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A FIRST PRINCIPLES SOLUTION TO THE NON-SINGULAR H^∞ CONTROL PROBLEM

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SUMMARY

This paper presents an elementary solution to the non-singular H^∞ control problem. In this control problem, the underlying linear system satisfies a set of assumptions which ensures that the solution can be obtained by solving just two algebraic Riccati equations of the game type. This leads to the central solution to the H^∞ control problem. The solution presented in this paper uses only elementary ideas beginning with the Bounded Real Lemma.

KEY WORDS H^∞ control Bounded real lemma Riccati equations

1. INTRODUCTION

Consider the standard H^∞ control problem depicted in Figure 1. In this problem, Σ is a time-invariant linear system described by the state equations

$$\begin{aligned}\dot{x} &= Ax + B_1 w + B_2 u\\ z &= C_1 x + D_{12} u\\ y &= C_2 x + D_{21} w\end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $w \in \mathbb{R}^p$ is the disturbance input, $u \in \mathbb{R}^m$ is the control input, $z \in \mathbb{R}^q$ is the error output and $y \in \mathbb{R}^l$ is the measured output. The system (1) is assumed to satisfy the following assumptions; see also [1].

A1. $D_{12}'D_{12} = E_1 > 0$.

A2. $D_{21}D_{21}' = E_2 > 0$.

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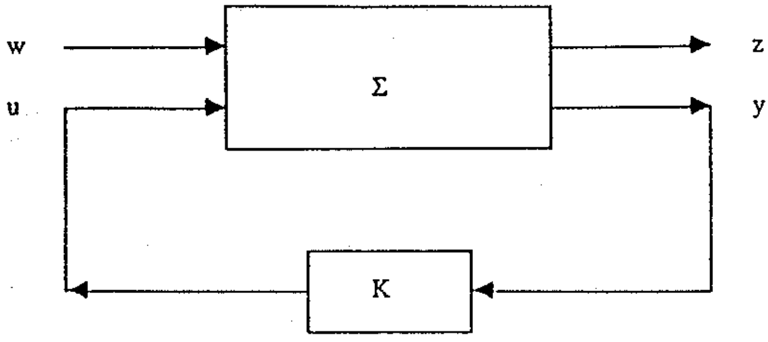

Figure 1.

A3. $\operatorname{rank}\begin{bmatrix} A-\mathrm{j}\omega I & B_2 \\ C_1 & D_{12}\end{bmatrix} = n+m$ for all $\omega \geqslant 0$.

A4. $\operatorname{rank}\begin{bmatrix} A-\mathrm{j}\omega I & B_1 \\ C_2 & D_{21}\end{bmatrix} = n+l$ for all $\omega \geqslant 0$.

The standard H^∞ control problem involves designing a compensator $K(s)$ which will internally stabilize the feedback system depicted above. Furthermore, it is required that the closed loop transfer function from disturbance input to error output (denoted $G(s)$) has infinity norm less than one; that is,* $\sup_{\omega\in\mathbb{R}} \sigma_{\max}[G(\mathrm{j}\omega)] =: \| G(s) \|_\infty < 1$. If assumptions A1–A4 are satisfied, we refer to such an H^∞ problem as a *non-singular H^∞ control problem.*

Current interest in the H^∞ control problem can be traced to the work of Zames.[2] In Reference 2, the H^∞ control problem was motivated by considering the sensitivity of a feedback control system. Following the work of Zames, numerous authors attacked the H^∞ control problem using frequency-domain methods; for example, see References 3–5 and their bibliographies.

Although the frequency-domain approach gave solutions to the H^∞ control problem in certain cases, subsequent research has shown that a more straightforward solution can be obtained in a state–space setting; for example, see References 1 and 6–23. In particular, the results of References 1 and 14 show that the above non-singular H^∞ control problem will have a solution if and only if the following Riccati equations

$$(A - B_2E_1^{-1}D_{12}'C_1)'X + X(A - B_2E_1^{-1}D_{12}'C_1) + X(B_1B_1' - B_2E_1^{-1}B_2')X + C_1'(I - D_{12}E_1^{-1}D_{12}')C_1 = 0 \quad (2)$$

$$(A - B_1D_{21}'E_2^{-1}C_2)Y + Y(A - B_1D_{21}'E_2^{-1}C_2)' + Y(C_1'C_1 - C_2'E_2^{-1}C_2)Y + B_1(I - D_{21}'E_2^{-1}D_{21})B_1' = 0 \quad (3)$$

have positive-semidefinite solutions X and Y such that

(i) $A - B_2E_1^{-1}D_{12}'C_1 + (B_1B_1' - B_2E_1^{-1}B_2')X$ is a stability matrix,

(ii) $A - B_1D_{21}'E_2^{-1}C_2 + Y(C_1'C_1 - C_2'E_2^{-1}C_2)$ is a stability matrix,

(iii) the matrix XY has a spectral radius strictly less than one.

