Automatica, Vol. 28, No. 2, pp. 417–421, 1992
Printed in Great Britain.

0005-1098/92 $5.00 + 0.00
Pergamon Press plc

Brief Paper

The Combined Sensitivity and Phase Margin Problem*

WEI-YONG YAN,†‡ BRIAN D. O. ANDERSON†
and ROBERT R. BITMEAD†

Key Words—Linear systems; robust control; interpolation; sensitivity analysis; phase control.

Abstract—Any two design performance indices used in control system design have the potential to conflict with each other and a good control system is often some kind of compromise which optimizes neither index but secures satisfactory values for both. The objective of this paper is to study the two indices of sensitivity and phase margin simultaneously and reveal how the indices affect each other. The combined sensitivity and phase margin problem is basically solved, and both upper and lower bounds are derived on the achievable values of one index subject to a constraint on the other. It is shown that for a minimum phase plant, the optimal sensitivity and the optimal phase margin can be achieved simultaneously in a closed-loop system. Another particularly important result is that for a nonminimum phase plant, phase margin maximization will lead to an arbitrarily large sensitivity while sensitivity minimization does not cause an arbitrarily small phase margin. All the results are confined to scalar linear time-invariant plants.

1. *Introduction and preliminaries*

In a great many control system designs, a number of performance measures are simultaneously in contemplation. A good design is not necessarily one which optimizes one performance measure (since the value assumed by another measure may then be quite unsatisfactory), but rather one which achieves a satisfactory trade-off between the different measures. It then becomes important to understand what are fundamental trade-offs possible, and in Yan and Anderson (1990) the authors have investigated this problem for the case of gain margin and sensitivity. There they found for example that optimizing (maximizing) the gain margin led to infinite sensitivity, and they described how a curve depicting the trade-off between gain margin and sensitivity could be defined. In this paper we will investigate another kind of trade-off problem—between sensitivity and phase margin, i.e. the combined sensitivity and phase margin (CSPM) problem. Briefly, the CSPM problem is to characterize all the constraints on sensitivity and on phase margins for a given plant which can be simultaneously satisfied in a closed-loop system with LTI compensation.

Our tool is to use some ideas of interpolation theory which have been applied to several problems of robust control by some investigators (e.g. Francis and Zames, 1984; Khargonekar and Tannenbaum, 1985; Kimura, 1987; Tannenbaum, 1980). In fact, many robust control problems in the scalar case can be reduced to special cases of a general interpolation problem: Given a simply connected domain X containing 0, 1, find a real proper rational function mapping the closed right half plane (or the outside of the open unit disk) to X and satisfying certain interpolation conditions. Moreover, the solution to this interpolation problem is just the sensitivity function associated with the corresponding control problem, so that knowing the plant and the sensitivity function, the controller transfer function is easily obtained. Of course, it is not always possible to express a solvability condition for the interpolation problem because of the requirements of both reality and rationality of solutions. Recently, it has been shown in Yan and Anderson (1990) that if the conformal equivalence from X to the unit disk can be constructed and is symmetric with respect to the real axis, then there is indeed an explicit necessary and sufficient condition available for the problem solvability.

Let $p(s)$ be a proper SISO, continuous-time LTI plant with closed right half plane (RHP) zeros $z_1, z_2, \ldots, z_m$ (including those zeros at infinity, if any) and closed RHP poles $p_1, p_2, \ldots, p_n$. Let C denote the complex plane, $\bar{H}$ the closed right half plane including the point at infinity, D the open unit disk and $\bar{D}$ the closed unit disk. Define

$$\alpha \triangleq \sup\{\gamma > 0;\ \text{there exists an analytic function } F(s): \bar{D} \to D \text{ satisfying (a) and (b)}\}, \quad (1.1)$$

(a) the zeros of $F(s)$ contain $\{a_1, a_2, \ldots, a_n\}$,
(b) the zeros of $F(s) - \gamma$ contain $\{a_{n+1}, a_{n+2}, \ldots, a_{n+m}\}$

where

$$a_i \triangleq \begin{cases} h(p_i), & i = 1, \ldots, n \\ h(z_{i-n}), & i = n+1, \ldots, n+m, \end{cases}$$

with $h(s) = (1-s)/(1+s)$.

