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Practical issues in multirate output controllers

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Multirate output controllers are a new type of controller which detect the ith output of the plant N_i times and change the plant input once during a period T_0. Important features of this type of controller include equivalent realization of state feedback and strong stabilization. Above all, they distinguish themselves from multirate input controllers in that they appear more suitable for industrial application. Nevertheless, there are some potential problems that can degrade the operation of the controller. In this paper, these problems are highlighted and approaches to avoid them are provided.

1. Introduction

Recently, it has been confirmed that periodically time-varying (PTV) controllers used in conjunction with a linear time-invariant (LTI) plant offer a new dimension of flexibility of the design process. In particular, they have been used to achieve equivalent state feedback without observers, pole assignment, zero assignment, gain margin improvement, strong and simultaneous stabilization and the removal of decentralized fixed modes in decentralized control (see Anderson and Moore 1981, Khargonekar *et al.* 1985). Evidently, PTV controllers can offer substantially more design freedom than conventional LTI controllers; also, a PTV digital controller can be implemented in practice without any significant difficulty since it does not violate the constraint of finite memory in a computer.

In the last decade, and particularly in the last several years, a number of results on PTV digital controllers have been reported. Chammas and Leondes (1978, 1979 a, b) proposed a certain type of periodically time-varying gain controller. Araki and Hagiwara (1985, 1986) proposed the use of multirate input controllers. Greschak and Verghese (1982), and Khargonekar *et al.* (1985) proposed another type of PTV controller and Mita *et al.* (1987) proposed the use of intersample data controllers. A general review of these controllers can be found in the work of Hagiwara and Araki (1988). In 1988, Hagiwara and Araki introduced the multirate output controller (MROC). MROCs are a new type of controller which detects the ith plant output at N_i uniformly spaced times and changes the plant input once during one frame period T_0. Incidentally, MROCs are the dual of the multirate input controllers proposed by Araki and Hagiwara (1985). They overcome a major drawback of other types of controllers in that the plant input does not take large positive and negative values during its transient response and they have the nice features of allowing implementation of arbitrary linear state feedback and strong stabilization.

In this paper, we seek to identify certain disadvantages of MROCs. We show that frame periods and output sampling periods must fulfill certain inequality

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constraints to avoid the gains in the controller in becoming very large. Large gains will have the effect of amplifying noise substantially, but not of introducing large controls (in the absence of noise or other non-ideal behaviour). For ease of explanation, the term 'frame period' T_0 is used to refer to the 'cycle' of the controllers and the term 'sampling period' is used to indicate the interval in which the plant outputs are detected or inputs are applied; often such sampling periods are multiples or submultiples of T_0.

Section 2 reviews the operation of MROCs. Section 3 highlights the potential problems via theory and examples and § 4 introduces how these can be avoided through appropriate choice of frame and sampling periods. Section 5 contains concluding remarks.

2. Review of operation of MROCs

The MROCs sampling mechanism involves detecting the ith plant output y_i at every T_i seconds where T_i is a submultiple of the so-called frame period T_0 as shown in Fig. 1. At time kT_0, all outputs are sampled and all inputs are changed simultaneously. The sampled values of the plant output obtained over $[kT_0, (k+1)T_0)$ are stored in a vector $\hat{y}(kT_0)$ as shown below:

$$\hat{y}(kT_0) = \begin{bmatrix} y_1(kT_0) \\ \vdots \\ y_1(kT_0 + (N_1 - 1)T_1) \\ \vdots \\ y_p(kT_0) \\ \vdots \\ y_p(kT_0 + (N_p - 1)T_p) \end{bmatrix} \tag{2.1}$$

In Fig. 1, $N_1 = 3$ and $N_2 = 2$. The input dimension m as well as the output dimension p is 2.

The vector $\hat{y}(kT_0)$ is used in the control law, which changes the value of $u(\cdot)$ every T_0 seconds. The nature of this control law will now be explained.

Suppose the LTI continuous-time plant is described by

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t) \tag{2.2}$$

$$y(t) = Cx(t) \tag{2.3}$$

where the state $x \in \mathbb{R}^n$, the plant input $u \in \mathbb{R}^m$, the plant output $y \in \mathbb{R}^p$ and

$$u(t) = u(kT_0) \quad kT_0 \leqslant t < (k+1)T_0 \tag{2.4}$$

We can express the basic formula of the MROC sampling mechanism in a vector-matrix form given by

$$\hat{C}x((k+1)T_0) = \hat{y}(kT_0) - \hat{G}u(kT_0) \tag{2.5}$$

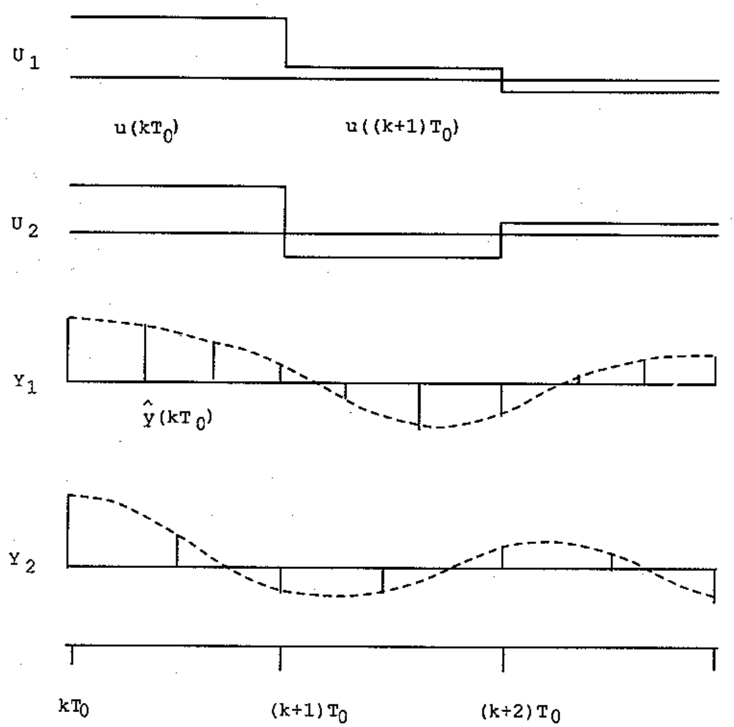

Figure 1. Multirate-output sampling mechanism ($m = p = 2$, $N_1 = 3$, $N_2 = 2$).

