not* swamp the values of tr [$\Sigma_f P^*$], etc.

Figure 4.2 shows the variation of tr [$\hat{C}^{-L}\hat{R}(\hat{C}^{-L})'P^*$] as a function of α/ω_s when the MROC law is implemented with $N_1 = 3$. The values of tr [$\Sigma_f P^*$] and tr [$\Sigma_p P^*$] are also shown in the graph for comparison. It is clear from the graph that the MROC law performs worse than LQG law I and LQG law II for all values of α. The reason for the minimum trace value when the MROC law is applied is the same as that for the LQG law.

5. Conclusion

In this paper, we have identified a drawback of the MROC under noise disturbances. We have shown that in a typical situation the MROC performs significantly worse than an LQG controller with performance measured by an LQG index. It is remarkable to note that despite the fact that the MROC has the advantage of sampling the output more frequently than LQG law I, the latter still outperforms the MROC law by an enormous magnitude in typical situation. This suggests that for practical purposes, the MROC law proposed by Hagiwara and Araki (1988) will often need to be modified in one way or another so that the performance of MROC can be significantly enhanced.

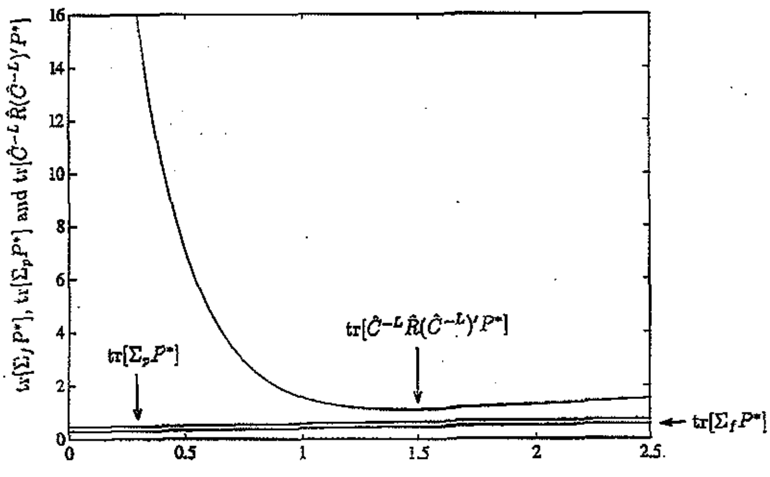

Figure 4.2. $\mathrm{tr}\,[\Sigma_f P^*]$, $\mathrm{tr}\,[\Sigma_p P^*]$ and $\mathrm{tr}\,[\hat{C}^{-L}\hat{R}(\hat{C}^{-L})'P^*]$ versus α/ω_s.

Appendix

Proof of Lemma 3.1

It is known that the Riccati equation associated with the state equation (2.7 *a*) and performance index (3.2) is given by

$$P(k) = Q + F'P(k+1)F - P^*(k+1) \tag{A 1 a}$$

$$P^*(k+1) = F'P(k+1)G[R + G'P(k+1)G]^{-1}G'P(k+1)F \tag{A 1 b}$$

$$P(N) = 0 \tag{A 1 c}$$

$$L(k) = [R + G'P(k+1)G]^{-1}G'P(k+1)F \tag{A 1 d}$$

Using (A 1 *d*) in (A 1 *b*) yields

$$P^*(k+1) = L'(k)[R + G'P(k+1)G]L(k) \tag{A 2}$$

We first compute the difference between $\tilde{x}'_d(k+1)P(k+1)\tilde{x}_d(k+1)$ and $\tilde{x}'_d(k)P(k)\tilde{x}_d(x)$. Using (2.7 *a*) in $\tilde{x}'_d(k+1)P(k+1)\tilde{x}_d(k+1)$, we have $\tilde{x}'_d(k+1)P(k+1)\tilde{x}_d(k+1)$

$$\begin{aligned} &= [F\tilde{x}_d(k) + Gu_d(k) + \omega_{ud}(k)]'P(k+1)[F\tilde{x}_d(k) + Gu_d(k) + \omega_{ud}(k)] \\ &= [F\tilde{x}_d(k) + Gu_d(k)]'P(k+1)[F\tilde{x}_d(k) + Gu_d(k)] \\ &\quad + [F\tilde{x}_d(k) + Gu_d(k)]'P(k+1)\omega_{ud}(k) \\ &\quad + \omega'_{ud}(k)P(k+1)[F\tilde{x}_d(k) + Gu_d(k)] + \omega'_{ud}(k)P(k+1)\omega_{ud}(k) \end{aligned} \tag{A 3}$$