The relevance of this quantity α will be seen below. The following mathematical problem proves to be an abstract of many robust control problems.

General problem. Let $G \subset \mathbb{C}$ be a given simply connected domain containing $0, 1$. Find (if possible) a real rational analytic function

$$S(s): \bar{H} \to G,$$

satisfying the interpolation conditions:

(i) the zeros of $S(s)$ contain $\{p_1, p_2, \ldots, p_n\}$,
(ii) the zeros of $S(s) - 1$ contain $\{z_1, z_2, \ldots, z_m\}$.

Lemma 1.1 (Yan and Anderson, 1990). Suppose that there exists a conformal equivalence $\varphi(s): G - D$ with

$$\overline{\varphi(\bar{s})} = \varphi(s) \quad \text{and} \quad \varphi(0) = 0. \quad (1.2)$$

Let α be defined as in (1.1). Then the general problem is solvable iff $|\varphi(1)| < \alpha$.

* Received 22 March, 1990; revised 1 March, 1991; received in final form 6 July, 1991. The original version of this paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor H. Kimura under the direction of Editor H. Kwakernaak.

† Department of Systems Engineering, Research School of Physical Sciences and Engineering, the Australian National University, GPO Box 4, Canberra, ACT 2601, Australia.

‡ Author to whom all correspondence and page proofs should be addressed.

2. *Two separate robust control problems*

As preparation for investigation of the combined sensitivity and phase margin problem, in this section we discuss two simple separate problems—the phase margin (PM) problem and the generalized sensitivity (GS) problem, as defined below.

Given a plant $P(s)$, we denote the set of all its proper LTI stabilizing controllers by Ω. Then it is a standard fact (Youla *et al.*, 1974) that $C(s)$ is in Ω iff its corresponding sensitivity function is analytic $\bar{H}$ and satisfies the interpolation conditions (i)–(ii) in the previous section. The PM problem is to find $C(s) \in \Omega$ which internally stabilizes $e^{i\phi}P(s)$ for all $\phi \in [-\theta_1, \theta_2]$, where $0 \leq \theta_i < \pi$, $i = 1, 2$. Although we shall only consider the case where $\theta_1 = \theta_2 = \theta$, at the moment we relax this assumption temporarily. The necessity of the assumption will be seen in the development to follow. The GS problem is to find $C(s) \in \Omega$ for given $r > 0$ and $x \in \mathbb{R}$ such that

$$\|(1+PC)^{-1} - x\| < r. \tag{2.1}$$

Both the problems will be reduced to a special case of the general problem stated in the previous section.

Let us first consider the PM problem. It is not hard to see that $C(s)$ is a solution to the PM problem iff $S(s) = (1+PC)^{-1}$ is real rational and satisfies the interpolation conditions (i)–(ii) and

$$1 + e^{i\phi}P(s)C(s) \neq 0,\ \forall\phi \in [-\theta_1, \theta_2] \quad\text{and}\quad \forall s \in \bar{H}. \tag{2.2}$$

Since (2.2) is equivalent to requiring that $S(s)$ be an analytic function from $\bar{H}$ to

$$G_1 \triangleq \{s \in \mathbb{C}: s \neq [1 - i\cot(\phi/2)]/2,\ \forall\phi \in [-\theta_1, \theta_2]\}, \tag{2.3}$$

the PM problem is a case of the general problem with $G = G_1$.

The conformal equivalence φ from $G = G_1$ to D can be derived as follows

$$\varphi(s) = i\frac{\sqrt{(s-s_1)/(s_2-s)} - \sqrt{-s_1/s_2}}{\sqrt{(s-s_1)/(s_2-s)} + \sqrt{-\bar{s}_1/\bar{s}_2}} \times \frac{1+\sqrt{-\bar{s}_1/\bar{s}_2}}{1+\sqrt{-s_1/s_2}}, \tag{2.4}$$

with

$$|\varphi(1)|^2 = \left|\frac{s_1\bar{s}_2 + \bar{s}_1 s_2 + 2|s_1|\,|s_2|}{4\bar{s}_1 s_2}\right|. \tag{2.5}$$

Note that in general $\varphi(s)$ does not possess the property $\varphi(s) = \varphi(\bar{s})$ except in the case $\theta_1 = \theta_2 = \theta$, which is the reason for assuming that $\theta_1 = \theta_2 = \theta$ in the phase margin problem. In this case, $s_2 = \bar{s}_1$, and an intricate calculation leads to

$$\varphi(s) = \frac{|s_1| - \sqrt{(s-s_1)(s-\bar{s}_1)}}{s-1} \quad\text{and}\quad |\varphi(1)| = \sin\frac{\theta}{2}. \tag{2.6}$$

By Lemma 1.1, the following result is established immediately.