Here, $\hat{C} \in \mathbb{R}^{\bar{N} \times n}$ and $\hat{G} \in \mathbb{R}^{\bar{N} \times m}$ are respectively given by

$$\hat{C} = \begin{bmatrix} c_1 \exp(-AN_1T_1) \\ \vdots \\ c_1 \exp(-AT_1) \\ \vdots \\ c_p \exp(-AN_pT_p) \\ \vdots \\ c_p \exp(-AT_p) \end{bmatrix} \tag{2.6}$$

$$\hat{G} = \begin{bmatrix} c_1 \int_0^{-N_1T_1} \exp(At)B\,dt \\ \vdots \\ c_1 \int_0^{-T_1} \exp(At)B\,dt \\ \vdots \\ c_p \int_0^{-N_pT_p} \exp(At)B\,dt \\ \vdots \\ c_p \int_0^{-T_p} \exp(At)B\,dt \end{bmatrix} \tag{2.7}$$

where c_i is the ith row of C.

The integer $\bar{N}$ is given by

$$\bar{N} = \sum_{i=1}^{p} N_i \tag{2.8}$$

(The reader is referred to Hagiwara and Araki 1988 for a detailed derivation.)

Equation (2.5) gives the relation of the vector $\hat{y}(kT_0)$ for the inputs at the beginning of each frame period and the final state of the frame period.

To facilitate the following discussion, the term 'observability index vector' is defined.

Definition

Consider an observable pair (A, C) where $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{p\times n}$. Expressing C as

$$C = [c_1^{\mathrm{T}} \quad \dots \quad c_p^{\mathrm{T}}]^{\mathrm{T}}$$

then a set of p integers $(n_1, \dots, n_p)$ is said to be an *observability index vector* (OIV) of the pair (A, C) if

$$\sum_{i=1}^{p} n_i = n \tag{2.9}$$

and

$$\operatorname{rank}[c_1', A'c_1', \dots, A'^{(n_1-1)}c_1', \dots, c_p', A'c_p', \dots, A'^{(n_p-1)}c_p'] = n \tag{2.10}$$

Consider the matrix $\hat{C}$ of the basic formula (2.5). Hagiwara and Araki (1988) have proved that the matrix $\hat{C}$ given by the (2.6) has full column rank ($=n$) for almost every frame period T_0 if the output multiplicities $(N_1, \dots, N_p)$ satisfy

$$N_i \geqslant n_i \quad (i = 1, \dots, p) \tag{2.11}$$

where $(n_1, \dots, n_p)$ is an OIV of the pair (A, C).

A related result, also proved in the work of Hagiwara and Araki (1988), is the following. Suppose that (A, C) is an observable pair and that

$$\operatorname{rank}\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = n + m \tag{2.12}$$

(which means that $p \geqslant m$, the plant is non-degenerate and the plant has no zero at the origin). Then the matrix $[\hat{C} \ \hat{G}]$ given by (2.6) and (2.7) has full column rank ($=n+m$) for almost every frame period T_0 if the output multiplicities $(N_1, \dots, N_p)$ satisfy

$$N_i \geqslant m_i \quad (i = 1, \dots, p) \tag{2.13}$$

where $(m_1, \dots, m_p)$ is an OIV of the augmented system pair

$$\left(\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}, [C \quad 0]\right)$$

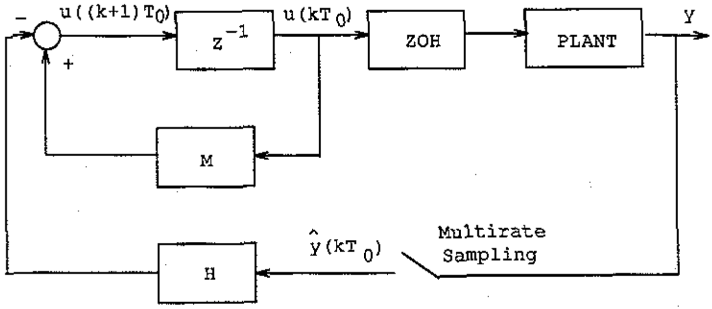

Figure 2. Closed-loop configuration with an MROC.

The control law of the MROC takes the general form

$$u((k+1)T_0) = Mu(kT_0) - H\hat{y}(kT_0) \tag{2.14}$$

where $M \in \mathbb{R}^{m \times m}$ and $H \in \mathbb{R}^{m \times \bar{N}}$ (see Fig. 2).

The above equation means that the control inputs for the $(k+1)$th frame period are determined based on the values of the control inputs for the kth frame period, $u(kT_0)$, and the sampled values of the outputs, $\hat{y}(kT_0)$, obtained during the kth frame period. The time available for the computation of $u((k+1)T_0)$ is evidently $\min\limits_{1 \leqslant i \leqslant p} T_i$.

