SIAM J. CONTROL AND OPTIMIZATION
Vol. 30, No. 2, pp. 262-283, March 1992

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A GAME THEORETIC APPROACH TO $\mathscr{H}^\infty$ CONTROL FOR TIME-VARYING SYSTEMS*

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Abstract. A representation formula for all controllers that satisfy an $\mathscr{L}^\infty$-type constraint is derived for time-varying systems. It is now known that a formula based on two indefinite algebraic Riccati equations may be found for time-invariant systems over an infinite time support (see [J. C. Doyle et al., IEEE *Trans. Automat. Control*, AC-34 (1989), pp. 831–847]; [K. Glover and J. C. Doyle, *Systems Control Lett.*, 11 (1988), pp. 167–172]; [K. Glover et al., SIAM *J. Control Optim.*, 29 (1991), pp. 283–324]; [M. Green et al., SIAM *J. Control Optim.*, 28 (1990), pp. 1350–1371]; [D. J. N. Limebeer et al., in *Proc.* IEEE *conf. on Decision and Control*, Austin, TX, 1988]; [G. Tadmor, *Math. Control Systems Signal Processing*, 3 (1990), pp. 301–324]). In the time-varying case, two indefinite Riccati differential equations are required. A solution to the design problem exists if these equations have a solution on the optimization interval. The derivation of the representation formula illustrated in this paper makes explicit use of linear quadratic differential game theory and extends the work in [J. C. Doyle et al., IEEE *Trans. Automat. Control*, AC-34 (1989), pp. 831–847] and [G. Tadmor, *Math. Control Systems Signal Processing*, 3 (1990), pp. 301–324]. The game theoretic approach is particularly simple, in that the background mathematics required for the sufficient conditions is little more than standard arguments based on "completing the square."

Key words. $\mathscr{H}^\infty$-optimal control, game theory, indefinite Riccati equations, four-block general distance problems, worst-case design

1. Introduction. Since the early 1980s, numerous techniques have been developed for solving $\mathscr{H}^\infty$-control problems. In addition to their individual interest, further insights have been gained by studying the interplay between these various methodologies, and researchers have now acquired a good understanding of several aspects of the theory. Over the last two years, there has been a flurry of activity that has had a significant impact on both the accessibility of the theoretical ideas and the ease of computation. Once an $\mathscr{H}^\infty$-norm bound has been decided, and provided that a solution exists, the computational burden associated with finding all $\mathscr{H}^\infty$ controllers is essentially the same as that required in solving a linear quadratic Gaussian regulator problem [9], [10], [11], [13], [20], [21], [27]. $\mathscr{H}^\infty$ problems in which perfect information is assumed may be solved using a single Riccati equation with dimension equal to that of the problem [18], [19], [23], while the output-feedback problem requires the solution of two Riccati equations.

The "two Riccati equation" formula for all stabilizing controllers satisfying a closed loop $\mathscr{H}^\infty$-norm constraint has been derived in a number of ways. A state-space solution via the "four-block" problem is reported in [11], [21], while an alternative approach reminiscent of classical linear quadratic theory may be found in [9]. A solution based on J-spectral factorization theory is given in [2], [13], while the related notion of conjugation is used in [20]. An interesting by-product of this activity has

* Received by the editors April 19, 1989; accepted for publication (in revised form) December 13, 1990.

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been the discovery of a number of new interconnections. In [19], [28], the authors note a connection between $\mathcal{H}^\infty$-control and game theory. The interplay between indefinite factorization and game theory, probably first noted by Banker [3], has been rediscovered in the more general setting of $\mathcal{H}^\infty$ control [10], [13], [22]. The connection between risk-sensitive optimal control [10], [30] and game theory, originally discovered by Jacobson in the perfect-information case [17], has also received renewed interest in the wider setting of $\mathcal{H}^\infty$ control [7], [10]. More recently still, Tadmor [27] has given results on the finite-horizon time-varying case using the maximum principle.

The aim of the present paper, which builds on the work given in [9] and [27], is to give a solution to the finite-horizon time-varying $\mathcal{H}^\infty$-control problems, which makes explicit use of the existing theory of linear quadratic games. This approach offers the advantage of introducing game theoretic intuition to the solution path, and also gives a game theoretic interpretation to the change of variable introduced in [9]. We also show that the main infrastructure of a time-varying $\mathcal{H}^\infty$-control theory may be developed using classical arguments based on completing the square. This simple solution is possible because of the linear quadratic (LQ) nature of the problem, and, as a final comment, we mention that we have been able to remove all the removable assumptions in [27].

In this paper we will consider the finite-dimensional linear time-varying plant

$$\begin{bmatrix} z\\ y\end{bmatrix}=\begin{bmatrix}\mathfrak{P}_{11} & \mathfrak{P}_{12}\\ \mathfrak{P}_{21} & \mathfrak{P}_{22}\end{bmatrix}\begin{bmatrix} w\\ u\end{bmatrix}, \tag{1.1a}$$

which has a time-varying state-space realization

$$\begin{array}{c} n\\ p\\ q\end{array}\begin{bmatrix}\dot{x}\\ z\\ y\end{bmatrix}(t)=\begin{array}{c}\begin{array}{ccc} n & \quad l & \quad m\end{array}\\ \begin{bmatrix} A & B_1 & B_2\\ C_1 & D_{11} & D_{12}\\ C_2 & D_{21} & D_{22}\end{bmatrix}\end{array}(t)\begin{bmatrix}x\\ w\\ u\end{bmatrix}(t), \qquad x(0)=0. \tag{1.1b}$$

The class of controls of interest is given by $u=\mathfrak{K}y$, where $\mathfrak{K}$ is a linear time-varying (LTV) controller. Eliminating u and y gives rise to the closed loop operator $z=\mathfrak{R}w$, where $\mathfrak{R}=\mathfrak{P}_{11}+\mathfrak{P}_{12}\mathfrak{K}(I-\mathfrak{P}_{22}\mathfrak{K})^{-1}\mathfrak{P}_{21}$. Our goals are (1) to give necessary and sufficient conditions for the existence of LTV controllers such that $\|z\|_2<\gamma\|w\|_2$ for all $w\neq 0$ and a given γ ($\|\cdot\|_2$ denotes the norm on $\mathcal{L}^2[0, T]$), and (2) to characterize all such controllers when they exist. We assume that the matrices in (1.1) have entries that are continuous functions of time, that D_{12} has full column rank m for all $t\in[0, T]$, and that D_{21} has full row rank q for all $t\in[0, T]$. The game theoretic nature of the problem is immediate: Roughly speaking, the w-player tries to maximize the energy in z, while the controller or u-player simultaneously seeks to minimize it.

Section 2 has a large tutorial component and deals with the time-varying problem in which perfect information is assumed. Section 2.1 establishes the necessary conditions for the existence of a solution to the $\mathcal{H}^\infty$-control problem via a conjugate point argument. We then show that $\mathcal{H}^\infty$-control problems with perfect information may be solved if and only if a solution to the Riccati differential equation (RDE) exists on $[0, T]$. Section 2.2 presents a representation formula for all solutions to the time-varying $\mathcal{H}^\infty$-control problem in the perfect-information case. A brief review of some pertinent properties of adjoint systems is given in § 2.3. Section 2.4 partially reconciles the more familiar time-invariant infinite-horizon $\mathcal{H}^\infty$-control problem with the time-varying finite-horizon case. This reconciliation calls for a connection between the limiting solution of the RDE and its algebraic counterpart.

Section 3 contains the main results of the paper. We begin with an analysis of problems in which both D_{12} and D_{21} are assumed to be square. A solution is found by calculating a plant inverse, and does not require the solution of any Riccati equations. It is interesting that the plant inverse has an observer structure that is indicative of the solution in the general case. Following that, we treat problems in which either D_{12} or D_{21} is square. Every problem of this type requires the solution of a single RDE. Finally, we treat the case in which neither D_{12} nor D_{12} is square, and we show that in these cases two RDEs are required. We give necessary and sufficient conditions for the existence of solutions and characterize all solutions (when they exist).

2. $\mathcal{H}^\infty$ control and linear quadratic differential games. Numerous variants of the linear quadratic differential game have been studied over the last twenty years. As an example of the extensive literature on the subject, we refer the interested reader to Basar and Olsder [4] and the references therein. The purpose of this section is to review those aspects of the existing game theory literature that are relevant to the solution of the time-varying $\mathcal{H}^\infty$-control problem, and to study the connections between the two. We begin by considering a system with input vectors $u(t)$ and $w(t)$, dynamical description

$$\dot{x}(t)=A(t)x(t)+B_1(t)w(t)+B_2(t)u(t), \qquad x(0)=0, \tag{2.1}$$

and output,

$$z(t)=\begin{bmatrix} C(t)x(t) \\ D(t)u(t) \end{bmatrix}. \tag{2.2}$$

We will assume that $D(t)$ has full column rank, and that it has been scaled so that

$$D'(t)D(t)=I. \tag{2.3}$$

Each of the matrices in (2.1) and (2.2) is assumed to have entries that are continuous functions of time. In the interests of notational compactness, this time dependence will not always be shown from now on.

With (2.1)-(2.3) given, the $\mathcal{H}^\infty$-control problem is to find a linear causal control $u(t)=K(x(s), w(s), t)$, $0\leqq s\leqq t$, such that $\|\mathcal{T}_{zw}\|=\sup_w\{\|\mathcal{T}_{zw}w\|_2\colon\ w\in\mathcal{L}^2[0,T],\ \|w\|_2\leqq 1\}<\gamma$ for some given $\gamma>0$. The operator $\mathcal{T}_{zw}$ maps w to z when the control $u(t)=K(\cdot,\cdot,\cdot)$ is in place. If $z=\mathcal{T}_{zw}w$, then $\|\mathcal{T}_{zw}\|<\gamma$ if and only if

$$J(K,w)=\int_0^T (z'z-\gamma^2 w'w)\,dt\leqq -\varepsilon\|w\|_2^2 \tag{2.4}$$

for all $w\in\mathcal{L}^2[0,T]$ and some positive ε.