Also, by (A 1 *a*), there holds $\tilde{x}'_d(k)P(k)\tilde{x}_d(k)$

$$= \tilde{x}'_d(k)[Q + F'P(k+1)F - P^*(k+1)]\tilde{x}_d(x) \tag{A 4}$$

Therefore, $\tilde{x}'_d(k+1)P(k+1)\tilde{x}_d(k+1) - \tilde{x}'_d(k)P(k)\tilde{x}_d(k)$

$$\begin{aligned} &= u'_d(k)G'P(k+1)Gu_d(k) + u'_d(k)G'P(k+1)F\tilde{x}_d(k) \\ &\quad + \tilde{x}'_d(k)F'P(k+1)Gu_d(k) + [u'_d(k)Ru_d(k) - u'_d(k)Ru_d(k)] \\ &\quad - \tilde{x}'_d(k)Q\tilde{x}_d(k) + \tilde{x}'_d(k)P^*(k+1)\tilde{x}_d(k) + \omega'_{ud}(k)P(k+1)\omega_{ud}(k) \\ &\quad + [F\tilde{x}_d(k) + Gu_d(k)]'P(k+1)\omega_{ud}(k) \\ &\quad + \omega'_{ud}(k)P(k+1)[F\tilde{x}_d(k) + Gu_d(k)] \end{aligned} \tag{A 5}$$

Straightforward rearrangement yields

$$\begin{aligned} &\tilde{x}'_d(k)Q\tilde{x}_d(k) + u'_d(k)Ru_d(k) + \tilde{x}'_d(k+1)P(k+1)\tilde{x}_d(k+1) - \tilde{x}'_d(k)P(k)\tilde{x}_d(k) \\ &\quad = [u_d(k) + L(k)\tilde{x}_d(k)]'[R + G'P(k+1)G][u_d(k) + L(k)\tilde{x}_d(k)] \\ &\qquad + \omega'_{ud}(k)P(k+1)\omega_{ud}(k) \\ &\qquad + [F\tilde{x}_d(k) + Gu_d(k)]'P(k+1)\omega_{ud}(k) \\ &\qquad + \omega'_{ud}(k)P(k+1)[F\tilde{x}_d(k) + Gu_d(\mathrm{k})] \end{aligned} \tag{A 6}$$

Next, observe that because $P(N) = 0$ by (A 1 *c*)

$$\begin{aligned} 0 &= \tilde{x}'_d(N)P(N)\tilde{x}_d(N) \\ &= \tilde{x}'_d(0)P(0)\tilde{x}_d(0) \\ &\quad + \sum_{k=0}^{N-1} [\tilde{x}'_d(k+1)P(k+1)\tilde{x}_d(k+1) \\ &\quad - \tilde{x}'_d(k)P(k)\tilde{x}_d(k)] \end{aligned} \tag{A 7}$$

Hence, J_N can be rewritten as

$$\begin{aligned} J_N = \frac{1}{N} E \Bigg\{ &\tilde{x}'_d(0)P(0)\tilde{x}_d(0) \\ &+ \sum_{k=0}^{N-1} [\tilde{x}'_d(k)Q\tilde{x}_d(k) + u'_d(k)Ru_d(k) \\ &+ \tilde{x}'_d(k+1)P(k+1)\tilde{x}_d(k+1) - \tilde{x}'_d(k)P(k)\tilde{x}_d(k)] \Bigg\} \end{aligned} \tag{A 8}$$

Using (A 6) in (A 8) yields

$$\begin{aligned} J_N = \frac{1}{N} \Bigg\{ &\tilde{x}'_d(0)P(0)\tilde{x}_d(0) \\ &+ \sum_{k=0}^{N-1} \{[u_d(k) + L(k)\tilde{x}_d(k)]'[R + G'P(k+1)G][u_d(k) + L(k)\tilde{x}_d(k)] \\ &+ \omega'_{ud}(k)P(k+1)\omega_{ud}(k)\} + 2[F\tilde{x}_d(k) + Gu_d(k)]'P(k+1)\omega_{ud}(k) \Bigg\} \end{aligned} \tag{A 9}$$

Because $\tilde{x}_d(0)$, $u_d(j)$ and $\omega_{ud}(j)$ are independent of $\tilde{x}_d(k)$ for all $j < k$, $\tilde{x}_d(k)$ is independent of $\omega_{ud}(k)$. Also, $u_d(k)$ is independent of $\omega_{ud}(k)$. Hence, the last summand has zero expectation. Consequently,

$$
\begin{aligned}
J_N = \frac{1}{N} E \Bigg\{ & \tilde{x}_d'(0) P(0) \tilde{x}_d(0) \\
& + \sum_{k=0}^{N-1} [u_d(k) + L(k)\tilde{x}_d(k)]' [R + G'P(k+1)G] [u_d(k) + L(k)\tilde{x}_d(k)] \Bigg\} \\
& + \sum_{k=0}^{N-1} \operatorname{tr} [\hat{Q} P(k+1)]
\end{aligned} \tag{A 10}
$$

References

ANDERSON, B. D. O., and MOORE, J. B., 1981, Time-varying feedback laws for decentralised control. *IEEE Transactions on Automatic Control*, **26**, 1133–1138.