Now, suppose that (A, C) is an observable pair and the output multiplicities $(N_1, ..., N_p)$ satisfy (2.11) where $(n_1, ..., n_p)$ is an OIV of the pair (A, C). Then if the matrix H can be chosen to satisfy

$$H\hat{C} = F \tag{2.15}$$

and if we set

$$M = H\hat{G} \tag{2.16}$$

we can make the control law equivalent to any state feedback control law

$$u(kT_0) = -Fx(kT_0) \quad k \geqslant 1 \tag{2.17}$$

This can be argued from (2.5) and (2.14).

Two separate cases can now be considered.

Case 1

Suppose that the output multiplicities $(N_1, ..., N_p)$ are set to their minimum values, i.e.

$$N_i = n_i \quad (i = 1, ..., p) \tag{2.18}$$

Then the matrix $\hat{C}$ becomes a square matrix and H is uniquely determined by

$$H = F\hat{C}^{-1} \tag{2.19}$$

Hence M is completely determined and this means that the stability of the open-loop controller (which is governed by the eigenvalues of M) depends solely (and directly) on the choice of the state feedback matrix F.

Case 2

Suppose that the output multiplicities $(N_1, ..., N_p)$ are chosen larger than their minimum values as

$$N_i > n_i \tag{2.20}$$

Then we can find, in general, infinitely many matrices H which satisfy (2.15) and it becomes plausible that we can select H so that the matrix M given by (2.16) becomes stable, i.e. all its eigenvalues are of magnitude less than one.

As an approach to achieving this, Hagiwara and Araki (1988) proceed as follows. Suppose that (A, C) is an observable pair and that

$$\operatorname{rank}\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = n + m \tag{2.21}$$

Further suppose that the output multiplicities $(N_1, ..., N_p)$ satisfy

$$N_i \geqslant m_i \quad (i = 1, ..., p) \tag{2.22}$$

where $(m_1, ..., m_p)$ is an OIV of the augmented system. Then for almost every frame period T_0, there exists a matrix $H \in \mathbb{R}^{m \times \bar{N}}$ such that (2.15) and (2.16) are both satisfied and where $F \in \mathbb{R}^{m \times n}$ is the desired state feedback and $M \in \mathbb{R}^{m \times m}$ is an *arbitrary* specified matrix corresponding to the desired state transition matrix of the controller itself. This is because $[\hat{C} \ \hat{G}]$ has full column rank under the stated assumption, and accordingly, H can be found to satisfy

$$H[\hat{C} \quad \hat{G}] = [F \quad M] \tag{2.23}$$

(We simply choose $H = [F \ \ M]E$ where E is a left inverse of $[\hat{C} \ \hat{G}]$.)

The above implies that we can equivalently realize any state feedback F by a multirate output controller possessing any prescribed degree of stability since we can choose the matrix M arbitrarily. The choice $M = 0$ is of course permissible.

The procedure for strong stabilization of the original plant boils down to choosing a stable feedback matrix F which makes $(\hat{A} - \hat{B}F)$ stable where

$$\hat{A} = \exp(AT_0) \tag{2.24}$$

$$\hat{B} = \int_0^{T_0} \exp(At)B \, dt \tag{2.25}$$

and then choosing a stable matrix M. Finally, it involves determining H by

$$H = [F \quad M][\hat{C} \quad \hat{G}]^{-L} \tag{2.26}$$

where $[\hat{C} \ \hat{G}]^{-L}$ is a left inverse of $[\hat{C} \ \hat{G}]$.

3. Potential problems of MROCs

In this section, we identify two situations giving rise to potential problems associated with the MROCs. Specifically, we point out two situations (Case 1 with $N_i = n_i$ and Case 2 with $N_i > n_i$) where the matrix $\hat{C}$ or $[\hat{C} \ \hat{G}]$ can approach a rank deficient matrix. The consequence of this is that for almost all desired feedback gains F, the gain matrix H of the controller will acquire extremely large entries. Although the ideal plant input $u(kT_0)$ will remain well-behaved, taking the value $-Fx(kT_0)$, the actual plant input will not remain well behaved, since any inaccuracies in the output, due, for example, to noise or non-linearity, will be amplified by H.

To fix ideas, assume that A has distinct eigenvalues; then we can always find an invertible matrix T such that

$$A_{\mathrm{d}} = T^{-1}AT \tag{3.1}$$

$$B_{\mathrm{d}} = T^{-1}B \tag{3.2}$$

$$C_{\mathrm{d}} = CT \tag{3.3}$$

with A_{d} in Jordan form.

For convenience, let us also assume that A has real eigenvalues; this keeps the algebra simpler. Further, we arrange A_{d} such that it is given as follows:

$$A_{\mathrm{d}} = \begin{bmatrix} A_1 & 0 \\ 0 & \alpha \end{bmatrix} \tag{3.4}$$

where $A_{\mathrm{d}} \in \mathbb{R}^{n \times n}$, $A_1 = \operatorname{diag}(\lambda_i)$ $(i = 1, \ldots, (n-1))$, $|\alpha| > |\lambda_i|$, $\alpha \in \mathbb{R}\backslash\{0\}$, and the λ_i's are distinct.

Let us also define

$$B_{\mathrm{d}} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{3.5}$$

where $B_{\mathrm{d}} \in \mathbb{R}^{n \times m}$, $B_1 \in \mathbb{R}^{(n-1) \times m}$, $B_2 \in \mathbb{R}^{1 \times m}$ and

$$C_{\mathrm{d}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{p1} & c_{p2} \end{bmatrix} \tag{3.6}$$

where $C_{\mathrm{d}} \in \mathbb{R}^{p \times n}$, $c_k \in \mathbb{R}^{1 \times n}$, $c_{k1} \in \mathbb{R}^{1 \times (n-1)}$ and $c_{k2} \in \mathbb{R}^{1 \times 1}$.