In the language of game theory, the $\mathcal{H}^\infty$-control problem is a leader-follower game and requires the control system designer to make the first move and select an admissible $u(t)=K(\cdot,\cdot,\cdot)$ that minimizes $J(K,w)$. A control functional $K(\cdot,\cdot,\cdot)$ is admissible if (1) $u(t)\in\mathcal{L}^2[0,T]$ for every $w(t)\in\mathcal{L}^2[0,T]$, and (2) $x(0)=0$ and $w(t)\equiv 0\Rightarrow u(t)\equiv 0$; we denote the set of admissible control functionals $\mathscr{C}$. After the designer's choice of control strategy has been made public, we assume that nature is malicious and selects that $w(t)\in\mathcal{L}^2[0,T]$, which maximizes (2.4). The $\mathcal{H}^\infty$-control problem therefore has a solution if and only if

$$\min_{K\in\mathscr{C}}\ \max_{w\in\mathcal{L}^2[0,T]}\{J(K,w)\}\leqq -\varepsilon\|w\|_2^2 \tag{2.5}$$

is satisfied. In the notation of (2.5), the w-player maximizes $J(K,w)$ over all $w\in\mathcal{L}^2[0,T]$ *after* the u-player has announced a choice of $K\in\mathscr{C}$.

Before studying the minimax problem (2.5) in detail, it is interesting to reconcile it with the classical linear quadratic optimal regulator. If we allow γ to increase without bound, then it is necessarily the case that the energy in $w(t)$ decreases to zero, rendering the w-player impotent. The game thus degenerates to the optimal regulator [1] as γ is increased without bound.

Our first result is based on a standard completion of squares argument [4, p. 290], and shows that the $\mathscr{H}^\infty$-control problem has a solution if a particular RDE has a solution on the time interval $[0, T]$.

THEOREM 2.1. *Suppose that the* RDE

$$-\mathring{P}=PA+A'P-P(B_2B_2'-\gamma^{-2}B_1B_1')P+C'C, \qquad P(T)=0 \tag{2.6}$$

has a solution on $[0, T]$. *Then*

$$u^*=-B_2'Px, \tag{2.7}$$

$$w^*=\gamma^{-2}B_1'Px \tag{2.8}$$

result in

$$\|z\|_2^2-\gamma^2\|w\|_2^2=\|u-u^*\|_2^2-\gamma^2\|w-w^*\|_2^2 \tag{2.9}$$

for any u (either open or closed loop) and $w\in\mathscr{L}^2[0, T]$ *in* (2.1), *noting that* $x(0)=0$. *With* $u=u^*$, *we have*

$$\|\mathscr{T}_{zw}\|<\gamma. \tag{2.10}$$

Proof. Since (2.6) is assumed to have a solution on $[0, T]$ with $P(T)=0$, we have, for any u and w, and $x(0)=0$,

$$\begin{aligned}J(u, w)&=\int_0^T\left\{z'z-\gamma^2w'w+\frac{d}{dt}(x'Px)\right\}dt\\&=\int_0^T\{x'C'Cx+u'u-\gamma^2w'w+(x'A'+w'B_1'+u'B_2')Px\\&\qquad+x'\mathring{P}x+x'P(Ax+B_1w+B_2u)\}\,dt.\end{aligned}$$

Substituting from (2.6) gives us

$$\begin{aligned}J(u, w)&=\int_0^T\left\{x'P(B_2B_2'-\gamma^{-2}B_1B_1')Px+u'u-\gamma^2w'w\right.\\&\qquad\left.+[w' \quad u']\begin{bmatrix}B_1'\\B_2'\end{bmatrix}Px+x'P[B_1 \quad B_2]\begin{bmatrix}w\\u\end{bmatrix}\right\}dt\\&=\int_0^T[u'+x'PB_2][u+B_2'Px]\,dt\\&\qquad-\int_0^T[w'-x'\gamma^{-2}PB_1]\gamma^2[w-\gamma^{-2}B_1'Px]\,dt\\&=\|u-u^*\|_2^2-\gamma^2\|w-w^*\|_2^2\end{aligned} \tag{2.11}$$

with w^*, u^* as in (2.7) and (2.8), which establishes (2.9). Suppose that $\mathscr{L}$ is the operator with realization

$$\mathring{x}=(A+B_2K)x+B_1w, \qquad (w-w^*)=-\gamma^{-2}B_1'Px+w,$$

which maps w to $(w-w^*)$. Since $\mathcal{L}^{-1}$ exists (and is given by $\mathring{x}=(A+B_2K+\gamma^{-2}B_1B_1'P)x+B_1(w-w^*)$, $w=\gamma^{-2}B_1'Px+(w-w^*)$), we can write

$$\|\mathcal{T}_{zw}w\|_2^2-\gamma^2\|w\|_2^2=-\gamma^2\|w-w^*\|_2^2=-\gamma^2\|\mathcal{L}w\|_2^2\leqq-\kappa\|w\|_2^2$$

for some positive κ. Thus $\|\mathcal{T}_{zw}\|<\gamma$, as required. □

Remark 2.1 (Positive semidefiniteness of $P(t)$). It is not hard to show that if (2.6) with $P(T)=0$ has a solution on $[0, T]$, then $P(t)\geqq 0$ for all $t\in[0, T]$. Consider system (2.1) with $0\leqq t_o\leqq t\leqq T$, and with $x(t_o)=x_o$. If

$$J_{t_o}(u, w):=\int_{t_o}^{T}(z'z-\gamma^2w'w)\,dt, \tag{2.12}$$

then we may complete the square as above to yield

$$J_{t_o}(u, w)-x_o'P(t_o)x_o=\|u-u^*\|_2^2-\gamma^2\|w-w^*\|_2^2,$$

in which x_o is regarded as an arbitrary initial condition for the running period $[t_o, T]\subseteq[0, T]$. Consequently,

$$J_{t_o}(u^*, 0)+\gamma^2\|w^*\|_2^2=x_o'P(t_o)x_o \tag{2.13}$$

for every x_o. Since $J_{t_o}(u^*, 0)\geqq 0$ for any x_o, it follows that $P(t)\geqq 0$ for all $t\in[0, T]$.

Note that the game Riccati equation (2.6) and the corresponding linear quadratic regulator Riccati equation "approach each other" as γ increases. Since the linear quadratic regulator Riccati equation always has a solution, we expect (2.6) to have a solution if γ is large enough. In the next section, we show that the existence of an admissible controller satisfying (2.5) is a necessary condition for (2.6) to have a solution on $[0, T]$.

2.1. The necessary conditions. The aim of this section is to show that if an admissible feedback control exists that solves the $\mathcal{H}^\infty$-control problem, then (2.6) has a solution on $[0, T]$. Our proof uses conjugate point arguments that are reminiscent of those found in [25], [26]. To begin, we introduce the two-point boundary value problem

$$\begin{bmatrix}\mathring{x}\\ \mathring{p}\end{bmatrix}=\begin{bmatrix}A & -(B_2B_2'-\gamma^{-2}B_1B_1')\\ -C'C & -A'\end{bmatrix}\begin{bmatrix}x\\ p\end{bmatrix},\qquad \begin{bmatrix}x(0)\\ p(T)\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \tag{2.14}$$

and its associated transition matrix, which is generated by the linear differential equation

$$\begin{bmatrix}\mathring{\Phi}_{11} & \mathring{\Phi}_{12}\\ \mathring{\Phi}_{21} & \mathring{\Phi}_{22}\end{bmatrix}(t, T)=\begin{bmatrix}A & -(B_2B_2'-\gamma^{-2}B_1B_1')\\ -C'C & -A'\end{bmatrix}\begin{bmatrix}\Phi_{11} & \Phi_{12}\\ \Phi_{21} & \Phi_{22}\end{bmatrix}(t, T), \tag{2.15}$$

with $\Phi(T, T)=I$. Next, we note that

$$\begin{aligned}-\frac{d}{dt}(\Phi_{21}\Phi_{11}^{-1})&=-\mathring{\Phi}_{21}\Phi_{11}^{-1}+\Phi_{21}\Phi_{11}^{-1}\mathring{\Phi}_{11}\Phi_{11}^{-1}\\ &=A'(\Phi_{21}\Phi_{11}^{-1})+(\Phi_{21}\Phi_{11}^{-1})A-(\Phi_{21}\Phi_{11}^{-1})(B_2B_2'-\gamma^{-2}B_1B_1')(\Phi_{21}\Phi_{11}^{-1})\\ &\quad+C'C.\end{aligned}$$

Thus, if $\Phi_{11}^{-1}(t, T)$ exists on $t\in[0, T]$, the RDE (2.6) has a solution on $[0, T]$ given by $P(t)=\Phi_{21}(t, T)\Phi_{11}^{-1}(t, T)$.

The existence of $\Phi_{11}^{-1}(t, T)$ is equivalent to a conjugate point condition, and this observation forms the basis of the necessity proof.

DEFINITION 2.1. Suppose that $t_o < t_f$. Then t_o and t_f are *conjugate points* if it is possible to find a *nontrivial* solution to (2.14) such that $x(t_o) = p(t_f) = 0$. We note also that $p(t_o) \neq 0$ and $x(t_f) \neq 0$, since otherwise (2.14) would only have the trivial solution $[x' \quad p'](t) \equiv 0$.

LEMMA 2.2. *The matrix $\Phi_{11}(t_o, t_f)$ defined by* (2.15) *is singular if and only if t_o and t_f are conjugate points.*

Proof. Suppose that t_o and t_f are conjugate points. Since

$$\begin{bmatrix} x(t_o) \\ p(t_o) \end{bmatrix} = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} (t_o, t_f) \begin{bmatrix} x(t_f) \\ p(t_f) \end{bmatrix},$$

and since t_o and t_f are conjugate points, we obtain

$$0 = \Phi_{11}(t_o, t_f)x(t_f). \tag{2.16}$$

Hence $\Phi_{11}(t_o, t_f)$ is singular, as $x(t_f)$ is nonzero.