Theorem 2.1. Let the plant $P(s)$ be given and α be defined as in (1.1). The phase margin problem for $P(s)$ is solvable iff $\sin(\theta/2) < 0$.

Corollary 2.1. The maximal achievable phase margin of $P(s)$ is equal to $2\sin^{-1}\alpha$ if $\alpha \leq 1$, and π if $\alpha > 1$, in the sense that there exists a compensator which yields a closed-loop phase margin arbitrarily approaching this maximal phase margin.

We now turn to the GS problem. Obviously, this problem includes the sensitivity problem and the complementary sensitivity problem as special cases: finding the infimum of all such positive numbers of r for which the generalized sensitivity problems with $x = 0$ and $x = 1$ are solvable, respectively. Another important case of the generalized sensitivity problem is where $x = 1/2$. In this case, the corresponding problem of finding the infimum $r > 0$ is called the problem of compromise sensitivity minimization, and $\|(1+PC)^{-1} - 1/2\|$ a compromise sensitivity. The physical implication of the compromise sensitivity minimization is that this minimization causes both the sensitivity function and the complementary sensitivity function to be as close to 1/2 as possible simultaneously. The conditions for the solvability of the GS problem are completely characterized in the following result, which is easily proved on noting that the problem amounts to the general problem with

$$G = G_2 \triangleq \{s \in \mathbb{C};\ |s - x| < r\}.$$

Theorem 2.2. Suppose the plant $P(s)$ is given with α be defined as in (1.1). Let $r > 0$ and $x \in \mathbb{R}$ be given.

(i) In case $P(s)$ has both unstable zeros and unstable poles, the GS problem for (r, x) is solvable iff

$$r > \frac{1/\alpha + \sqrt{(2x-1)^2 - 1 + 1/\alpha^2}}{2}; \tag{2.7}$$

and in particular, the GS problem for $r > 0$ and $x = 1/2$ is solvable iff

$$r > \beta \triangleq \frac{1+\sqrt{1-\alpha^2}}{2\alpha}. \tag{2.8}$$

(ii) In case $P(s)$ has no unstable zeros but has at least one unstable pole, the GS problem for (r, x) is solvable iff $r > |x|$.

(iii) In case $P(s)$ has no unstable poles but has at least one unstable zero, the GS problem for (r, x) is solvable iff $r > |x - 1|$.

(iv) In case $P(s)$ has neither unstable zeros nor unstable poles, the GS problem is always solvable for all $r > 0$ and $x \in \mathbb{R}$.

The following corollary is direct from the above theorem.

Corollary 2.2. With the same assumption as in Theorem 2.2. It is further assumed that the plant $P(s)$ has both unstable zeros and unstable poles. There hold

1. $\inf_{C(s)\in\Omega} \|(1+PC)^{-1} - 1/2\| = \beta.$

2. $\inf_{C(s)\in\Omega} \|(1+PC)^{-1} - 1/2\| < \|(1+PC)^{-1} - x\|, \quad \forall x \neq 1/2$ and $C(s) \in \Omega$.

3. *The combined sensitivity and phase margin problem*

In the previous section, all the constraints on phase margin or on sensitivity for a given plant have been characterized which can be satisfied in a closed-loop system using LTI compensation. Of course, maximizing a phase margin is one thing while minimizing a sensitivity is another thing. In practice, one might like to both maximize the phase margin and also minimize the sensitivity. Intuitively, it is understandable that maximization of phase margin does not always result in minimization of sensitivity, and vice versa. Therefore, one naturally wants to know what happens to one index when one optimizes another index, whether it is possible to simultaneously optimize both the indices, what the best achievable value for one index is subject to a constraint on the other, and so on.