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* In this expression, $\sigma_{\max}[\cdot]$ denotes the maximum singular value of a matrix.

Furthermore, if these conditions are satisfied then the required compensator $K(s)$ can be constructed from the following state–space realization:

$$\begin{aligned} \dot{\hat{x}} &= A_K\hat{x} + B_K y \\ u &= C_K\hat{x} \end{aligned} \tag{4}$$

where

$$\begin{aligned} A_K &= A + B_2C_K - B_KC_2 + (B_1 - B_KD_{21})B_1'X \\ B_K &= L = (I - YX)^{-1}(YC_2' + B_1D_{21}')E_2^{-1} \\ C_K &= K = -E_1^{-1}(B_2'X + D_{12}'C_1) \end{aligned} \tag{5}$$

This compensator is referred to as the *central solution* to the above H^{∞} control problem.

The purpose of this paper is to present an elementary proof of this solution to the non-singular H^{∞} control problem. The motivation for presenting such an elementary proof derives from the fact that the proof presented in Reference 14 is quite involved and requires advanced concepts from linear systems and differential game theory. Furthermore, alternative proofs such as those found in References 18, 23 and 24 require a stronger set of assumptions than those required in this paper. These stronger assumptions, although allowing for a simplified proof, may not hold in practical problems. Thus, there is call for an elementary proof of the above solution to the non-singular H^{∞} control problem in which the minimal set of assumptions A1–A4 are required.

The proof presented in this paper is closely related to the algebraic proof presented in Reference 21. However, it should be noted that algebraic ideas similar to those used in Reference 21 were previously published in Reference 13. Furthermore in contrast to this paper, the result given in Reference 21 does not give the solution to the non-singular H^{∞} control problem in terms of the Riccati equations (2) and (3) but rather in terms of corresponding Riccati inequalities. In particular, this means that the results of Reference 21 do not given the central solution to the H^{∞} control problem. However, the central solution to the H^{∞} control problem is known to play an important role in the theory of H^{∞} control. For example it is shown in Reference 25 that the central solution to the H^{∞} control problem minimizes a certain entropy integral and, thus, the central solution would be the preferred solution to use in any practical application of H^{∞} control. Hence, the proof presented in this paper adds significant insight into the H^{∞} control problem, over and above the results of Reference 21.

The structure of the paper is as follows. Section 2 presents the Bounded Real Lemma which underlies our approach. Section 3 contains our main result which is a solution to the non-singular H^{∞} control problem together with a solution to the state feedback non-singular H^{∞} control problem. This section begins with an outline of our approach followed by a number of preliminary results. These preliminary results are then combined to give an elementary proof of the main result. The state feedback result is then given as a corollary to the proof of the main result.

Notation

In the sequel, we will use the following notation: $\sigma_{\max}[\cdot]$ denotes the maximum singular value of a matrix and for a stable matrix transfer function $G(s)$, $\| G(s) \|_{\infty}$ denotes the infinity norm of $G(s)$; that is $\| G(s) \|_{\infty} := \sup_{\omega \in \mathbb{R}} \sigma_{\max}[G(j\omega)]$. For a matrix M, $\| M \|_2$ denotes the induced norm of the matrix; that is $\| M \|_2 = \sigma_{\max}[M]$. Furthermore $\rho(M)$ denotes the spectral radius of M. Also $\mathcal{N}(M)$ denotes the null space of the matrix M. For symmetric matrices P

Q, $P > 0$ denotes the fact that the matrix P is positive-definite and $P \geqslant 0$ denotes the fact that the matrix P is positive-semidefinite. Moreover $P > Q$ denotes $P - Q > 0$ and $P \geqslant Q$ denotes $P - Q \geqslant 0$.

2. THE STRICT BOUNDED REAL LEMMA

In this section we present a strengthened version of the Bounded Real Lemma given in Reference 26. This result relates to non-minimal realizations and gives a strict H^∞ norm bound. This forms the basis of our solution to the non-singular H^∞ control problem. We begin by presenting a definition.

Definition 2.1

Consider an algebraic Riccati equation written in the form

$$A'X + XA - XMX + N = 0 \tag{6}$$

A symmetric matrix X which satisfies this Riccati equation is said to be a *stabilizing solution* if $A - MX$ is stable. A symmetric matrix X which satisfies (6) is said to be a *strong solution* if the matrix $A - MX$ has no eigenvalues in the open right half plane.