If, on the other hand, $\Phi_{11}(t_o, t_f)$ is singular, there exists a vector $g \neq 0$ such that $\Phi_{11}(t_o, t_f)g = 0$. By considering the final value problem with $x(t_f) = g$ and $p(t_f) = 0$, we see that $x(t_o) = 0$, which establishes that t_o and t_f are conjugate. □

THEOREM 2.3. *Consider the linear system* (2.1). *If there exists a closed loop control $\bar{K} \in \mathscr{C}$ such that*

$$\|\mathscr{T}_{zw}\| < \gamma, \tag{2.17}$$

then $\Phi_{11}(t, T)$ is nonsingular for all $t \in [0, T]$. Consequently, the RDE (2.6), *with boundary condition $P(T) = 0$, has a solution on $[0, T]$.*

Proof. We suppose for contradiction that $t^* \in [0, T]$ is the largest time such that $\Phi_{11}(t^*, T)$ is singular. Since $\Phi_{11}(T, T) = I$, $t^* < T$ by continuity. It follows from Lemma 2.2 that t^* and T are conjugate points, giving $x(t^*) = p(T) = 0$. Next, we suppose that

$$\bar{w}(t) = \begin{cases} \gamma^{-2}B_1'p(t), & t^* \leqq t \leqq T, \\ 0, & 0 \leqq t < t^*. \end{cases}$$

Since (2.14) has a nontrivial solution, $p(t) \not\equiv 0$ $t \in [t^*, T]$. Furthermore, if $B_1'p(t) \equiv 0$, then (under this assumption) (2.14) becomes the two-point boundary problem for the LQ regulator; this can have no conjugate points [1]. Thus $\bar{w}(t) \not\equiv 0$. Next, we see that $J(\bar{K}, \bar{w}) \geqq J_{t^*}(\bar{K}, \bar{w})$, since $\bar{w}(t) = 0$ for $t < t^*$. Let $\bar{u}(t)$ be the function generated by $x(t^*) = 0$, $\bar{K}$ and $\bar{w}(t)$. Then it follows that

$$J_{t^*}(\bar{K}, \bar{w}) = J_{t^*}(\bar{u}, \bar{w}) \geqq \min_{u(t)} \int_{t^*}^{T} \{\tilde{x}'C'C\tilde{x} + u'u - \gamma^2\bar{w}'\bar{w}\}\, dt, \tag{2.18a}$$

subject to

$$\mathring{\tilde{x}} = A\tilde{x} + B_1\bar{w} + B_2u, \qquad \tilde{x}(t^*) = 0. \tag{2.18b}$$

The tilde is used to distinguish the state trajectory associated with (2.14) from that associated with the minimization problem in (2.18). The initial condition $\tilde{x}(t^*) = 0$ is a consequence of (i) $x(0) = 0$, (ii) $\bar{w} \equiv 0$ for all $t \in [0, t^*)$, and (iii) $K \in \mathscr{C}$. The minimization on the right-hand side of (2.18a) subject to (2.18b), with $u(\cdot)$ being sought in *open-loop* form is almost a standard LQ control problem and is solved as follows: Form

$$H(t, \tilde{x}, \lambda, u) = \tfrac{1}{2}(\tilde{x}'C'C\tilde{x} + u'u - \gamma^2\bar{w}'\bar{w}) + \lambda(A\tilde{x} + B_1\bar{w} + B_2u).$$

Then

$$\frac{\partial H}{\partial u} = 0 \Rightarrow u_{\min} = -B_2'\lambda$$

and

$$-\mathring{\lambda}=\frac{\partial H}{\partial x} \Rightarrow -\mathring{\lambda}=C'C\tilde{x}+A'\lambda, \qquad \lambda(T)=0,$$

giving

$$\begin{bmatrix}\mathring{\tilde{x}}\\ \mathring{\lambda}\end{bmatrix}=\begin{bmatrix}A & -B_2B_2'\\ -C'C & -A'\end{bmatrix}\begin{bmatrix}\tilde{x}\\ \lambda\end{bmatrix}+\begin{bmatrix}B_1\bar{w}\\ 0\end{bmatrix}, \qquad \begin{bmatrix}\tilde{x}(t^*)\\ \lambda(T)\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}. \tag{2.19}$$

Using $\bar{w}=\gamma^{-2}B_1'p$, and subtracting (2.19) from (2.14), gives us

$$\begin{bmatrix}\mathring{\tilde{x}}-\mathring{x}\\ \mathring{\lambda}-\mathring{p}\end{bmatrix}=\begin{bmatrix}A & -B_2B_2'\\ -C'C & -A'\end{bmatrix}\begin{bmatrix}\tilde{x}-x\\ \lambda-p\end{bmatrix}, \qquad \begin{bmatrix}(\tilde{x}-x)(t^*)\\ (\lambda-p)(T)\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}. \tag{2.20}$$

Since there are no conjugate points associated with the LQ boundary problem in (2.20), $\tilde{x}(t)\equiv x(t)$ and $\lambda(t)\equiv p(t)$ for all $t\in[t^*, T]$. Consequently,

$$\begin{aligned}\min_{u(t)}\int_{t^*}^{T}\{\tilde{x}'C'C\tilde{x}+u'u-\gamma^2\bar{w}'\bar{w}\}\,dt &= \int_{t^*}^{T}\{\tilde{x}'C'C\tilde{x}+\lambda'B_2B_2'\lambda-\gamma^{-2}p'B_1B_1'p\}\,dt\\ &=\int_{t^*}^{T}\{x'C'Cx+p'B_2B_2'p-\gamma^{-2}p'B_1B_1'p\}\,dt\\ &=-\int_{t^*}^{T}\{x'(\mathring{p}+A'p)+p'(\mathring{x}-Ax)\}\,dt \text{ by (2.14)}\\ &=-\int_{t^*}^{T}\frac{d}{dt}\{x'p\}\,dt=0,\end{aligned}$$

which contradicts (2.17). □

Remark 2.2. In our later work, we will need to consider problems in which the output is given by

$$z(t)=C(t)x(t)+D(t)u(t), \tag{2.21}$$

rather than (2.2). This change in output has the effect of introducing a cross-coupling term between u and x in the functional $J(K, w)$. This may be removed by the change of variable $u=\hat{u}-D'Cx$. In the case of an output given by (2.21), the Riccati equation (2.6) becomes

$$\begin{aligned}-\mathring{P}&=P(A-B_2D'C)+(A-B_2D'C)P-P(B_2B_2'-\gamma^{-2}B_1B_1')P\\ &\quad+C(I-DD')C, \qquad P(T)=0,\end{aligned}$$

and the corresponding minimizing feedback control is given by $u^*(t)=-(D'C+B_2'P)(t)x(t)$, rather than (2.7).

2.2. A representation formula for all solutions. In Theorem 2.1 we found one feedback control that solves the time-varying $\mathcal{H}^\infty$-control problem. In the infinite-horizon case, it is well known that there are usually many feedback controls that result in $\|\mathcal{T}_{zw}\|<\gamma$. The aim of this section is to show how to construct all linear full-information (access to w and x) feedback controls such that $\|\mathcal{T}_{zw}\|<\gamma$. Our analysis is based on the output equation (2.21), rather than (2.2).

THEOREM 2.4. *Suppose that*

$$\mathring{x}(t)=A(t)x(t)+B_1(t)w(t)+B_2(t)u(t), \qquad x(0)=0, \tag{2.22a}$$

$$z(t)=C(t)x(t)+D(t)u(t) \tag{2.22b}$$

is given. Then there exists a control $\tilde{K} \in \mathcal{C}$ *such that* $\|\mathcal{T}_{zw}\| < \gamma$ *if and only if*

$$-\dot{P} = (A - B_2D'C)'P + P(A - B_2D'C) - P(B_2B_2' - \gamma^{-2}B_1B_1')P + C'(I - DD')C \tag{2.23}$$

has a solution on $[0, T]$ *with* $P(T) = 0$. *In this case, all closed-loop systems* $\mathcal{T}_{zw}$ *generated by controls of the form*

$$u(t) = -(\mathfrak{L}_1x + \mathfrak{L}_2w)(t), \tag{2.24}$$

where $\mathfrak{L}_1$ *and* $\mathfrak{L}_2$ *are arbitrary causal linear operators such that*

$$\|\mathcal{T}_{zw}\| < \gamma$$

are also generated by

$$u(t) = u^*(t) + (\mathfrak{U}(w - w^*))(t), \tag{2.26}$$

where

$$u^*(t) = -F_\infty x = -(D'C + B_2'P)x, \tag{2.27a}$$

$$w^*(t) = \gamma^{-2}B_1'Px, \tag{2.27b}$$

and $\gamma^{-1}\mathfrak{U}$ *is an arbitrary linear causal strictly contractive operator on* $\mathcal{L}^2[0, T]$ (*i.e.,* $\|\mathfrak{U}\| < \gamma$).

Equivalently, every $\mathfrak{L}_1$ *and* $\mathfrak{L}_2$ *may be parametrized by*

$$\mathfrak{L}_1 = F_\infty + \gamma^{-2}\mathfrak{U}B_1'P, \tag{2.28a}$$

$$\mathfrak{L}_2 = -\mathfrak{U}. \tag{2.28b}$$

Equations (2.22), (2.26), *and* (2.27) *may be represented diagrammatically, as in Fig.* 1.

Proof. The first part, that the existence of a controller such that $\|\mathcal{T}_{zw}\| < \gamma$ is necessary and sufficient for the existence of $P(t)$ is just Theorems 2.1 and 2.3 for the cross-coupled game.

Now suppose that the RDE (2.23) with $P(T) = 0$ has a solution on $[0, T]$. We need to show (1) that there exists a causal $\mathfrak{U}$ in (2.26) corresponding to every control of the form given in (2.24), and (2) that $\|\mathcal{T}_{zw}\| < \gamma$ if and only if $\gamma^{-1}\mathfrak{U}$ is strictly contractive.