ARAKI, M., and HAGIWARA, T., 1985, Pole assignment by multirate sampled-data output feedback. *Proceedings of the 24th IEEE Conference on Decision Control*, pp. 189–193; 1986, Pole assignment by multirate sampled-data output feedback. *International Journal of Control*, **44**, 1661–1673.

ARAKI, M., and YAMAMOTO, K., 1986, Multivariable multirate sampled-data systems: State-space description, transfer characteristics, and Nyquist criterion. *IEEE Transactions on Automatic Control*, **33**, 145–154.

ÅSTRÖM, K. J., and WITTENMARK, B., 1990, *Computer-controlled Systems-theory and Design*, second edition (Englewood Cliffs, NJ: Prentice Hall).

BERG, M. C., AMIT, N., and POWELL, J., 1988, Multirate digital control system design. *IEEE Transactions on Automatic Control*, **33**, 1139–1150.

BOYKIN, W. H., and FRAZIER, B. D., 1975, Analysis of multiloop multirate sampled-data systems. *AIAA Journal*, **13**, 453–456.

CHAMMAS, A. B., and LEONDES, C. T., 1978, On the design of linear time-invariant systems by periodic output feedback. Parts I and II. *International Journal of Control*, **27**, 885–903; 1979 a, Pole assignment by piecewise constant output feedback. *Ibid.*, **29**, 31–38; 1979 b, On the finite time control of linear systems by piecewise constant output feedback. *Ibid.*, **30**, 227–234.

ER, M. J., and ANDERSON, B. D. O., 1991, Practical issues in multirate output controllers. *International Journal of Control*, **53**, 1005–1020.

FRANCIS, B. A., and GEORGIOU, T. T., 1988, Stability theory for linear time-invariant plants with periodic digital controllers. *IEEE Transactions on Automatic Control*, **33**, 820–832.

FRANKLIN, G. F., and POWELL, J. D., 1980, *Digital Control of Dynamic Systems* (Reading, Mass: Addison-Wesley).

FRANKLIN, G. F., POWELL, J. D., and WORKMAN, M. L., 1990, *Digital Control of Dynamic Systems*, second edition (Reading, Mass: Addison-Wesley).

GLASSON, D. P., 1983, Development and application of multirate digital control. *Control Systems Magazine*, **1**–**8**.

GRESCHAK, J. P., and VERGHESE, G. C., 1982, Periodically varying compensation of time-invariant systems. *Systems and Control Letters*, **2**, 88–93.

HAGIWARA, T., and ARAKI, M., 1988, Design of a stable state feedback controller based on the multirate sampling of the plant output. *IEEE Transactions on Automatic Control*, **33**, 812–819.

JURY, E. I., 1958, *Sampled Data Control Systems* (New York: Wiley).

KALMAN, R. E., and BERTAM, J. E., 1959, A unified approach to the theory of sampling systems. *Journal of the Franklin Institute*, **267**, 405–436.

KHARGONEKAR, P. P., POOLLA, K., and TANNENBAUM, A., 1985, Robust control of linear time-invariant plants using periodic compensation. *IEEE Transactions on Automatic Control*, **30** (11).

KRANC, G. M., 1957, Input–output analysis of multirate feedback systems. *IRE Transactions on Automatic Control*, **3**, 21–28.

MEYER, R. A., and BURRUS, C. S., 1975, A unified analysis of multirate and periodically time-invarying digital filters. *IEEE Transactions on Circuits and Systems*, **22**, 162–168.

MIDDLETON, R. H., and GOODWIN, G. C., 1990, *Digital Control and Estimation—a Unified Approach* (Englewood Cliffs, NJ: Prentice Hall).
MITA, T., PANG, B. C., and LIU, K. Z., 1987, Design of optimal strongly stable digital control systems and application to output feedback control of mechanical systems. *International Journal of Control*, **45**, 2071–2082.
OGATA, K., 1987, *Discrete-Time Control Systems* (Englewood Cliffs, NJ: Prentice-Hall).
RAGAZZINI, J. R., and FRANKLIN, G. F., 1958, *Sampled-Data Control Systems* (New York: McGraw-Hill).