Thus we have

$$\hat{C} = \begin{bmatrix} c_1 \exp(-A_{\mathrm{d}} N_1 T_1) \\ \vdots \\ c_1 \exp(-A_{\mathrm{d}} T_1) \\ \vdots \\ c_p \exp(-A_{\mathrm{d}} N_p T_p) \\ \vdots \\ c_p \exp(-A_{\mathrm{d}} T_p) \end{bmatrix} = \begin{bmatrix} c_{11} \exp(-A_1 N_1 T_1) & c_{12} \exp(-\alpha N_1 T_1) \\ \vdots & \vdots \\ c_{11} \exp(-A_1 T_1) & c_{12} \exp(-\alpha T_1) \\ \vdots & \vdots \\ c_{p1} \exp(-A_1 N_p T_p) & c_{p2} \exp(-\alpha N_p T_p) \\ \vdots & \vdots \\ c_{p1} \exp(-A_1 T_p) & c_{p2} \exp(-\alpha T_p) \end{bmatrix}$$

$$= \begin{bmatrix} c_{11} \exp(-A_1 T_0) & c_{12} \exp(-\alpha T_0) \\ \vdots & \vdots \\ c_{11} \exp\left(-\dfrac{A_1 T_0}{N_1}\right) & c_{12} \exp\left(-\dfrac{\alpha T_0}{N_1}\right) \\ \vdots & \vdots \\ c_{p1} \exp(-A_1 T_0) & c_{p2} \exp(-\alpha T_0) \\ \vdots & \vdots \\ c_{p1} \exp\left(-\dfrac{A_1 T_0}{N_p}\right) & c_{p2} \exp\left(-\dfrac{\alpha T_0}{N_p}\right) \end{bmatrix} \tag{3.7}$$

(Recall that $T_i = T_0/N_i$.)

$$\hat{G} = \begin{bmatrix} c_1 \int_0^{-N_1 T_1} \exp(A_d t) B_d \, dt \\ \vdots \\ c_1 \int_0^{-T_1} \exp(A_d t) B_d \, dt \\ \vdots \\ c_p \int_0^{-N_p T_p} \exp(A_d t) B_d \, dt \\ \vdots \\ c_p \int_0^{-T_p} \exp(A_d t) B_d \, dt \end{bmatrix} = \begin{bmatrix} c_1 A_d^{-1}[\exp(-A_d N_1 T_1) - I] B_d \\ \vdots \\ c_1 A_d^{-1}[\exp(-A_d T_1) - I] B_d \\ \vdots \\ c_p A_d^{-1}[\exp(-A_d N_p T_p) - I] B_d \\ \vdots \\ c_p A_d^{-1}[\exp(-A_d T_p) - I] B_d \end{bmatrix}$$

$$= \begin{bmatrix} [c_{11} \quad c_{12}] \begin{bmatrix} A_1^{-1} & 0 \\ 0 & \dfrac{1}{\alpha} \end{bmatrix} \begin{bmatrix} \exp(-A_1 T_0) - I & 0 \\ 0 & \exp(-\alpha T_0) - 1 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ \vdots \\ [c_{11} \quad c_{12}] \begin{bmatrix} A_1^{-1} & 0 \\ 0 & \dfrac{1}{\alpha} \end{bmatrix} \begin{bmatrix} \exp\left(-\dfrac{A_1 T_0}{N_1}\right) - I & 0 \\ 0 & \exp\left(-\dfrac{\alpha T_0}{N_1}\right) - 1 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ \vdots \\ [c_{p1} \quad c_{p2}] \begin{bmatrix} A_1^{-1} & 0 \\ 0 & \dfrac{1}{\alpha} \end{bmatrix} \begin{bmatrix} \exp(-A_1 T_0) - I & 0 \\ 0 & \exp(-\alpha T_0) - 1 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ \vdots \\ [c_{p1} \quad c_{p2}] \begin{bmatrix} A_1^{-1} & 0 \\ 0 & \dfrac{1}{\alpha} \end{bmatrix} \begin{bmatrix} \exp\left(-\dfrac{A_1 T_0}{N_p}\right) - I & 0 \\ 0 & \exp\left(-\dfrac{\alpha T_0}{N_p}\right) - 1 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} c_{11} A_1^{-1}(\exp(-A_1 T_0) - I) B_1 + \dfrac{c_{12}}{\alpha}(\exp(-\alpha T_0) - 1) B_2 \\ \vdots \\ c_{11} A_1^{-1}\left(\exp\left(-\dfrac{A_1 T_0}{N_1}\right) - I\right) B_1 + \dfrac{c_{12}}{\alpha}\left(\exp\left(-\dfrac{\alpha T_0}{N_1}\right) - 1\right) B_2 \\ \vdots \\ c_{p1} A_1^{-1}(\exp(-A_1 T_0) - I) B_1 + \dfrac{c_{p2}}{\alpha}(\exp(-\alpha T_0) - 1) B_2 \\ \vdots \\ c_{p1} A_1^{-1}\left(\exp\left(-\dfrac{A_1 T_0}{N_p}\right) - I\right) B_1 + \dfrac{c_{p2}}{\alpha}\left(\exp\left(-\dfrac{\alpha T_0}{N_p}\right) - 1\right) B_2 \end{bmatrix} \tag{3.8}$$

Case 1 *for* $N_i = n_i$

In the previous section, we mentioned that when $N_i = n_i$, the state transition matrix M of the controller cannot be freely chosen. In this case, $\hat{C}$ is square and the design procedure is as follows.