Substituting (2.24) and (2.27b) into the dynamical equation (2.22a) gives us

$$\dot{x} = (A + \gamma^{-2}B_1B_1'P - B_2(\mathfrak{L}_1 + \gamma^{-2}\mathfrak{L}_2B_1'P))x + (B_1 - B_2\mathfrak{L}_2)(w - w^*), \qquad x(0) = 0,$$

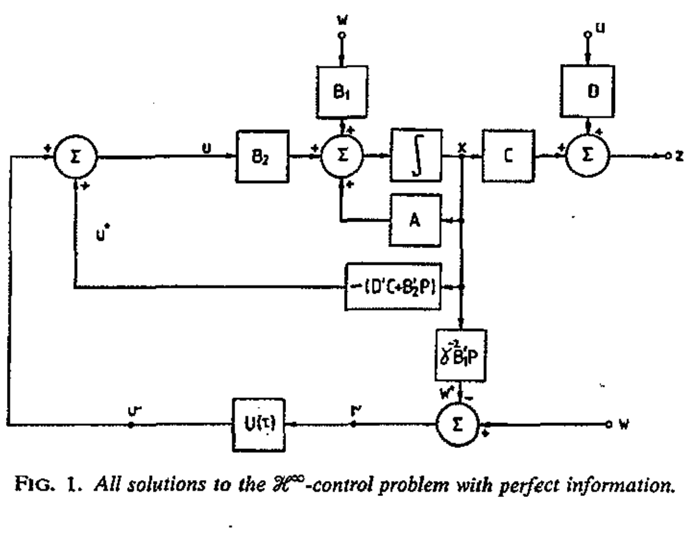

FIG. 1. *All solutions to the* $\mathcal{H}^\infty$*-control problem with perfect information.*

which shows that x depends causally on $(w-w^*)$, and so $x(t)=\mathfrak{L}_3(w-w^*)$, in which $\mathfrak{L}_3$ is a linear causal operator. Substituting (2.27) into (2.24) gives us

$$\begin{aligned} u-u^* &= -(\mathfrak{L}_1-F_\infty-\gamma^{-2}\mathfrak{L}_2B_1'P)x-\mathfrak{L}_2(w-w^*) \\ &= [-(\mathfrak{L}_1-F_\infty-\gamma^{-2}\mathfrak{L}_2B_1'P)\mathfrak{L}_3-\mathfrak{L}_2](w-w^*) \\ &= \mathfrak{U}(w-w^*) \end{aligned}$$

for some linear causal $\mathfrak{U}$. This establishes the existence of the causal mapping in (2.26). The proof is completed by noting that $\gamma^{-1}\mathfrak{U}$ is strictly contractive as follows:

$$\begin{aligned} \|z\|_2^2-\gamma^2\|w\|_2^2 &= \|u-u^*\|_2^2-\gamma^2\|w-w^*\|_2^2 \\ &= \|\mathfrak{U}(w-w^*)\|_2^2-\gamma^2\|w-w^*\|_2^2 \\ &= \|\mathfrak{U}\mathscr{L}w\|_2^2-\gamma^2\|\mathscr{L}w\|_2^2 \\ &< 0 \text{ for all } w\neq 0\in\mathfrak{L}^2[0,T] \Leftrightarrow \gamma^{-1}\mathfrak{U} \text{ is strictly contractive.} \end{aligned}$$

In the above, $\mathscr{L}$ is the causal and causally invertible operator linking w and $(w-w^*)$. The invertibility of $\mathscr{L}$ was established in the proof of Theorem 2.1. □

At this point it is interesting to reexamine Fig. 1. By reviewing (2.26), we see that $r=(w-w^*)$, and that $u=(v+u^*)$ drives B_2 with $v=\mathfrak{U}r$. If $w=w^*$, there is no signal into the $\mathfrak{U}$ parameter, and the corresponding control is given by u^* (irrespective of $\mathfrak{U}$). If $w\neq w^*$, we do not have to use the control u^*. Thus for the purpose of solving the $\mathscr{H}^\infty$-control problem, the u-player only has to play well enough to ensure that $J(K,w)<0$ for a given $w\neq 0\in\mathscr{L}^2[0,T]$.

2.3. Adjoint systems. In § 3, where we derive a representation formula for all solutions to the time-varying $\mathscr{H}^\infty$ problem with output feedback, we will require an elementary property of adjoint systems [8], [14].

Let $\mathfrak{L}: X\to Y$ be a linear operator between two Hilbert spaces X and Y. Then $\mathfrak{L}^*: Y\to X$, the adjoint of $\mathfrak{L}$, is the unique linear operator such that, for all $z\in Y$ and $w\in X$, $\langle z,\mathfrak{L}w\rangle=\langle\mathfrak{L}^*z,w\rangle$ [14, Thm. 2, p. 39]. Note also that $\|\mathfrak{L}\|=\|\mathfrak{L}^*\|$. Now suppose that $\mathfrak{L}$ is described by the state-space equations

$$\dot{x}(t)=A(t)x(t)+B(t)w(t),\qquad x(0)=0, \tag{2.29a}$$

$$y(t)=C(t)x(t)+D(t)w(t), \tag{2.29b}$$

or, equivalently,

$$\begin{bmatrix} \frac{d}{dt}I-A(t) & -B(t) \\ C(t) & D(t) \end{bmatrix}\begin{bmatrix} x(t) \\ w(t)\end{bmatrix}=\begin{bmatrix} 0 \\ y(t)\end{bmatrix},\qquad x(0)=0. \tag{2.30}$$

Thus

$$\begin{aligned} \langle z,\mathfrak{L}w\rangle &= \left\langle \begin{bmatrix} p \\ z\end{bmatrix}, \begin{bmatrix} \frac{d}{dt}I-A & -B \\ C & D\end{bmatrix}\begin{bmatrix} x \\ w\end{bmatrix}\right\rangle \\ &= \int_0^T \left(p'\frac{dx}{dt}\right)dt-\int_0^T p'(Ax+Bw)\,dt+\int_0^T z'(Cx+Dw)\,dt. \end{aligned}$$

Integrating the first term by parts gives us

$$
\begin{aligned}
&= p'(t_1)x(t_1) - \int_0^T \left(\frac{dp'}{dt}\right) x\, dt - \int_0^T p'(Ax+Bw)\, dt \\
&\quad + \int_0^T z'(Cx+Dw)\, dt \\
&= \left\langle \begin{bmatrix} -\left(\frac{d}{dt}I + A'\right) & C' \\ -B' & D' \end{bmatrix} \begin{bmatrix} p \\ z \end{bmatrix}, \begin{bmatrix} x \\ w \end{bmatrix} \right\rangle + p'(T)x(T) \\
&= \langle \mathfrak{L}^* z, w \rangle,
\end{aligned}
$$

where the adjoint $\mathfrak{L}^*$ is a linear system with realization

$$-\mathring{p}(t) = A'(t)p(t) + C'(t)z(t), \qquad p(T) = 0, \tag{2.31a}$$

$$q(t) = B'(t)p(t) + D'(t)z(t). \tag{2.31b}$$

In the following, we will require the solution of a dual LQ game system, which is obtained by applying Theorem 2.4 to its associated adjoint system.

2.4. The asymptotic properties of the game equation. This section represents a digression from the main stream of the time-varying finite-horizon $\mathcal{H}^\infty$-control problem, and its purpose is to make some connections with the more familiar time-invariant, infinite interval results [9]. In particular, we show that in the limit as $T \to \infty$, the solution of (2.6) approaches the smallest nonnegative solution of an algebraic Riccati equation (if such a solution exists). We also show that this smallest nonnegative solution is stabilizing in the sense alluded to in [9]. We begin by connecting the properties of the solution of the RDE (2.6) as $T \to \infty$ (when the various coefficient matrices are assumed to be constant) to the solution of the algebraic Riccati equation

$$0 = PA + A'P - P(B_2B_2' - \gamma^{-2}B_1B_1')P + C'C. \tag{2.32}$$

In doing this, we will suppose that each of the matrices in (2.1) and (2.2) is time-invariant and that (A, B_2, C) is stabilizable and observable (it is not hard to remove the observability assumption, but we will not address this issue here).

If (2.6) is solved backward from the terminal condition $P(T) = Q$, then we will refer to the solution as $P(t, T, Q)$. If a limiting solution to (2.6) exists, we will call it $\bar{P}(t) = \lim_{T\uparrow\infty} P(t, T, Q)$. The first result of this section is standard and shows that $P(t)$ is nonincreasing if $P(T) = 0$.

LEMMA 2.5. *If* (2.6) *has a solution on* $[0, T]$ *with* $P(T) = 0$, *then* $P(t)$ *is nonincreasing* (*in the sense of semidefinite matrices*).

Proof. Differentiating (2.6) gives us

$$-\mathring{\mathring{P}} = \mathring{P}(A - (B_2B_2' - \gamma^{-2}B_1B_1')P) + (A - (B_2B_2' - \gamma^{-2}B_1B_1')P)'\mathring{P}, \qquad \mathring{P}(T) = -C'C,$$

which has solution [8, Thm. 1, p. 59] $\mathring{P}(t) = -\Phi(t, T)'C'C\Phi(t, T)$, where $\Phi(t, T)$ is the transition matrix associated with $(A - (B_2B_2' - \gamma^{-2}B_1B_1')P)'$. It is evident that $\mathring{P}(t) \leqq 0$ for all t, so $P(t)$ is nonincreasing. $\square$

If (2.32) has at least one nonnegative solution, then we will show that the smallest of these solutions, Y say, is an upper bound for $P(t)$ provided that $Y \geqq P(T)$.

LEMMA 2.6. *Suppose that* $P(t, T, Q)$ *exists* $[0, T]$ *and that* $Y = Y' \geqq 0$ *satisfies* (2.32) *with* $Y \geqq Q$. *Then* $Y \geqq P(t)$ *for all* $t \in [0, T]$.

Proof. Completing the square using (2.32) gives us

$$J_t(u, w) = \int_t^T [(u + B_2'Yx)'(u + B_2'Yx) - \gamma^2(w - \gamma^{-2}B_1'Yx)'(w - \gamma^{-2}B_1'Yx)]\, d\tau$$
$$+ x'(t)Yx(t) - x'(T)Yx(T)$$

for any u and w. In the same way,

$$J_t(u, w) = \int_t^T [(u + B_2'Px)'(u + B_2'Px) - \gamma^2(w - \gamma^{-2}B_1'Px)'(w - \gamma^{-2}B_1'Px)]\, d\tau$$
$$+ x'(t)P(t)x(t) - x'(T)P(T)x(T)$$

by invoking (2.6). Subtracting and setting $\tilde{u} := -B_2'Yx$ and $\tilde{w} := \gamma^{-2}B_1'Px$ yields

$$x'(t)[Y - P(t)]x(t) = x'(T)[Y - P(T)]x(T) + \gamma^2 \int_t^T (\tilde{w} - \gamma^{-2}B_1'Yx)'(\tilde{w} - \gamma^{-2}B_1'Yx)\, d\tau$$
$$+ \int_t^T (\tilde{u} + B_2'Px)'(\tilde{u} + B_2'Px)\, d\tau \geqq 0$$

for all $x(t)$. □

We remark that a variation on the above argument, in conjunction with Lemma 2.5, will establish the following result.