The purpose of this section is to answer all these questions by studying a basic problem—the combined sensitivity and phase margin (CSPM) problem: Given a plant model $P(s)$, find a proper LTI compensator which stabilizes $e^{i\phi}P(s)$ for all $\phi \in [-\theta, \theta]$ such that the closed-loop sensitivity is less than r, where $r > 1$ and $0 \leq \theta < \pi$. Obviously, this problem is solvable iff there exists one compensator simultaneously solving the sensitivity problem for r and the phase margin problem for θ. From this, it can be further seen that the problem is equivalent to the general problem with

$$G = G_3 \triangleq \{s \in \mathbb{C};\ |s| < r \text{ and } s \neq 1[1 + i\cot(\phi/2)]/2,\ \forall\phi \in [-\theta, \theta]\}.$$

The conformal equivalence ψ from G_3 to D can be expressed as a composition of three conformal mappings, i.e. $\psi = \psi_3 \circ \psi_2 \circ \psi_1$, where

$$\psi_1(s) = -\sqrt{\frac{r}{4r-1}\,\frac{2s-1-i\sqrt{4r-1}}{2s-1+i\sqrt{4r-1}}},$$

$$\psi_3(s) = \frac{s - \psi_2(\psi_1(0))}{s - \psi_2(\psi_1(0))},$$

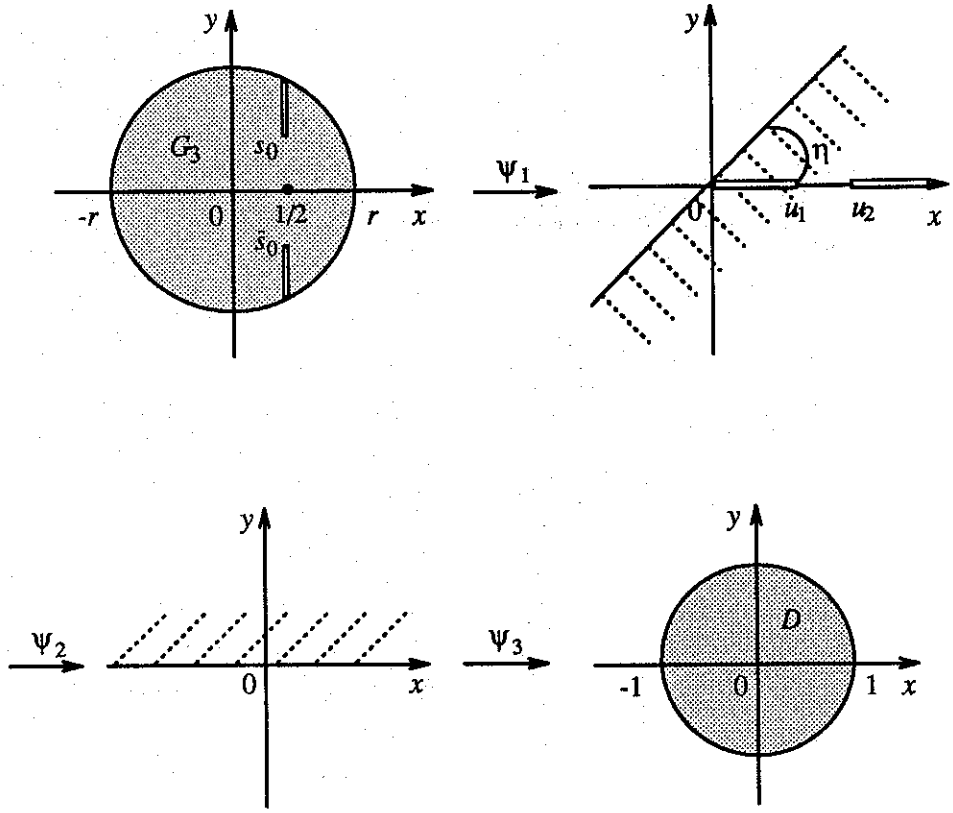

FIG. 1. Construction of conformal mapping from G_3 to D.