Step 1. Choose the closed-loop poles to be assigned and calculate the state feedback matrix F which realizes those poles.

Step 2. Determine H uniquely by

$$H = F\hat{C}^{-1} \tag{3.9}$$

Step 3. Determine the state transition matrix M of the controller by

$$\begin{aligned} M &= H\hat{G} \\ &= F\hat{C}^{-1}\hat{G} \end{aligned} \tag{3.10}$$

Now, we demonstrate conditions under which $\hat{C}$ approaches a singular matrix. Observe first that when $T_0 \to 0$,

$$\hat{C} \to \begin{bmatrix} c_{11} & c_{12} \\ \vdots & \vdots \\ c_{11} & c_{12} \\ \vdots & \vdots \\ c_{p1} & c_{p2} \\ \vdots & \vdots \\ c_{p1} & c_{p2} \end{bmatrix}$$

Evidently, $\hat{C} \in \mathbb{R}^{\bar{N} \times \bar{N}}$ and we have $\sum_{i=1}^{p}(N_i - 1)$ rows identical. Now $N_i > 1$ for at least one i, or else we do not have different input and output sampling rates; hence the limiting matrix is singular.

Next, suppose that $\alpha > 0$, i.e. the plant is open-loop unstable. (The conclusion above made no assumption concerning stability or instability.) As $T_0 \to \infty$, the last column of $\hat{C}$ in (3.7) tends to zero and the limiting matrix is singular. We can also observe that as $\alpha \to \infty$, then

$$\hat{C} \to \begin{bmatrix} c_{11}\exp(-A_1 T_0) & 0 \\ \vdots & \vdots \\ c_{11}\exp\left(-\dfrac{A_1 T_0}{N_1}\right) & 0 \\ \vdots & \vdots \\ c_{p1}\exp(-A_1 T_0) & 0 \\ \vdots & \vdots \\ c_{p1}\exp\left(-\dfrac{A_1 T_0}{N_p}\right) & 0 \end{bmatrix}$$

and once again, we see that the limiting matrix is singular.

Let us indicate a subtle point regarding this result. Consider an SISO system such that

$$c'(sI-A)^{-1}b = \frac{n(s)}{d_1(s)(s+\alpha)}$$

with all coefficients of $n(s)$, $d_1(s)$ fixed and α variable. Then an easy calculation shows that $\|c_{11}\|$ depends inversely on α and $\|c_{12}\|$ is independent of α, if b_{11}, b_{12} are chosen independently of α. The whole transfer function goes to zero as $\alpha \to \infty$. If on the other hand,

$$c'(sI-A)^{-1}b = \frac{[n_1(s)+\alpha n_2(s)]}{d_1(s)(s+\alpha)}$$

with $n_2(s)$ such that the transfer function does not go to zero as $\alpha \to \infty$, then $\|c_{12}\| \to \infty$, again assuming b_{11}, b_{12} are chosen independently of α. In either case when $\alpha \to \infty$, viz. $n_2(s) = 0$ or $n_2(s)$ are non-zero, the conclusion that $\hat{C}$ approaches a singular matrix as $\alpha \to \infty$ remains valid, despite the dependence of c_{11}, c_{12} on α. The conclusion obviously also applies to the multivariable case.

Case 2 for $N_i > n_i$

For $N_i > n_i$, the state transition matrix M can be freely chosen. The design procedure is different from the previous case when $N_i = n_i$. Here, the procedure is as follows.

Step 1. Choose the stable transition matrix of the controller M so that it is stable.

Step 2. Choose the closed-loop poles to be assigned and calculate the state feedback matrix F which realizes these poles.

Step 3. Determine H by the following matrix equation:

$$H = [F \quad M][\hat{C} \quad \hat{G}]^{-L} \tag{3.11}$$

Now, we study situations where $[\hat{C} \quad \hat{G}]$ approaches a matrix with deficient column rank. First, when $T_0 \to 0$,

$$[\hat{C} \quad \hat{G}] \to \begin{bmatrix} c_{11} & c_{12} & 0 \\ \vdots & \vdots & \vdots \\ c_{11} & c_{12} & 0 \\ \vdots & \vdots & \vdots \\ c_{p1} & c_{p2} & 0 \\ \vdots & \vdots & \vdots \\ c_{p1} & c_{p2} & 0 \end{bmatrix}$$

Obviously, the limiting matrix fails to have full column rank.

Next, suppose that $\alpha > 0$ (so that the plant is open-loop unstable). As $T_0 \to \infty$, the last column of $[\hat{C}]$ goes to zero while other columns of $[\hat{C} \ \hat{G}]$ may tend to infinity or zero or remain finite and non-zero. Thus one can anticipate that a left inverse of the matrix could become unbounded as $T_0 \to \infty$ and this is borne out by a later example.

Also as $\alpha \to \infty$,

$$[\hat{C} \quad \hat{G}] \to \begin{bmatrix} c_{11}\exp(-A_1T_0) & 0 & c_{11}A_1^{-1}(\exp(-A_1T_0)-I)B_1+B_2 \\ \vdots & \vdots & \vdots \\ c_{11}\exp\left(-\dfrac{A_1T_0}{N_1}\right) & 0 & c_{11}A_1^{-1}\left(\exp\left(-\dfrac{A_1T_0}{N_1}\right)-I\right)B_1+B_2 \\ \vdots & \vdots & \vdots \\ c_{p1}\exp(-A_1T_0) & 0 & c_{p1}A_1^{-1}(\exp(-A_1T_0)-I)B_1+B_2 \\ \vdots & \vdots & \vdots \\ c_{p1}\exp\left(-\dfrac{A_1T_0}{N_p}\right) & 0 & c_{p1}A_1^{-1}\left(\exp\left(-\dfrac{A_1T_0}{N_p}\right)-I\right)B_1+B_2 \end{bmatrix}$$

The loss of column rank for the limiting matrix is evident.

