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TIME-VARYING VERSION OF THE LEMMA OF LYAPUNOV

The lemma of Lyapunov gives the necessary and sufficient condition for the stability of a time-invariant system in terms of the existence of a positive-definite symmetric matrix. In this letter, the lemma is generalised for time-varying systems, in which it is shown that additional conditions are required for the lemma to hold. The conditions involve boundedness of certain quantities and uniform complete observability of a pair of matrixes.

This letter generalises, for time-varying systems, the well known lemma of Lyapunov, which provides a necessary and sufficient condition for the stability of the system

$$\dot{x} = Fx \qquad (1)$$

the matrix F being constant. This lemma, in the modified form of Kalman,[1] is as follows:

Lemma: Consider the system of eqn. 1, and let L be a matrix such that $[F, L]$ is completely observable.[2] Then F is asymptotically stable (i.e. all the eigenvalues of F have negative real parts) if, and only if, there exists a positive-definite symmetric matrix P such that

$$PF + F'P = -L'L \qquad (2)$$

Moreover, if F is stable, P is given by

$$P = \int_0^\infty e^{F't}L'Le^{Ft}dt \qquad (3)$$

The quadratic expression

$$V(x) = x'(t)Px(t) \qquad (4)$$

is a Lyapunov function for the system of eqn. 1, and, although $\dot{V}$ is not negative-definite in general, being given by

$$\dot{V} = -x'L'Lx \qquad (5)$$

the complete observability of $[F, L]$ is sufficient to ensure that along any trajectory of the system of eqn. 1, V is strictly decreasing in any nonzero interval.

The generalisation of the lemma to time-varying systems, which permits singular L matrixes to be considered (in contrast to Reference 3, in which L is nonsingular), involves the use of the concept of uniform complete observability.[4]

Denoting the transition matrix associated with the system of eqn. 1 by $\Phi(t, \tau)$, where F is now permitted to be time varying, and using the notation $X > Y[X \geqslant Y]$ to mean that $X - Y$ is positive-(semi)definite, the pair $[F, L]$ is said to be uniformly completely observable if two of the following three conditions hold:

(*a*) $$||\Phi(t, \tau)|| \leqslant \alpha_1(|t-\tau|) \qquad (6)$$

with

$$M(t, t+\sigma) = \int_t^{t+\sigma} \Phi'(\lambda, t)L'(\lambda)L(\lambda)\Phi(\lambda, t)d\lambda \qquad (7)$$

(*b*) $$0 < \alpha_2(\sigma)I \leqslant M(t, t+\sigma) \leqslant \alpha_3(\sigma)I < \infty \text{ for some } \sigma \qquad (8)$$

(*c*) $$0 < \alpha_4(\sigma)I \leqslant \Phi'(t, t+\sigma)M(t, t+\sigma)\Phi(t, t+\sigma) \leqslant \alpha_5(\sigma)I < \infty \text{ for some } \sigma \qquad (9)$$

If any two of (*a*), (*b*) and (*c*) hold, the third is implied.[4]

The checking of uniform complete observability, given F and L, will, in general, not be straightforward. However, if F and L have certain particular structures, uniform complete observability obtains, and expressions 6, 8 and 9 need not be checked.

In the sequel, F will be restricted to being bounded, i.e.

$$||F|| \leqslant K_1 \quad \text{for all } t \qquad (10)$$

for some positive constant K_1. This is a well known condition guaranteeing expression 6. A second consequence of expression 10 is that if the system of eqn. 1 is asymptotically stable, in the sense that

$$\Phi(t, t_0)x_0 \to 0 \quad \text{as } t \to \infty \qquad (11)$$

for all x_0, t_0, eqn. 1 is exponentially asymptotically stable' . for all t, τ with $t \geqslant \tau$:

$$||\Phi(t, \tau)|| \leqslant K_2 \exp -K_3(t-\tau) \qquad (12)$$

for some positive constants K_2, K_3. See Reference 3 for proof.