and ψ_2 is the inverse of the following Schwarz–Christoffel conformal transformation

$$\xi(s) = \int_0^s (v - v_1)(v - v_2)(v - v_3)^{-1}(v - v_4)^{1+(\eta/\pi)}\, dv + c, \tag{3.1}$$

with $\eta = \tan^{-1}(\sqrt{4r-1})$. The construction is depicted in Fig. 1. In the above, the parameters $c, v_i, i = 1, \ldots, 4$ are determined by the relations

$$\zeta(v_1) = u_1 \triangleq \sqrt{\frac{r}{4r-1}\frac{\sqrt{4r-1} - \cot(\theta/2)}{\sqrt{4r-1} + \cot(\theta/2)}},$$

$$\zeta(v_2) = u_2 \triangleq \sqrt{\frac{r}{4r-1}\frac{\sqrt{4r-1} + \cot(\theta/2)}{\sqrt{4r-1} - \cot(\theta/2)}},$$

$$\zeta(v_3) = 0, \quad \xi(v_4) = \infty, \quad \xi(\infty) = \infty,$$

which appear very difficult to solve. More unfortunately, it is generally impossible to express explicitly the inverse of $\zeta(\cdot)$, and therefore evaluation of $|\zeta(1)|$ is effectively impossible. Nevertheless, we can still come up with two partial results as follows. The first result is readily obtained upon noting by the Riemann Mapping Theorem the existence of the required conformal equivalence from G_3 to D.

Theorem 3.1. Consider a plant $P(s)$ with α defined as in (1.1). If $\alpha = 1$, the CSPM problem is solvable for any $r > 1$ and $0 \le \theta < \pi$; if $\alpha = \infty$, the CSPM problem is solvable for any $r > 0$ and $0 \le \theta < \pi$.

It is known that the minimal achievable sensitivity equals $1/\alpha$ in the sense that for any given $r > 1/\alpha$ there always exists a compensator such that the closed-loop sensitivity is less than r, i.e. the sensitivity problem is solvable for $r > 1/\alpha$. Recall that the sensitivity problem is equivalent to the general problem with $G = \{s \in \mathbb{C}; |s| < r\}$. Now it can be seen that $G_3 = \{s \in \mathbb{C}; |s| < r\}$ iff $\theta \le 2\sin^{-1}(1/2r)$. This means that the CSPM problem reduces to the sensitivity problem when $\theta \le 2\sin^{-1}(1/2r)$. This observation immediately yields the following result.

Theorem 3.2. Given a nonminimum phase plant $P(s)$ with α defined as in (1.1), the CSPM problem is always solvable for any $r > 1/\alpha$ and $\theta \le 2\sin^{-1}(1/2r)$; moreover, a closed-loop sensitivity of less than r implies that the closed-loop phase margin is at least $2\sin^{-1}(1/2r)$.

To discuss further the CSPM problem, it is convenient to deal instead with the modified combined sensitivity and phase margin (MCSPM) problem: given $r > 1/2$ and $0 \le \theta < \pi$, find a proper compensator $C(s)$ both stabilizing $e^{i\phi}P(s)$ for all $\phi \in [-\theta, \theta]$ and satisfying $\|(1+PC)^{-1} - 1/2\| < r$. One can immediately see that the MCSPM problem is equivalent to the general problem with

$$G = G_4 \triangleq \{s \in \mathbb{C}; |s - 1/2| < r \text{ and } s \ne [1 + i\cot(\phi/2)]/2, \forall \phi \in [-\theta, \theta]\}.$$

Theorem 3.3. Suppose that $P(s)$ is a nonminimum phase plant. Let $r > 1/2$ and $0 \le \theta < \pi$, and define $x_\theta \triangleq \frac{1}{2}\cot(\theta/2)$. Then the MCSPM problem for (r, θ) is solvable iff $\sin\theta/2 < \alpha$ and $r \in \Gamma_\theta$, where

$$\Gamma_\theta \triangleq (\beta, x_\theta) \cup (\max\{x_\theta, \sqrt{f(x_\theta)}\}, \infty), \tag{3.2}$$

with

$$\beta \triangleq \frac{1 + \sqrt{1-\alpha^2}}{2\alpha} \quad \text{and} \quad f(x) \triangleq \frac{x(2\sqrt{1-\alpha^2}x + \alpha)}{2(2\alpha x - \sqrt{1-\alpha^2})}. \tag{3.3}$$