Examples

To illustrate the above observations, we provide results for some stable and unstable plants in which T_0 tends to zero and infinity and α tends to infinity for each of the two cases.

Figure 3 is for the stable plant

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \quad B = \begin{bmatrix} -1\cdot5 \\ 1 \end{bmatrix} \quad C = [1 \quad 4\cdot5]$$

with transfer function given by $3(s-0\cdot5)/(s+1)(s+4)$. Since the plant has an observability index of two, let the output multiplicity be $N_1 = 2$. The graph shows that when T_0 tends to zero, entries of the gain vector H blow up. The entries remain finite when T_0 becomes large.

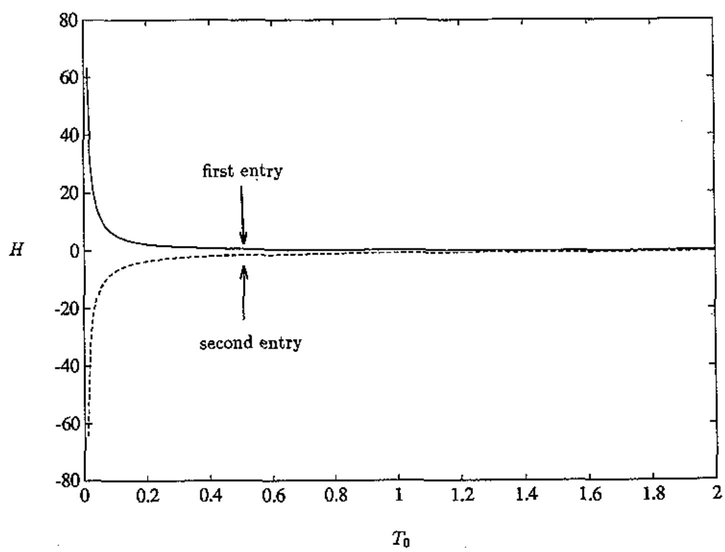

Figure 3. Entries of gain vector H versus frame period T_0 for transfer function $3(s-0\cdot5)/(s+1)(s+4)$.

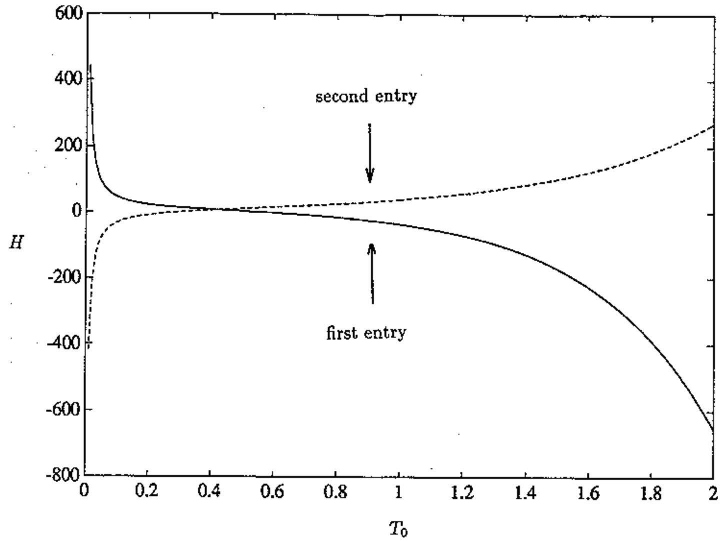

Figure 4. Entries of gain vector H versus frame period T_0 for transfer function $3(s-0{\cdot}5)/(s-1)(s-4)$.

Figure 4 is for the unstable plant

$$A=\begin{bmatrix}1 & 0\\ 0 & 4\end{bmatrix} \quad B=\begin{bmatrix}-0{\cdot}5\\ 1\end{bmatrix} \quad C=[1 \quad 3{\cdot}5]$$

with transfer function given by $3(s-0{\cdot}5)/(s-1)(s-4)$. Again, since the plant has an observability index of two, let the output multiplicity be $N_1=2$ for Case 1. The graph shows that when T_0 is too small *or* too big, entries of the gain vector H blow up.

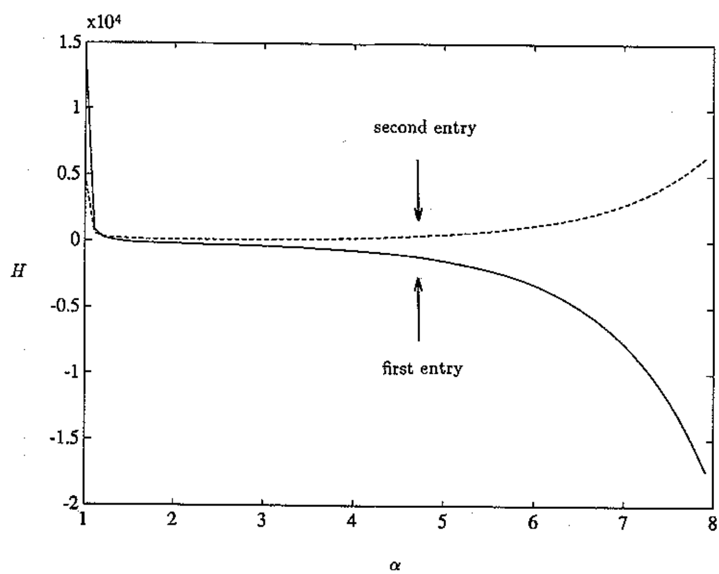

Figure 5. Entries of gain vector H versus largest mode α for transfer function $(\alpha-1)(s-0{\cdot}5)/(s-1)(s-\alpha)$.