COROLLARY 2.7. *Assume that $Y = Y' \geqq 0$ satisfies (2.32) with $Y \geqq Q \geqq 0$. Then $P(t, T, Q)$ exists for all $t \leqq T$ and $Y \geqq P(t, T, Q) \geqq P(t, T, 0) \geqq 0$.*

If $\bar{P}(t)$ exists for some Q, it is easy to argue that it satisfies (2.6). Suppose that $t \leqq T_1 \leqq T$. Then $P(t, T, Q) = P(t, T_1, P(T_1, T, Q))$, and thus

$$\bar{P}(t) = \lim_{T\uparrow\infty} P(t, T, Q) = \lim_{T\uparrow\infty} P(t, T_1, P(T_1, T, Q)).$$

For any fixed T_1, the solution $P(t, T_1, \tilde{Q})$ depends continuously on $\tilde{Q}$ and therefore

$$\bar{P}(t) = P(t, T_1, \lim_{T\uparrow\infty} P(T_1, T, Q)) = P(t, T_1, \bar{P}(T_1)), \tag{2.33}$$

which shows that $\bar{P}(t)$ is a solution to (2.6) for all t. Since the system and output matrices have been assumed to be *time-invariant, the value of the game* $J_t(K^*, w^*) = x'(t)P(t)x(t)$ is invariant under time translations. Consequently, $\bar{P}(t) = \bar{P}$ is constant and therefore satisfies the algebraic equation (2.32).

If a nonnegative stabilizing (i.e., $(A - (B_2B_2' - \gamma^{-2}B_1B_1')P)$ asymptotically stable) solution to (2.32) exists, it is the smallest of the nonnegative solutions.

THEOREM 2.8. *Suppose that $[A, C]$ is observable and that $X \geqq 0$ and $Y \geqq 0$ satisfy*

$$A'P + PA - PDP + C'C = 0, \tag{2.34}$$

with $A - DX$ asymptotically stable. Then $Y \geqq X$.

Proof. Due to the observability of $[A, C]$, every solution to (2.32) is nonsingular. We can therefore write $X(A - DX)X^{-1} = -(A' + C'CX^{-1})$, which shows that $A' + C'CX^{-1}$ is completely unstable. Now

$$AX^{-1} + X^{-1}A' - D + X^{-1}C'CX^{-1} = 0 \quad \text{and} \quad AY^{-1} + Y^{-1}A' - D + Y^{-1}C'CY^{-1} = 0.$$

Subtracting gives us

$$A(X^{-1} - Y^{-1}) + (X^{-1} - Y^{-1})A' + X^{-1}C'CX^{-1} - Y^{-1}C'CY^{-1} = 0$$
$$\Rightarrow (A + X^{-1}C'C)(X^{-1} - Y^{-1}) + (X^{-1}Y^{-1})(A + X^{-1}C'C)'$$
$$= (X^{-1} - Y^{-1})C'C(X^{-1} - Y^{-1}).$$

Since $(A+X^{-1}C'C)$ is completely unstable, we must have $(X^{-1}-Y^{-1})\geqq 0$ and so $X\leqq Y$. $\square$

Remark 2.3. Theorem 2.8 remains valid when the observability assumption on $[A, C]$ is weakened to one of detectability. Suppose that

$$A=\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22}\end{bmatrix}, \qquad C=[C_1 0],$$

with $[A_{11}, C_1]$ observable, then argue as before on the smaller system $[A_{11}, C_1]$.

3. Solution to the time-varying $\mathscr{H}^\infty$-control problem: Main results. Now that the background game theory is in place, we are in a position to solve the general time-varying $\mathscr{H}^\infty$-control problem. As mentioned in the Introduction, the plant is described by

$$\dot{x}(t)=A(t)x(t)+B_1(t)w(t)+B_2(t)u(t), \qquad x(0)=0, \tag{3.1a}$$

$$z(t)=C_1(t)x(t)+D_{12}(t)u(t), \tag{3.1b}$$

$$y(t)=C_2(t)x(t)+D_{21}(t)w(t), \tag{3.1c}$$

in which we assume that all the matrices have entries that are continuous functions of time, and that D_{12} and D_{21} have full column and row rank, respectively, for all $t\in[0, T]$. Under this assumption, we may scale the problem so that $D_{12}'D_{12}=I$ and $D_{21}D_{21}'=I$. By a corollary of Doležal's theorem [29, Cor. 3, p. 70], we know that there exist continuous extensions $D_\perp$ and $\tilde{D}_\perp$ to D_{12} and D_{21}, respectively, such that $[D_{12}\quad D_\perp]$ and $[D_{21}^*\quad \tilde{D}_\perp^*]$ are orthogonal for all $t\in[0, T]$. Note that by generalizing the loop-shifting transformations in [11], [24] to the time-varying case, there is no loss of generality in the assumption that D_{11} and D_{22} are zero. Details of the scaling and loop-shifting transformations required in the time-varying case may be found in the Appendix.

3.1. Problems of the first kind. We begin with a preliminary result in which we assume that D_{12} and D_{21} are square; we call such problems problems of the first kind. Under this assumption, scaling arguments allow us to assume, without loss of generality, that $D_{12}=I$ and $D_{21}=I$. As we will now show, finite-horizon $\mathscr{H}^\infty$ problems of the first kind may be solved by inverting the plant, and the resulting controllers have a remarkably simple observer structure.

THEOREM 3.1. *Consider the generalized plant described by*

$$\dot{x}(t)=A(t)x(t)+B_1(t)w(t)+B_2(t)u(t), \qquad x(0)=0, \tag{3.2a}$$

$$z(t)=C_1(t)x(t)+u(t), \tag{3.2b}$$

$$y(t)=C_2(t)x(t)+w(t). \tag{3.2c}$$

Then (i) *the set of all linear causal output-feedback control laws such that* $\|\mathscr{F}_{zw}\|<\gamma$ *is parametrized by*

$$\dot{\hat{x}}(t)=(A-B_1C_2-B_2C_1)(t)\hat{x}(t)+B_1(t)y(t)+B_2(t)v(t), \qquad \hat{x}(0)=0, \tag{3.3a}$$

$$u(t)=v(t)-C_1(t)\hat{x}(t), \tag{3.3b}$$

$$r(t)=y(t)-C_2(t)\hat{x}(t), \tag{3.3c}$$

$$v(t)=(\mathfrak{U}r)(t), \tag{3.3d}$$

in which $\mathfrak{U}$ *is a causal strictly contractive operator on* $\mathscr{L}^2[0, T]$, *and* (ii) *there exists a linear causal controller such that the closed-loop mapping from w to z is identically zero.*

Proof. Eliminating $y(t)$ and $u(t)$ from (3.2) and (3.3) gives us

$$\begin{bmatrix} \mathring{\hat{x}} \\ \mathring{x} \end{bmatrix} = \begin{bmatrix} A-B_1C_2-B_2C_1 & B_1C_2 \\ -B_2C_1 & A \end{bmatrix}\begin{bmatrix} \hat{x} \\ x \end{bmatrix} + \begin{bmatrix} B_1 & B_2 \\ B_1 & B_2 \end{bmatrix}\begin{bmatrix} w \\ v \end{bmatrix}$$

and

$$\begin{bmatrix} z \\ r \end{bmatrix} = \begin{bmatrix} -C_1 & C_1 \\ -C_2 & C_2 \end{bmatrix}\begin{bmatrix} \hat{x} \\ x \end{bmatrix} + \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} w \\ v \end{bmatrix}.$$

Consequently,

$$\begin{bmatrix} \mathring{\hat{x}} \\ \mathring{x}-\mathring{\hat{x}} \end{bmatrix} = \begin{bmatrix} A-B_2C_1 & B_1C_2 \\ 0 & A-B_1C_2 \end{bmatrix}\begin{bmatrix} \hat{x} \\ x-\hat{x} \end{bmatrix} + \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} w \\ v \end{bmatrix} \tag{3.4}$$

and

$$\begin{bmatrix} z \\ r \end{bmatrix} = \begin{bmatrix} 0 & C_1 \\ 0 & C_2 \end{bmatrix}\begin{bmatrix} \hat{x} \\ x-\hat{x} \end{bmatrix} + \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} w \\ v \end{bmatrix}.$$

The second row of (3.4) implies that the observations error $e(t)=(x-\hat{x})(t)$ is identically zero, thereby establishing the observer property.[1] In addition, it is clear that $[\begin{smallmatrix} z \\ r \end{smallmatrix}](t)=[\begin{smallmatrix} v \\ w \end{smallmatrix}](t)$ for all $t\in[0, T]$, which shows that the controller is plant inverting.

For the first part, we note that if $\mathcal{R}$ is any strictly contractive mapping from $w(t)$ to $z(t)$, then $\mathcal{R}$ will be generated by setting $\mathfrak{U}=\mathcal{R}$ in (3.3d). Setting $\mathfrak{U}=0$ proves the second part. □

The controller given in Theorem 3.1 is presented diagrammatically in Fig. 2, and is seen to have an observer structure with observer gain matrix $-B_1$ and state estimate feedback C_1.

3.2. Problems of the second kind. In this section we show that all problems in which either D_{12} or D_{21} is square, which we will refer to as *problems of the second kind*, have controllers that may be characterized by the solution of a single RDE. Simple plant inversion is no longer possible, since this requires that both D_{12} and D_{21} be square. Nevertheless, a solution is still possible and involves the game theory of § 2. If D_{21} is assumed to be square, we can replace (3.1c) with

$$y(t)=C_2(t)x(t)+w(t) \tag{3.1d}$$

by an appropriate scaling operation.

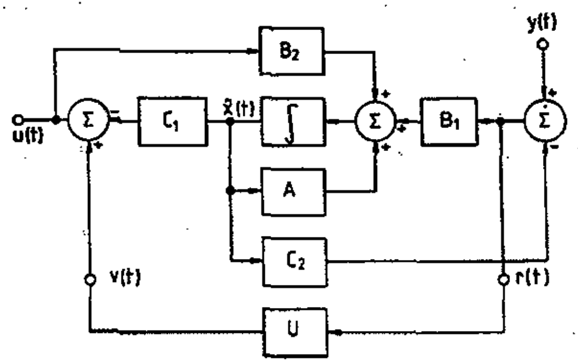

FIG. 2. *The plant inverse as an observer.*

[1] Since we are dealing with a finite time problem, the question of stability does not arise.