Proof. In view of Theorem 2.1, we might as well assume that $\theta < \sin^{-1}\alpha$, which is equivalent to

$$x_\theta > \frac{\sqrt{1-\alpha^2}}{2\alpha}. \tag{3.4}$$

Now consider one case where $r \le x_\theta$ and note that $G_4 = \{s \in \mathbb{C}; |s - 1/2| < r\}$. Thus from Theorem 2.2, the MCSPM problem is solvable iff $r > \beta$. For the other case where $r > x_\theta$, the conformal equivalence $\varphi(s)$ from G_4 to D

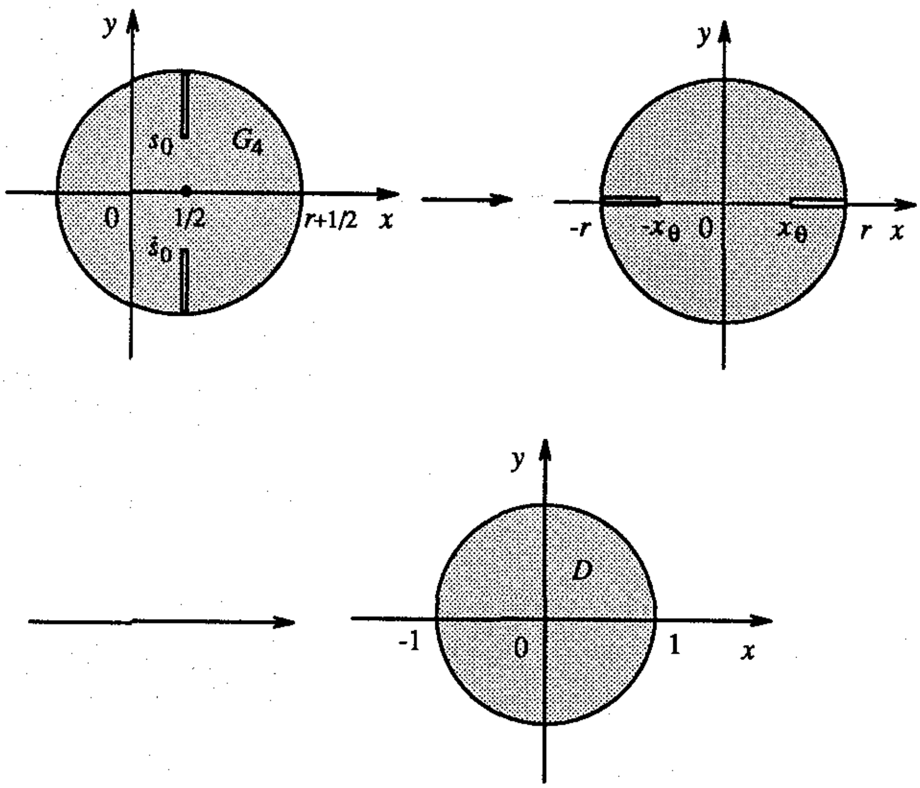

FIG. 2. Construction of conformal mapping from G_4 to D.

with $\varphi(0) = 0$ turns out to be

$$\varphi(s) = i\frac{\phi(s)\sqrt{(x_\theta - i/2)(r^2 - ix_\theta/2)} - \overline{\phi(\bar{s})}\sqrt{(x_\theta + i/2)(r^2 + ix_\theta/2)}}{\phi(s)\sqrt{(x_\theta + i/2)(r^2 + ix_\theta/2)} + \overline{\phi(\bar{s})}\sqrt{(x_\theta - i/2)(r^2 - ix_\theta/2)}}, \tag{3.5}$$

where

$$\phi(s) = \sqrt{[x_\theta - i(s - 1/2)][r^2 - ix_\theta(s - 1/2)]}.$$

For details about the construction, refer to Fig. 2 and (Yan and Anderson, 1990). By Lemma 1.1, the rest of the proof is straightforward and omitted.

Remark 3.1. Since $G_4 = \{s \in \mathbb{C}; |s - 1/2| < r\}$ iff $r \le x_\theta$, a compromise sensitivity of less than r automatically leads to a phase margin of at least $2\cot^{-1} 2r$.

Corollary 3.1. Let $P(s)$ be a nonminimum phase plant, and let $r > 1, 0 \le \theta < \pi$. Assume that $\theta < 2\sin^{-1}\alpha$. Then the necessary condition and the sufficient condition for the solvability of the CSPM problem for (r, θ) are $r + 1/2 \in \Gamma_\theta$ and $r - 1/2 \in \Gamma_\theta$, respectively.