Theorem 2.1 (the Strict Bounded Real Lemma with non-minimal realizations)

The following statements are equivalent:

(i) A is stable and $\| C(sI - A)^{-1}B\|_\infty < 1$;
(ii) There exists a matrix $\tilde{P} > 0$ such that

$$A'\tilde{P} + \tilde{P}A + \tilde{P}BB'\tilde{P} + C'C < 0 \tag{7}$$

(iii) The Riccati equation

$$A'P + PA + PBB'P + C'C = 0 \tag{8}$$

has a stabilizing solution $P \geqslant 0$.

Furthermore, if these statements hold then $P < \tilde{P}$.

In order to establish this result, we first prove the following preliminary lemma.

Lemma 2.1

Suppose A is stable and the Riccati equation

$$A'\tilde{P} + \tilde{P}A + \tilde{P}BB'\tilde{P} + \tilde{Q} = 0 \tag{9}$$

has a symmetrical solution $\tilde{P}$. Furthermore, suppose $\tilde{Q} \geqslant Q \geqslant 0$. Then the Riccati equation

$$A'P + PA + PBB'P + Q = 0 \tag{10}$$

will have a unique strong solution P and moreover $0 \leqslant P \leqslant \tilde{P}$.

Proof. Let $\tilde{K} = -\tilde{P}$. Hence Riccati equation (9) can be rewritten as $A'\tilde{K} + \tilde{K}A - \tilde{K}BB'\tilde{K} - \tilde{Q} = 0$. Furthermore, since A is stable, the pair (A, B) must be stabilizable. Hence using a standard result on the monotonicity of Riccati solutions (see Theorems 2.1 and 2.2 of Reference 27), it follows that the Riccati equation $A'K + KA - KBB'K - Q = 0$ will have a unique strong solution $K \geqslant \tilde{K}$. We now let $P = -K$. It follows immediately that $P \leqslant \tilde{P}$ is the unique strong solution to (10). Moreover, using a standard result on Lyapunov equations, it follows from (10) that $P \geqslant 0$; see Lemma 12.1 of Reference 28. □

Proof of Theorem 2.1

We first establish the equivalence of statements (i)–(iii).

(i) ⇒ (ii): It follows from condition (i) that there exists and $\varepsilon \geqslant 0$ such that

$$C(\mathrm{j}\omega - A)^{-1}BB'(-\mathrm{j}\omega I - A')^{-1}C' \leqslant (1-\varepsilon)I \tag{11}$$

for all $\omega \geqslant 0$. Let $\mu := \| C(sI-A)^{-1} \|_\infty$. Hence,

$$\frac{\varepsilon}{2\mu^2} C(\mathrm{j}\omega I - A)^{-1}(-\mathrm{j}\omega I - A')^{-1}C' \leqslant \frac{\varepsilon}{2} I \tag{12}$$

for all $\omega \geqslant 0$. Adding equations (11) and (12), it follows that given any $\omega \geqslant 0$,

$$C(\mathrm{j}\omega I - A)^{-1}\tilde{B}\tilde{B}'(-\mathrm{j}\omega I - A')^{-1}C' \leqslant \left(1 - \frac{\varepsilon}{2}\right) I \tag{13}$$

where $\tilde{B}$ is a non-singular matrix defined by $\tilde{B}\tilde{B}' = BB' + \varepsilon/2\mu^2 I$. Furthermore, (13) implies

$$\tilde{B}'(-\mathrm{j}\omega I - A')^{-1}C'C(\mathrm{j}\omega I - A)^{-1}\tilde{B} \leqslant \left(1 - \frac{\varepsilon}{2}\right) I \tag{14}$$

for all $\omega \geqslant 0$. Now let $\eta := \| (sI-A)^{-1}\tilde{B} \|_\infty$. Hence,

$$\frac{\varepsilon}{2\eta^2} \tilde{B}'(-\mathrm{j}\omega I - A')^{-1}(\mathrm{j}\omega I - A)^{-1}\tilde{B} \leqslant \frac{\varepsilon}{2} I \tag{15}$$

for all $\omega \geqslant 0$. Adding equations (14) and (15), it follows that given any $\omega \geqslant 0$,

$$\tilde{B}'(-\mathrm{j}\omega I - A')^{-1}\tilde{C}'\tilde{C}(\mathrm{j}\omega I - A)^{-1}\tilde{B} \leqslant I$$