The previous two graphs show the effect of T_0 on stable and unstable plants with fixed α. Figure 5 shows the effect of varying α with fixed T_0 for

$$A=\begin{bmatrix}1 & 0\\ 0 & \alpha\end{bmatrix} \quad B=\begin{bmatrix}-0\cdot5\\ 1\end{bmatrix} \quad C=[1 \quad \alpha-0\cdot5]$$

with transfer function given by $(\alpha-1)(s-0\cdot5)/(s-1)(s-\alpha)$. Again, since the plant has an observability index of two, we choose the output multiplicity as $N_1=2$ for Case 1. When α is too large, entries of H blow up. The graph also shows that when α is close to one, entries of H blow up. This is due to the fact that the plant loses controllability when $\alpha=1$ and so pole shifting becomes impossible.

For Case 2, Figs 6, 7 and 8 illustrate the effect of varying T_0 and α for stable and unstable plants.

Figure 6 is for

$$A=\begin{bmatrix}-1 & 0\\ 0 & -4\end{bmatrix} \quad B=\begin{bmatrix}-1\cdot5\\ 1\end{bmatrix} \quad C=[1 \quad 4\cdot5]$$

with transfer function given by $3(s-0\cdot5)/(s+1)(s+4)$. Since the plant has an observability index of two, let the output multiplicity be $N_1=3$. The graph shows that when T_0 tends to zero, entries of the gain vector H blow up.

Figure 7 is for

$$A=\begin{bmatrix}1 & 0\\ 0 & 4\end{bmatrix} \quad B=\begin{bmatrix}-0\cdot5\\ 1\end{bmatrix} \quad C=[1 \quad 3\cdot5]$$

with transfer function given by $3(s-0\cdot5)/(s-1)(s-4)$. Since the plant has an observability index of two, we take the output multiplicity as $N_1=3$. The graph shows that when T_0 tends to zero or a very large value, entries of the gain vector H again blow up.

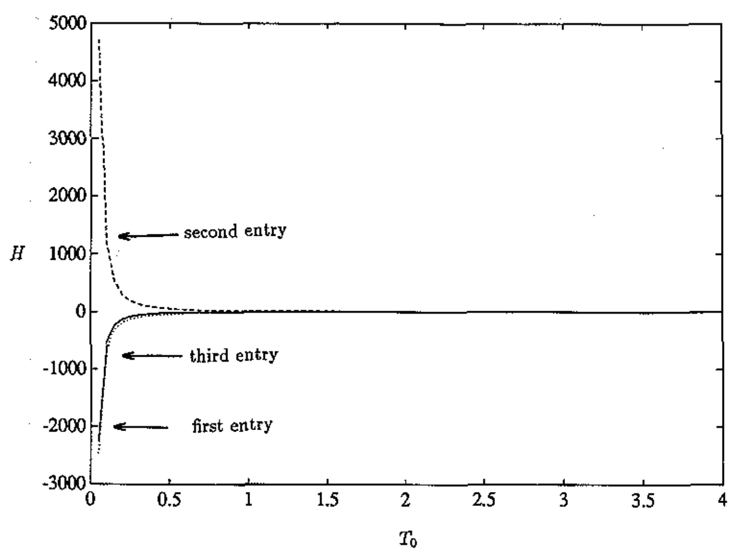

Figure 6. Entries of gain vector H versus frame period T_0 for transfer function $3(s-0\cdot5)/(s+1)(s+4)$.

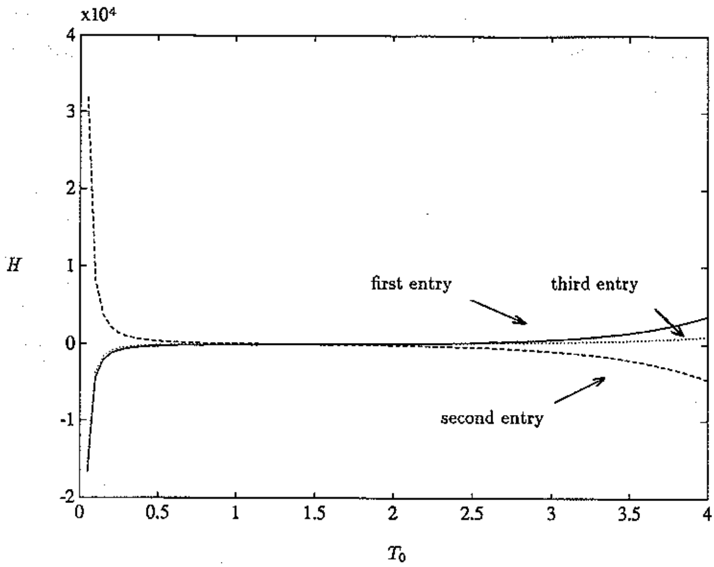

Figure 7. Entries of gain vector H versus frame period T_0 for transfer function $3(s-0{\cdot}5)/(s-1)(s-4)$.

Figure 8 is for

$$A=\begin{bmatrix}1 & 0\\ 0 & \alpha\end{bmatrix}\quad B=\begin{bmatrix}-0{\cdot}5\\ 1\end{bmatrix}\quad C=[1 \quad \alpha-0{\cdot}5]$$

with transfer function given by $(\alpha-1)(s-0{\cdot}5)/(s-1)(s-\alpha)$. Take $N_1=3$ with observability index of two. The graph shows that when α tends to a very large value, entries of the gain vector H blow up.