We now consider the possibility of obtaining a controller via a combination of an observer for x, and a linear feedback law as computed in § 2. Since the controllers that we are considering do not have access to w, the observer can only be driven by u and y. Suppose that

$$\mathring{\hat{x}} = A\hat{x} + B_2 u + K(C_2\hat{x} - y), \qquad \hat{x}(0) = 0. \tag{3.5}$$

Subtracting (3.1) from (3.5) and substituting (3.1d) gives us

$$(\mathring{\hat{x}} - \mathring{x}) = A(\hat{x} - x) + K(C_2\hat{x} - C_2 x - w) - B_1 w = (A + KC_2)(\hat{x} - x) - (K + B_1)w. \tag{3.6}$$

Setting $K = -B_1$ as before, and $e := \hat{x} - x$ gives us

$$\mathring{e} = (A - B_1 C_2)e, \qquad e(0) = 0 \tag{3.7}$$

$\Rightarrow e(t) \equiv 0$ and so $\hat{x}(t) \equiv x(t)$. Again, the stability of $A - B_1 C_2$ is not an issue for finite terminal times.

With the required observer property established, we may now find all linear closed-loop controls $K \in \mathscr{C}$ such that $\|\mathscr{T}_{zw}\| < \gamma$ as if full state information were available. As before,

$$J(u, w) = \int_0^T (z'z - \gamma^2 w'w)\, dt. \tag{3.8}$$

Using the results of § 2, we introduce the Riccati equation

$$\begin{aligned} -\mathring{X}_\infty &= (A - B_2 D_{12}' C_1)' X_\infty + X_\infty (A - B_2 D_{12}' C_1) - X_\infty (B_2 B_2' - \gamma^{-2} B_1 B_1') X_\infty \\ &\quad + C_1' D_\perp D_\perp' C_1 \end{aligned} \tag{3.9}$$

with terminal condition $X_\infty(T) = 0$, and define

$$u^* = -(D_{12}' C_1 + B_2' X_\infty)x = -F_\infty x, \tag{3.10}$$

$$w^* = \gamma^{-2} B_1' X_\infty x, \tag{3.11}$$

which gives us

$$J(-F_\infty x, w) \leqq -\varepsilon \|w\|_2^2 \tag{3.12}$$

for some $\varepsilon > 0$.

By combining the observer given in (3.5) with $K = -B_1$, the equilibrium strategies (3.10), (3.11), and the characterization of all solutions in § 2.2, we obtain the controller configuration illustrated in Fig. 3 and the main result of this section.

THEOREM 3.2. *Given*

$$\mathring{x}(t) = A(t)x(t) + B_1(t)w(t) + B_2(t)u(t), \qquad x(0) = 0, \tag{3.13a}$$

$$z(t) = C_1(t)x(t) + D_{12}(t)u(t), \tag{3.13b}$$

$$y(t) = C_2(t)x(t) + w(t), \tag{3.13c}$$

then (i) *an output-feedback controller exists such that* $\|\mathscr{T}_{zw}\| < \gamma$ *if and only if the* RDE (3.9) *has a solution on* $[0, T]$ *with terminal condition* $X_\infty(T) = 0$, *and* (ii) *every closed-loop operator* $\mathscr{T}_{zw}$ *with* $\|\mathscr{T}_{zw}\| < \gamma$, *corresponding to an output feedback control* $u = \mathscr{R}y$, *is generated by*

$$\mathring{\hat{x}}(t) = (A - B_1 C_2 - B_2 F_\infty)(t)\hat{x}(t) + B_1(t)y(t) + B_2(t)v(t), \qquad \hat{x}(0) = 0, \tag{3.14a}$$

$$u(t) = v(t) - F_\infty(t)\hat{x}(t), \tag{3.14b}$$

$$r(t) = y(t) - (C_2 + \gamma^{-2} B_1' X_\infty)(t)\hat{x}(t), \tag{3.14c}$$

$$v(t) = (\mathfrak{U} r(t), \tag{3.14d}$$

in which $\gamma^{-1}\mathfrak{U}$ *is a causal, strictly contractive operator on* $\mathscr{L}^2[0, T]$.

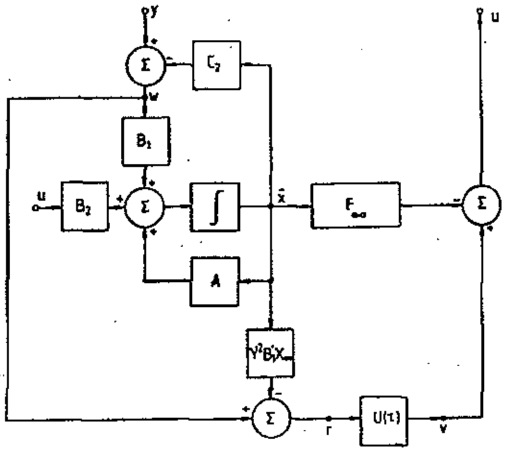

FIG. 3. *All controllers for problems of the second kind.*

Proof. (i) Suppose that a control of the form $u=\mathfrak{R}y$ exists such that $\|\mathcal{T}_{zw}\|<\gamma$. Then setting $u=\mathfrak{R}[C_2 \quad I][{x \atop w}]$ in Theorem 2.3 proves the "only if" part. To prove the "if" part, we suppose that a solution to (3.9) exists. We then invoke the observer property developed in (3.5) to (3.7) and the completing-the-square argument given in Theorem 2.1.

Part (ii) is a consequence of Theorem 2.4 on substituting $\hat{x}(t)$ for $x(t)$. □

A solution for problems of the second kind in which D_{12}, rather than D_{21}, is square is now given without proof. This particular result is needed in the next section in solutions to problems of the third kind and is solved via the dual game associated with the operator $[\mathfrak{P}_{11}^* \quad \mathfrak{P}_{21}^*]$ (see (1.1) for a definition).

THEOREM 3.3. *Given*

$$\dot{x}(t)=A(t)x(t)+B_1(t)w(t)+B_2(t)u(t), \qquad x(0)=0, \tag{3.15a}$$

$$z(t)=C_1(t)x(t)+u(t), \tag{3.15b}$$

$$y(t)=C_2(t)x(t)+D_{21}w(t); \tag{3.15c}$$

then (i) *a feedback controller exists such that* $\|\mathcal{T}_{zw}\|<\gamma$ *if and only if the* RDE

$$\begin{aligned}\dot{Y}_\infty=(A-B_1D_{21}'C_2)Y_\infty+Y_\infty(A-B_1D_{21}'C_2)'-Y_\infty(C_2'C_2-\gamma^{-2}C_1'C_1)Y_\infty\\ +B_1\tilde{D}_\perp'\tilde{D}_\perp B_1'\end{aligned} \tag{3.16}$$

has a solution on $[0, T]$ *with* $Y_\infty(0)=0$, *and* (ii) *every closed-loop operator* $\mathcal{T}_{zw}$ *with* $\|\mathcal{T}_{zw}\|<\gamma$, *corresponding to an output feedback control* $u=\mathcal{R}y$, *is generated by*

$$\dot{\hat{x}}(t)=(A-B_2C_1-H_\infty'C_2)(t)\hat{x}(t)-H_\infty'(t)y(t)-(B_2+\gamma^{-2}Y_\infty C_1')(t)v(t), \tag{3.17a}$$

$$u(t)=C_1(t)\hat{x}(t)+v(t), \tag{3.17b}$$

$$r(t)=C_2(t)\hat{x}(t)+y(t), \tag{3.17c}$$

$$v(t)=(\mathfrak{U}r)(t), \tag{3.17d}$$

in which $\gamma^{-1}\mathfrak{U}$ *is a causal, strictly contractive operator on* $\mathcal{L}^2[0, T]$ *and* $H_\infty=D_{21}B_1'+C_2Y_\infty$.

Proof. Apply Theorem 3.2 to the adjoint system associated with (3.15). □

3.3. Problems of the third kind. In this section we treat the case in which neither D_{21} nor D_{12} is square. We call such problems problems of the third kind. In the case where D_{21} has fewer rows than columns, the observer analysis in § 3.2 breaks down because the equation $\mathfrak{R}D_{21}(t)=-B_1(t)$ need not be soluble for $\mathfrak{R}$. What is required is a norm-preserving transformation to a new problem for which state reconstruction is possible. The desired transformation, which was first suggested in [9] in the time-invariant case, is immediate from identity (2.9). Suppose that

$$r=w-\gamma^{-2}B_1'X_\infty x, \tag{3.18}$$

$$v=u+F_\infty x. \tag{3.19}$$

Then (3.18), (3.19), and the completion-of-squares argument resulting in (2.9) gives us

$$\int_0^T (z'z-\gamma^2w'w)\,dt=\int_0^T (v'v-\gamma^2r'r)\,dt, \tag{3.20}$$

or, what is the same,

$$\|z\|_2^2-\gamma^2\|w\|_2^2=\|v\|_2^2-\gamma^2\|r\|_2^2 \tag{3.21}$$

for any r and v. Substituting (3.18) and (3.19) into (3.1) yields

$$\dot{x}=(A+\gamma^{-2}B_1B_1'X_\infty)x+B_1r+B_2u, \qquad x(0)=0, \tag{3.22a}$$

$$v=F_\infty x+Iu, \tag{3.22b}$$

$$y=(C_2+\gamma^{-2}D_{21}B_1'X_\infty)x+D_{21}r. \tag{3.22c}$$

LEMMA 3.4. *Suppose that the* RDE (3.9) *with* $X_\infty(T)=0$ *has a solution on* $[0, T]$. *Suppose also that* $\mathfrak{R}$ *is any controller, and that the control law* $u=\mathfrak{R}y$ *is applied to the systems given in* (3.1) *and* (3.22)[2]. *Then* $\|\mathscr{T}_{zw}\|<\gamma$ *if and only if* $\|\mathscr{T}_{vr}\|<\gamma$.