Theorem 3.3. Allows us to identify in the following theorem the smallest sensitivity consistent with a prescribed phase margin and the largest phase margin consistent with a prescribed sensitivity.

Theorem 3.4. Adopt the same hypotheses and notation as in Theorem 3.3 and assume that $\theta < 2\sin^{-1}\alpha$. Set

$$\bar{r}_\theta \triangleq \inf\{\|(1 + PC)^{-1}1/2\|; \quad C(s) \text{ stabilizes } e^{i\phi}P(s) \text{ for all } \phi[-\theta, \theta]\}$$

$$\bar{\theta}_r \triangleq \sup\{0 \le \theta < \pi; \quad \text{the MCSPM problem is solvable for } (r, \theta)\}.$$

Then

$$\bar{r}_\theta = \begin{cases} \sqrt{f(x_\theta)} & \text{if } \sin^{-1}\alpha < \theta < 2\sin^{-1}\alpha \\ \beta & \text{if } \theta \le \sin^{-1}\alpha \end{cases} \tag{3.6}$$

$$\bar{\theta}_r = \begin{cases} 0 & \text{if } r \le b \\ 2\cot^{-1}\dfrac{(4r^2 - 1)\alpha - \sqrt{\alpha^2(1 + 4r^2)^2 - 16r^2}}{2\sqrt{1 - \alpha^2}} & \text{if } r > \beta \end{cases}, \tag{3.7}$$

where $\sin^{-1}\alpha \in (0, \pi/2)$.

Proof. The proof is omitted due to limitations of space.

Remark 3.2. Notice that

$$\lim_{\theta \to \sin^{-1}\alpha} \sqrt{f(x_\theta)} = \beta \quad \text{and} \quad \lim_{\theta \to 2\sin^{-1}\alpha} \sqrt{f(x_\theta)} = \infty.$$

So the three intervals over which $\bar{r}_\theta$ is defined give connecting values.

The general shape of the curve relating θ and $\bar{r}_\theta$ is shown in Fig. 3. Quite evidently, the MCSPM problem for (θ, r) is solvable iff the point (θ, r) is above the curve $r = \bar{r}_\theta$. Similarly, from Corollary 3.1, the CSPM problem for (θ, r) is solvable if the point (θ, r) is above the curve $r = \bar{r}_\theta + 1/2$ and is not solvable if the point (θ, r) is below the curve $r = \bar{r}_\theta - 1/2$.

Again, we can infer some less tight conclusions regarding the CSPM problem in the following result which is easy to prove using Theorems 3.2 and 3.4.

Corollary 3.2. Suppose that $P(s)$ is a nonminimum phase plant. Given $r > 1$ and $0 \le \theta < \pi$ with x_θ defined as in Theorem 3.3, define

$$r_\theta \triangleq \inf\{\|(1 + PC)^{-1}\|; \quad C(s) \text{ stabilizes } e^{i\phi}P(s) \text{ for all } \phi[-\theta, \theta]\}$$

$$\theta_r \triangleq \sup\{0 \le \theta < \pi; \quad \text{the MCSPM problem is solvable for } (r, \theta)\}.$$

Then
(i) $\sqrt{f(x_\theta)} - 1/2 \le r_\theta \le \sqrt{f(x_\theta)} + 1/2$ if $\sin^{-1}\alpha < \theta < 2\sin^{-1}\alpha$,
(ii) $\beta - 1/2 \le r_\theta \le \beta + 1/2$ if $\theta \le \sin^{-1}\alpha$,
(iii) $\bar{\theta}_{r-1/2} \le \theta_r \le \bar{\theta}_{r+1/2}$,
(iv) $\theta_r > 2\sin^{-1}(1/2r)$ if $r > 1/\alpha$,