The main result can now be stated.

Theorem: Consider the system of eqn. 1, and let L be a matrix such that $[F, L]$ is uniformly completely observable. Suppose further that F and L are bounded. It follows that:

(i) If F is asymptotically stable, there exists a matrix P defined by

$$P(t) = \lim_{T\to\infty} \Pi(t, T) \qquad (13)$$

where $\Pi(t, T)$ in turn is defined by

$$-\dot{\Pi} = \Pi F + F'\Pi + L'L \qquad (14a)$$

with boundary condition

$$\Pi(T, T) = 0 \qquad (14b)$$

Moreover,

$$V(x, t) = x(t)P(t)x(t) \qquad (15)$$

is a Lyapunov function for eqn. 1, and finally P is given by

$$P(t) = \lim_{T\to\infty}\int_t^T \Phi'(\lambda, t)L'(\lambda)L(\lambda)\Phi(\lambda, t)d\lambda \qquad (16)$$

(ii) If there exists a matrix P and positive constants β_1 and β_2 such that, for all t,

$$0 < \beta_1 I \leqslant P(t) \leqslant \beta_2 I < \infty \qquad (17)$$

and such that

$$-\dot{P} = PF + F'P + L'L \qquad (18)$$

F is asymptotically stable and $V = x'Px$ is a Lyapunov function.

Proof of (i): Eqn. 14*a*, with the boundary condition (eqn. 14*b*), possesses a solution which can readily be verified to be

$$\Pi(t, T) = \int_t^T \Phi'(\lambda, t)L'(\lambda)L(\lambda)\Phi(\lambda, t)d\lambda \qquad (19)$$

Since F is asymptotically stable, expression 12 holds. This and the boundedness of L thus imply the existence of both the limit

$$P(t) = \lim_{T\to\infty} \Pi(t, T) \quad . . . . (13)$$

and eqn. 16, together with the inequality

$$P(t) \leqslant \beta_2 I < \infty \quad . . . . (20)$$

This is a necessary requirement for V to be a Lyapunov function (Reference 3). The requirement

$$P(t) \geqslant \beta_1 I > 0 \quad . . . . (21)$$

follows from eqn. 8 and the observation that

$$\int_t^{t+\sigma} \Phi'(\lambda, t) L'(\lambda) L(\lambda) \Phi(\lambda, t) d\lambda \leqslant \int_t^{\infty} \Phi'(\lambda, t) L'(\lambda) L(\lambda) \Phi(\lambda, t) d\lambda \quad (22)$$

because the integrands are nonnegative-definite matrixes. With V as in eqn. 15 and P as in eqn. 16, $\dot{V}$ may be readily evaluated as

$$\dot{V} = -x'L'Lx \quad . . . . (23)$$

which is plainly nonpositive. This proves (*i*).

Proof of (*ii*): Eqn. 17 guarantees that V, as distinct from $\dot{V}$, satisfies the necessary requirements for it to be a Lyapunov function. Eqn. 18 also leads to

$$\dot{V} = -x'L'Lx \quad . . . . (23)$$

which certainly establishes the stability, as distinct from asymptotic stability, of F. To establish asymptotic stability, compute the change in V along a length σ of trajectory. Thus

$$\Delta V \Big|_t^{t+\sigma} = \int_t^{t+\sigma} \dot{V} dt$$

$$= -x'(t) \int_t^{t+\sigma} \phi'(\lambda, t) L'(\lambda) L(\lambda) \Phi(\lambda, t) d\lambda x(t) \quad (24)$$

Using the uniform complete observability condition b, this implies

$$\Delta V \Big|_t^{t+\sigma} < -\alpha_2(\sigma) x'(t) x(t) \quad . . . . (25)$$

where $\alpha_2(\sigma)$ is positive. Simple extensions of standard arguments then establish the asymptotic stability of F.

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