Proof. First, note that the relationship between r and w in (3.18) is causally invertible. The result now follows directly from (3.21). □

Lemma 3.4 shows that the tasks of designing controllers for the systems in (3.1) and (3.22) are interchangeable. The key feature of (3.22) is that D_{12} is square, and thus this system description is a problem of the second kind, which makes state reconstruction possible along the lines of § 3.2. If we define

$$F\left(\begin{bmatrix}\mathfrak{P}_{11} & \mathfrak{P}_{12}\\ \mathfrak{P}_{21} & \mathfrak{P}_{22}\end{bmatrix}, \mathfrak{R}\right):=\mathfrak{P}_{11}+\mathfrak{P}_{12}\mathfrak{R}(I-\mathfrak{P}_{22}\mathfrak{R})^{-1}\mathfrak{P}_{21}, \tag{3.23}$$

then it is immediate that

$$\left\{F\left(\begin{bmatrix}\mathfrak{P}_{11} & \mathfrak{P}_{12}\\ \mathfrak{P}_{21} & \mathfrak{P}_{22}\end{bmatrix}, \mathfrak{R}\right)\right\}^*=F\left(\begin{bmatrix}\mathfrak{P}_{11}^* & \mathfrak{P}_{21}^*\\ \mathfrak{P}_{12}^* & \mathfrak{P}_{22}^*\end{bmatrix}, \mathfrak{R}^*\right) \tag{3.24}$$

and that

$$\left\|F\left(\begin{bmatrix}\mathfrak{P}_{11} & \mathfrak{P}_{12}\\ \mathfrak{P}_{21} & \mathfrak{P}_{22}\end{bmatrix}, \mathfrak{R}\right)\right\|_\infty=\left\|F\left(\begin{bmatrix}\mathfrak{P}_{11}^* & \mathfrak{P}_{21}^*\\ \mathfrak{P}_{12}^* & \mathfrak{P}_{22}^*\end{bmatrix}, \mathfrak{R}^*\right)\right\|_\infty. \tag{3.25}$$

A direct application of the results of § 2.3 (see (2.31)) shows that a state-space model for the adjoint operator associated with (3.22) is given by

$$-\dot{p}=(A+\gamma^{-2}B_1B_1'X_\infty)'p+F_\infty'u_1+(C_2+\gamma^{-2}D_{21}B_1'X_\infty)'u_2, \qquad p(T)=0, \tag{3.26a}$$

$$y_1=B_1'p+D_{21}'u_2, \tag{3.26b}$$

$$y_2=B_2'p+u_1. \tag{3.26c}$$

[2] For the purposes of this result, $\mathfrak{R}$ need not be linear.

If we now substitute (3.26) into (3.14), using (3.9) to (3.11), we obtain a representation formula for $\mathfrak{R}^*$ and thus also for $\mathfrak{R}$. This calculation forms the basis of the next result.

THEOREM 3.5. *Given*

$$\dot{x}(t)=A(t)x(t)+B_1(t)w(t)+B_2(t)u(t), \qquad x(0)=0, \tag{3.27a}$$

$$z(t)=C_1(t)x(t)+D_{12}(t)u(t), \tag{3.27b}$$

$$y(t)=C_2(t)x(t)+D_{21}(t)w(t), \tag{3.27c}$$

then (i) *a causal output-feedback control of the form* $u=\mathfrak{R}y$ *exists such that* $\|\mathcal{T}_{zw}\|<\gamma$ *if and only if the* RDEs (3.9) *and* (3.30a) (*given below*) *have solutions on* $[0, T]$, *and* (ii) *every closed-loop operator* $\mathcal{T}_{zw}$ *with* $\|\mathcal{T}_{zw}\|<\gamma$, *corresponding to an output feedback control* $u=\mathcal{R}y$, *is generated by*

$$\dot{x}_k(t)=A_k(t)x_k(t)+B_{k1}(t)y(t)+B_{k2}(t)v(t), \qquad x_k(0)=0, \tag{3.28a}$$

$$u(t)=C_{k1}(t)x_k(t)+v(t), \tag{3.28b}$$

$$r(t)=C_{k2}(t)x_k(t)+y(t), \tag{3.28c}$$

$$v(t)=(\mathfrak{U}r)(t), \tag{3.28d}$$

in which $\gamma^{-1}\mathfrak{U}$ *is a causal, strictly contractive operator on* $\mathcal{L}^2[0, T]$ *and*

$$\begin{aligned} A_k :=\; & A+\gamma^{-2}B_1B_1'X_\infty-B_2F_\infty-(B_1D_{21}'+Z_\infty(C_2'+\gamma^{-2}X_\infty B_1D_{21}')) \\ & \cdot(C_2+\gamma^{-2}D_{21}B_1'X_\infty), \end{aligned} \tag{3.29a}$$

$$\begin{bmatrix} C_{k1} \\ C_{k2} \end{bmatrix} := \begin{bmatrix} F_\infty \\ C_2+\gamma^{-2}D_{21}B_1'X_\infty \end{bmatrix}, \tag{3.29b}$$

$$[B_{k1} \quad B_{k2}] := [-B_1D_{21}'-Z_\infty(C_2+\gamma^{-2}D_{21}B_1'X_\infty)' \mid -B_2-\gamma^{-2}Z_\infty F_\infty'], \tag{3.29c}$$

with

$$\begin{aligned} & \dot{Z}_\infty=A_zZ_\infty+Z_\infty A_z'-Z_\infty(C_{k2}'C_{k2}-\gamma^{-2}C_{k1}'C_{k1})Z_\infty+B_1\tilde{D}_\perp'\tilde{D}_\perp B_1', \\ & Z_\infty(0)=0, \end{aligned} \tag{3.30a}$$

in which

$$A_z=A-B_1D_{21}'C_2+\gamma^{-2}B_1\tilde{D}_\perp'\tilde{D}_\perp B_1'X_\infty. \tag{3.30b}$$

Proof. (i) Suppose that a control given by $u=\mathfrak{R}y$ exists such that $\|\mathcal{T}_{zw}\|<\gamma$. Then $u=\mathfrak{R}[C_2 \quad D_{21}][\begin{smallmatrix} x \\ w \end{smallmatrix}]$ together with Theorem 2.3 implies that (3.9) has a solution on $[0, T]$. Since $X_\infty(t)$ exists on $[0, T]$, we may apply the control $u_2=\mathfrak{R}^*y_2$ to (3.26) to obtain $\|y_1\|_2<\gamma\|u_1\|_2$ for all $u_1\neq 0$ by Lemma 3.4 and (3.26). We may now use $u_2=\mathfrak{R}^*[B_2' \quad I][\begin{smallmatrix} p \\ u_1 \end{smallmatrix}]$ together with the "only if" part of Theorem 3.3 to establish the existence of $Z_\infty(t)$ on $[0, T]$. Sufficiency is immediate from Lemma 3.4 and Theorem 3.3.

(ii) The set of all closed-loop operators corresponding to (3.27) and $u=\mathfrak{R}y$ is the same as the set of all closed-loop operators corresponding to (3.26) and $u_2=\mathfrak{R}^*y_2$ by Lemma 3.4. Since (3.26) is a problem of the second kind, we may invoke Theorem 3.3 to complete the proof. □

In the last part of this section, we replace Z_∞ with Y_∞, which is the solution to the RDE introduced in Theorem 3.3. As we will show, Y_∞ is the dual of X_∞, and there is a close connection between X_∞, Y_∞, and Z_∞. Before supplying this connection, we give a more general version of the connection between a Riccati equation and the Hamiltonian matrix associated with the two-point boundary value problem.

LEMMA 3.6. *The* RDE

$$\dot{P} = A'P + PA - PDP + Q, \qquad P(0) = M \tag{3.31}$$

has a solution on $[0, T]$ *if and only if there exists an* X *such that the boundary value problem*

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} X - \begin{bmatrix} A & -D \\ -Q & -A' \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \tag{3.32}$$

has a solution on $[0, T]$ *with* $P_1(t)$ *nonsingular for all* $t \in [0, T]$ *and* $P_2(0)P_1(0)^{-1} = M$. *In this case,* $P(t) = P_2(t)P_1(t)^{-1}$ *is a solution to* (3.31) *and*

$$\begin{bmatrix} I \\ P \end{bmatrix}(A - DP) - \begin{bmatrix} A & -D \\ -Q & -A' \end{bmatrix}\begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} 0 \\ \dot{P} \end{bmatrix}. \tag{3.33}$$

Proof. Suppose that the RDE has a solution $P(t)$. Then (3.33) is immediate, and the result follows.

Conversely, with $P = P_2P_1^{-1}$ we have from (3.32),

$$\begin{aligned}[P \ \ -I]\begin{bmatrix} A & -D \\ -Q & -A' \end{bmatrix}\begin{bmatrix} I \\ P \end{bmatrix} &= [P \ \ -I]\left\{\begin{bmatrix} I \\ P \end{bmatrix} P_1 X - \frac{d}{dt}\begin{bmatrix} P_1 \\ P_2 \end{bmatrix}\right\} P_1^{-1} \\ &= \dot{P}_2 P_1^{-1} - P_2P_1^{-1}\dot{P}_1P_1^{-1} = \dot{P}. \end{aligned}$$

□

THEOREM 3.7. *Suppose that* (3.9) *has a solution* $X_\infty(t)$ *on* $[0, T]$. *Then* (3.30) *has a solution* $Z_\infty(t)$ *on* $[0, T]$ *if and only if* (3.16) *has a solution* $Y_\infty(t)$ *on* $[0, T]$ *and* $\rho(X_\infty Y_\infty)(t) < \gamma^2$ *for all* $t \in [0, T]$. *Furthermore,* $Z_\infty(t) = Y_\infty(I - \gamma^{-2}X_\infty Y_\infty)^{-1}(t)$.

Proof. A straightforward calculation using (3.9) shows that

$$\begin{bmatrix} I & -\gamma^{-2}X_\infty \\ 0 & I \end{bmatrix} H_Y \begin{bmatrix} I & \gamma^{-2}X_\infty \\ 0 & I \end{bmatrix} = H_Z + \begin{bmatrix} 0 & \gamma^{-2}\dot{X}_\infty \\ 0 & 0 \end{bmatrix}, \tag{3.34}$$

where H_Y and H_Z are the Hamiltonian matrices from the boundary value problems associated with the RDEs for Y_∞ and Z_∞, respectively.