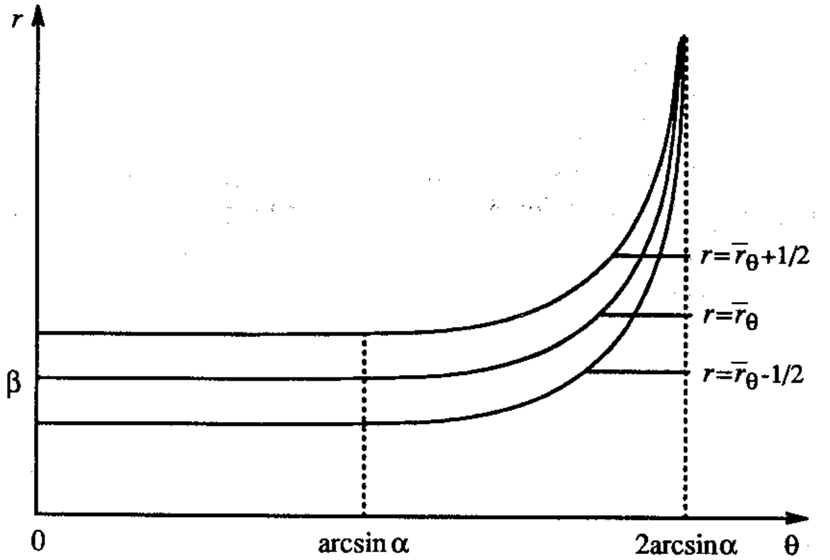

FIG. 3. Curve of $\bar{r}_\theta$ depending on θ.

(v) $\theta_r = 0$ if $r < 1/\alpha$,
(vi) $\lim_{\theta \to 2\sin^{-1}\alpha} \theta_r = \infty$ and $\liminf_{r \to 1/\alpha^+} \theta_r \geq 2\sin^{-1}(\alpha/2)$.

(vi) of the above result shows that phase margin maximization will lead to an infinite closed-loop sensitivity while sensitivity minimization is compatible with requiring a closed-loop phase margin greater than $2\sin^{-1}(\alpha/2)$, which is greater than $2/\pi \sin^{-1}\alpha$, i.e. the maximal phase margin over π. It can similarly be concluded that the same phenomena exist in the discrete-time case. However, it has been proved (Yan, 1990) that if a discrete-time plant is proper but nonstrictly proper with all its unstable poles simple, then periodic compensation can achieve an arbitrarily large phase margin with a bounded closed-loop sensitivity at the same time.

4. *Conclusions*

This paper has been concerned with the problem of designing an LTI controller for a SISO LTI plant so that the resulting closed-loop system simultaneously possesses a prescribed sensitivity and a prescribed phase margin. Both a necessary condition and a sufficient condition for the solvability of this problem have been obtained; in particular, the problem is always solvable for a minimum phase plant. The formula for the maximal achievable phase margin has been derived for a given plant. Furthermore, it has been revealed that phase margin maximization will lead to an infinite closed-loop sensitivity for a nonminimum phase plant with LTI compensation. Recall (Yan and Anderson, 1990) that gain margin maximization also has this effect on sensitivity.

It is not yet clear to us whether the weighted case can also be dealt with using the same method.

References

Francis, B. A. and G. Zames (1984). On H^∞-optimal sensitivity theory for SISO feedback systems. *IEEE Trans. Aut. Control,* **AC-29,** 9–16.

Khargonekar, P. P. and A. Tannenbaum (1985). Non-Euclidian metrics and the robust stabilization of systems with parameter uncertainty. *IEEE Trans. Aut. Control,* **AC-30,** 1005–1013.

Kimura, H. (1987). Directional interpolation approach to H^∞-optimization and robust stabilization. *IEEE Trans. Aut. Control,* **AC-32,** 1085–1093.

O'Young, S. D. and B. A. Francis (1985). Sensitivity tradeoffs for multivariable plants. *IEEE Trans. Aut. Control,* **AC-30,** 625–632.

Tannenbaum, A. (1980). Feedback stabilization of plants with uncertainty in the gain margin. *Int. J. Control,* **32,** 1–16.

Yan, W.-Y. and B. D. O. Anderson (1990). The simultaneous optimization problem for sensitivity and gain margin. *IEEE Trans. Aut. Control,* **AC-35,** 558–563.

Yan, W.-Y. (1990). Robust Control and Periodic Control, Ph.D. Thesis, Dept. of Systems Engineering, the Australian National University, Canberra, Australia.

Youla, D. C., J. J. Bongiorno and C. N. Lu (1974). Single-loop feedback-stabilization of linear multivariable dynamical plants. *Automatica,* **10,** 159–172.