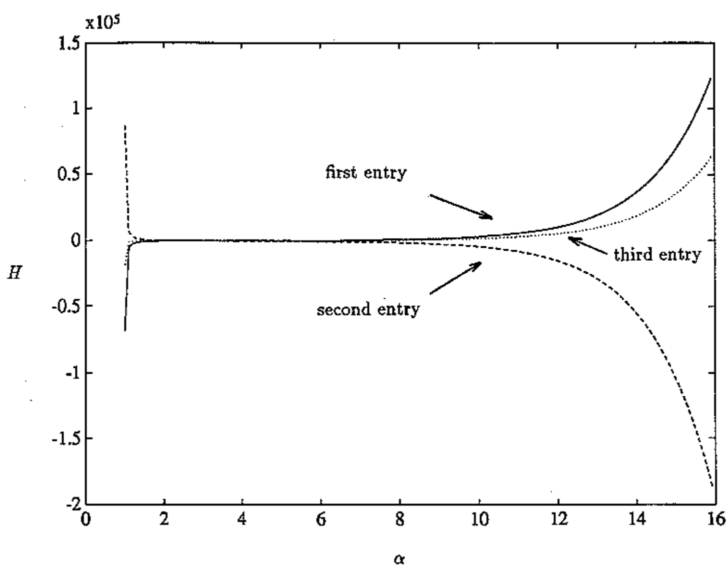

Figure 8. Entries of gain vector H versus largest mode α for transfer function $(\alpha-1)(s-0{\cdot}5)/(s-1)(s-\alpha)$.

4. Approaches to avoid problems

In this section, we indicate some rules of thumb that will ensure that excessive gain values are avoided for Case 1 and 2.

4.1. *Case* 1 *for* $N_i = n_i$

4.1.1. *Effect of* $T_0 \to 0$. Suppose (without loss of generality) that $\alpha > |\lambda_i|$, $i = 1, 2, ..., (n-1)$. When

$$T_0 < \frac{1}{20\alpha} \tag{4.1}$$

then $\exp(\alpha T_0) \simeq 1$ and the rest of the exponential terms $\exp(\alpha T_0/N_i)$ and $\exp(-A_1 \mu T_0/N_i)$, $\mu = 1, ..., (N_i - 1)$ appearing in $\hat{C}$ in (3.7) will also be approximately one, i.e. $\hat{C}$ will be close to singular. Therefore, for proper operation of MROCs

$$T_0 \geqslant \frac{1}{20\alpha}$$

$$\Leftrightarrow \omega_0 \leqslant 40\pi\alpha \tag{4.2}$$

Note that this gives an upper bound on the sampling frequency while the sampling theorem gives a lower bound in that ω_0 must be at least twice the closed-loop bandwidth. In practice, a larger multiple value must be assumed.)

4.1.2. *Effect of* $T_0 \to \infty$ *for* $\alpha > 0$. Examination of the last column of (3.7) shows that it will have very small entries when $T_0 > 4N_{i\max}/\alpha$. Here,

$$N_{i\max} = \max_{1 \leqslant i \leqslant p} N_i$$

Accordingly, to avoid problems, we want

$$T_0 < \frac{4N_{i\max}}{\alpha} \tag{4.3}$$

$$\Leftrightarrow \omega_0 > \frac{\pi\alpha}{2N_{i\max}} \tag{4.4}$$

This means that there are two constraints setting an underbound for ω_0, viz. the sampling theorem constraint requires that ω_0 exceed twice the closed-loop bandwidth as well as (4.4). Notice that (4.4) can also be regarded as a statement concerning a fast sampling frequency, viz. the sampling frequency $\omega_i = N_i\omega_0$ associated with the output for which N_i is maximum; calling this frequency $\omega_{i\max}$, (4.4) is equivalent to

$$\omega_{i\max} > \frac{\pi|\alpha|}{2} \tag{4.5}$$

which is again a sort of sampling theorem.

4.1.3. *Effect of* $\alpha \to \infty$. When $\alpha > 4N_{i\max}/T_0$, $\exp(-\alpha T_0) \simeq 0$ and the last column of $\hat{C}$ in (3.7) will also be approximately zero. Thus for unstable plants with a fixed

T_0, to ensure proper operation of the plant

$$\alpha < \frac{4N_{i\,\max}}{T_0} \tag{4.6}$$

This is the same constraint as (4.3).

4.2. *Case* 2

Examination of the cases $T_0 \to 0$, $T_0 \to \infty$ for stable and unstable plants respectively and $\alpha \to \infty$ for unstable plants reveals that exactly the same constraint on T_0 or ω_0 and α apply for Case 2 as apply for Case 1.

To summarize, (4.4) sets a lower bound on T_0 in ters of the mode furthest from the origin, whether or not the plant is stable, while (4.3) sets an upper-bound on the product αT_0 when α is an unstable mode; this latter bound involves the maximum of the N_i's.

5. Conclusion

In this paper, the potential problems of the MROC are identified and approaches to avoid the problems are provided. For proper operation of the MROC, the authors of Hagiwara and Araki (1988) have pointed out that N_i should be chosen sufficiently large. Here, we show that in addition to this, the frame period of the MROC, T_0, has to be chosen sufficiently large, but not too large in the case of unstable plants. Violation of the guidelines will mean that controller gains will be unacceptably large, leading to such problems as noise amplification in the controller. Section 4 contains quantitative guidelines.

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