Suppose that Z_∞ exists. Since $X_\infty, Z_\infty \geqq 0$, it follows that $(I + \gamma^{-2}X_\infty Z_\infty)(t)$ is nonsingular for all $t \in [0, T]$. Also, $Z_\infty(I + \gamma^{-2}X_\infty Z_\infty)^{-1}(0) = 0$. Using (3.30a) and (3.34), we see that

$$\begin{aligned} H_Y\begin{bmatrix} I + \gamma^{-2}X_\infty Z_\infty \\ Z_\infty \end{bmatrix} &= \begin{bmatrix} I + \gamma^{-2}X_\infty Z_\infty \\ Z_\infty \end{bmatrix}(A_z - Z_\infty(C_{k2}'C_{k2} - \gamma^{-2}C_{k1}'C_{k1}))' \\ &\quad - \frac{d}{dt}\begin{bmatrix} I + \gamma^{-2}X_\infty Z_\infty \\ Z_\infty \end{bmatrix}. \end{aligned} \tag{3.35}$$

So $Y_\infty = Z_\infty(I + \gamma^{-2}X_\infty Z_\infty)^{-1}$ is a solution to (3.16) by Lemma 3.6. Furthermore,

$$\rho[X_\infty Y_\infty] = \rho[X_\infty Z_\infty(I + \gamma^{-2}X_\infty Z_\infty)^{-1}] = \gamma^2 \frac{\rho(X_\infty Z_\infty)}{\gamma^2 + \rho(X_\infty Z_\infty)} < \gamma^2. \tag{3.36}$$

Conversely, if Y_∞ exists and $\rho(X_\infty Y_\infty) < \gamma^2$, then $I - \gamma^2 X_\infty Y_\infty$ is nonsingular on $[0, T]$, $Y_\infty(I - \gamma^{-2}X_\infty Y_\infty)^{-1}(0) = 0$, and from (3.16) we have that

$$\begin{aligned} H_Z\begin{bmatrix} I - \gamma^{-2}X_\infty Y_\infty \\ Y_\infty \end{bmatrix} &= \begin{bmatrix} I - \gamma^{-2}X_\infty Y_\infty \\ Y_\infty \end{bmatrix}(A - B_1D_{21}'C_2 - Y_\infty(C_2'C_2 - \gamma^{-2}C_1'C_1))' \\ &\quad - \frac{d}{dt}\begin{bmatrix} I - \gamma^{-2}X_\infty Y_\infty \\ Y_\infty \end{bmatrix}. \end{aligned}$$

Thus $Z_\infty = Y_\infty(I - \gamma^{-2}X_\infty Y_\infty)^{-1}$ is a solution to (3.30a). □

Substituting $Z_\infty = Y_\infty(I-\gamma^{-2}X_\infty Y_\infty)^{-1} = (I-\gamma^{-2}Y_\infty X_\infty)^{-1}Y_\infty$ into Theorem 3.5 gives our final result.

THEOREM 3.8. *Given*

$$\dot{x}(t) = A(t)x(t) + B_1(t)w(t) + B_2(t)u(t), \qquad x(0)=0, \tag{3.37a}$$

$$z(t) = C_1(t)x(t) + D_{12}(t)u(t), \tag{3.37b}$$

$$y(t) = C_2(t)x(t) + D_{21}(t)w(t), \tag{3.37c}$$

then (i) *an output feedback* $u = \mathfrak{R}y$ *exists such that* $\|\mathcal{T}_{zw}\| < \gamma$ *if and only if* (a) *the Riccati equation* (3.9) *has a solution* X_∞ *on* $[0, T]$, (b) *the Riccati equation* (3.16) *has a solution* Y_∞ *on* $[0, T]$, *and* (c) $\rho(X_\infty Y_\infty) < \gamma^2$ *for all* $t \in [0, T]$; *and* (ii) *every closed-loop operator* $\mathcal{T}_{zw}$ *with* $\|\mathcal{T}_{zw}\| < \gamma$, *corresponding to an output feedback control* $u = \mathcal{R}y$, *is generated by*

$$\dot{x}_k(t) = A_k(t)x_k(t) + B_{k1}(t)y(t) + B_{k2}(t)v(t), \qquad x_k(0)=0, \tag{3.38a}$$

$$u(t) = C_{k1}(t)x_k(t) + v(t), \tag{3.38b}$$

$$r(t) = C_{k2}(t)x_k(t) + y(t), \tag{3.38c}$$

$$v(t) = (\mathfrak{U}r)(t), \tag{3.38d}$$

in which $\gamma^{-1}\mathfrak{U}$ *is a causal, strictly contractive operator on* $\mathcal{L}^2[0, T]$ *and*

$$\begin{bmatrix} C_{k1} \\ C_{k2} \end{bmatrix} := \begin{bmatrix} F_\infty \\ C_2 + \gamma^{-2}D_{21}B_1'X_\infty \end{bmatrix}, \tag{3.39a}$$

$$[B_{k1} \quad B_{k2}] := -(I-\gamma^{-2}Y_\infty X_\infty)^{-1}[H_\infty' \,|\, B_2 + \gamma^{-2}Y_\infty C_1'D_{12}], \tag{3.39b}$$

$$A_k := A + \gamma^{-2}B_1B_1'X_\infty - B_2F_\infty + B_{k1}C_{k2}. \tag{3.39c}$$

Proof. This follows from Theorems 3.5 and 3.7. □

As a check, we note that (3.38) and (3.39) reduce to those in [9], [11], [13], and [21] in the time-invariant case. In the infinite-horizon case, we must establish an internal stability property; this has already been done in the case of time-invariant problems [9], [11], [13], [21].

Appendix. Suppose that the design problem is described by the equation

$$\begin{array}{c} \\ n \\ p \\ q \end{array} \begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \begin{array}{c} \begin{array}{ccc} n & l & m \end{array} \\ \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \end{array} \begin{bmatrix} x \\ w \\ u \end{bmatrix}, \tag{A.1}$$

in which each of the matrices has entries that are continuous functions of time. We will also assume that D_{12} and D_{21} have, respectively, full column and row rank for all $t \in [0, T]$. The controller measures y and generates the control u via $u = \mathfrak{R}y$ and has the task of minimizing the worst-case energy gain between z and w.

It is the aim of this appendix to establish that, without loss of generality, we may assume that

$$\text{(i)} \quad D_{12}'D_{12} = I_m, \tag{A.2}$$

$$\text{(ii)} \quad D_{21}D_{21}' = I_q, \tag{A.3}$$

$$\text{(iii)} \quad D_{11} = 0, \tag{A.4}$$

$$\text{(iv)} \quad D_{22} = 0. \tag{A.5}$$

If one prefers, it is possible to have (i) and (ii) replaced by

(A.2)′ (i)′ $D'_{12}=[I_m \quad 0_{p-m}]$,

(A.3)′ (ii)′ $D_{21}=[I_q \quad 0_{l-q}]$,

which are clearly special cases of (i) and (ii).

To begin, we consider the closed-loop configuration in Fig. A.1, which contains the direct feedthrough matrix from (A.1), the controller, and four scaling matrices. S_1 and S_2 will be chosen to enforce (A.2) and (A.3), while S_3 and S_4 ensure that (A.2)′ and (A.3)′ are satisfied (should we wish to include them).

Suppose that

(A.6) $$D'_{12}D_{12}=N'N$$

and

(A.7) $$D_{21}D'_{21}=MM'$$

are the Cholesky factorizations. Then the entries of M and N are continuous, since they are expressible in terms of the entries of D_{12} and D_{21} [12, p. 88]. It is easy to check that $S_1=M^{-1}$ and $S_2=N^{-1}$ will achieve the desired orthogonalization of D_{21} and D_{12}, and that these scaling matrices will not destroy the continuity properties assumed for the original problem. It is evident from Fig. A.1 that we would design the controller $\tilde{K}$ for the scaled problem, and then back substitute through S_1 and S_2 for K. Next, we introduce the orthogonal extensions $D_\perp$ and $\tilde{D}_\perp$ such that $[D_{12} \quad D_\perp]$ and $[D'_{21} \quad \tilde{D}'_\perp]$ are orthogonal. It is a consequence of Doležal's theorem that the extensions $D_\perp$ and $\tilde{D}_\perp$ exist and are continuous [29, Cor. 3, p. 70]. Setting

(A.8) $$S'_3=[D_{12} \quad D_\perp], \qquad S_4=[D'_{21} \quad \tilde{D}'_\perp]$$

in Fig. A.1 leads to a direct feedthrough matrix of the form

(A.9) $$\begin{bmatrix} D_{1111} & D_{1112} & I_m \\ D_{1121} & D_{1122} & 0_{p-m} \\ I_q & 0_{l-q} & D_{22} \end{bmatrix}.$$

The partitioning of D_{11} into D_{11ij} $i,j=1,2$ is induced by (A.2)′ and (A.3)′. Since S_3 and S_4 are orthogonal and thus norm preserving, these scale factors do not change the set of admissible controllers and therefore the controller representation formula.

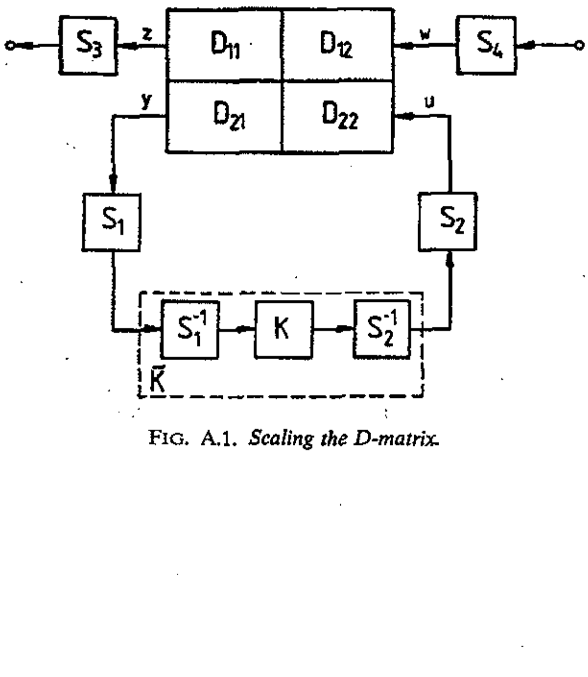

FIG. A.1. *Scaling the D-matrix.*

We are now in a position to invoke a loop-shifting argument that eliminates both D_{11} and D_{22}, and leads to a significant simplification in the main analysis [11], [24]. Defining

$$\gamma_p = \max\left(\sup_{t\in[0,T]} \|D_{1121} \quad D_{1122}\|_2, \sup_{t\in[0,T]} \|D'_{1112} \quad D'_{1122}\|_2\right), \tag{A.10}$$

$$Q_\infty = -(D_{1111} + D_{1112}(\gamma_p^2 I - D'_{1122}D_{1122})^{-1} D'_{1122}D_{1121}), \tag{A.11}$$

$$F_\infty = (I + Q_\infty D_{22})^{-1} Q_\infty, \tag{A.12}$$

allows us to mimic the arguments in [11], [24], which remove D_{11} and D_{22}. In conclusion, we mention that the Cholesky factors in the Julia operator of [11] always exist and have continuous entries. □

Addendum. After the completion of the first version of this paper, we became aware of [5] and [28]. Reference [28] discusses the connections between $\mathscr{H}^\infty$ control and game theory, while [5] applies discrete game theory to the state-feedback $\mathscr{H}^\infty$-control problem.

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