where $\tilde{C}$ is a non-singular matrix defined so that $\tilde{C}'\tilde{C} = C'C + (\varepsilon/2\eta^2)I$. Thus, we have $\| \tilde{C}(sI-A)^{-1}\tilde{B} \|_\infty \leqslant 1$. Furthermore, since $\tilde{B}$ and $\tilde{C}$ are non-singular, the triple $(A, \tilde{B}, \tilde{C})$ is minimal. Hence, using the standard Bounded Real Lemma given in Reference 26, it follows that there exists at $\tilde{P} > 0$ such that $A'\tilde{P} + \tilde{P}A + \tilde{P}\tilde{B}\tilde{B}'\tilde{P} + \tilde{C}'\tilde{C} = 0$. That is

$$A'\tilde{P} + \tilde{P}A + \tilde{P}BB'\tilde{P} + C'C + \frac{\varepsilon}{2\mu^2}\tilde{P}^2 + \frac{\varepsilon}{2\eta^2} I = 0$$

Hence, condition (ii) holds.

(ii) ⇒ (iii): It follows from (ii) that there exists matrices $\tilde{P} > 0$ and $\tilde{R} > 0$ such that

$$A'\tilde{P} + \tilde{P}A + \tilde{P}BB'\tilde{P} + C'C + \tilde{R} = 0 \tag{16}$$

Hence, using a standard Lyapunov stability result, it follows that A is stable; see Lemma 12.2 of Reference 28. Furthermore if we compare Riccati equations (16) and (8), it follows from Lemma 2.1 that (8) will have a unique strong solution $P \leqslant \tilde{P}$. Moreover, using the fact that A is stable, it follows from a standard property of Lyapunov equations that $P \geqslant 0$; see Lemma 12.1 of Reference 28.

We must now establish that P is in fact the stabilizing solution to (8). First let $S := \tilde{P} - P \geqslant 0$ and $\bar{A} := A + BB'P$. It follows from (8) and (16) that

$$\bar{A}'S + S\bar{A} + SBB'S + \tilde{R} = 0 \tag{17}$$

We will use this equation to show that $A + BB'P$ has no eigenvalues on the imaginary axis. Indeed, suppose $\bar{A}$ has an imaginary axis eigenvalue $j\omega$ with corresponding eigenvalue x. That is $\bar{A}x = j\omega x$. It follows from (17) that $-j\omega x^*Sx + j\omega x^*Sx + x^*SBB'Sx + X^*\tilde{R}x = 0$ and hence $x^*\tilde{R}x = 0$. However, this contradicts the fact that $\tilde{R} > 0$. Thus, we can conclude that $A + BB'P$ is stable and therefore $P \geqslant 0$ is the stabilizing solution (8). Hence, we have established condition (iii).

(iii) $\Rightarrow$ (i): Suppose condition (iii) holds and let $\tilde{C} := B'P$. It follows that $A + B\tilde{C} = A + BB'P$ is stable and hence the pair $(A, \tilde{C})$ is detectable. Furthermore it follows from (8) that $A'P + PA + \tilde{C}'\tilde{C} \leqslant 0$. Hence using a standard Lyapunov stability result, we can conclude that A is stable; see Lemma 12.2 of Reference 28.

In order to show that $\| C(sI - A)^{-1}B \|_\infty < 1$, we first observe that equation (8) implies

$$\begin{aligned} B'(-j\omega I - A')^{-1}C'C(j\omega I - A)^{-1}B \\ = I - [I - B'P(-j\omega I - A)^{-1}B]'[I - B'P(j\omega I - A)^{-1}B] \end{aligned} \tag{18}$$

for all $\omega \geqslant 0$. It follows that $\| C(sI - A)^{-1}B \|_\infty \leqslant 1$. Furthermore, note that $C(j\omega I - A)^{-1}B \to 0$ as $\omega \to \infty$. Now suppose that there exists an $\bar{\omega} \geqslant 0$ such that $\| C(j\bar{\omega} I - A)^{-1}B \|_2 = 1$. It follows from (18) that there exists a vector z such that $[I - B'P(j\bar{\omega} I - A)^{-1}B]z = 0$. Hence, $\det[I - B'P(j\bar{\omega} I - A)^{-1}B] = 0$. However, using a standard result on determinants, it follows that $\det[j\bar{\omega} I - A - BB'P] = \det[j\bar{\omega} I - A]$ $\det[I - B'P(j\bar{\omega} I - A)^{-1}B]$; for example see Section A.11 of Reference 29. Thus $\det[j\bar{\omega} I - A - BB'P] = 0$. However, this contradicts the fact that P is the stabilizing solution to (8). Hence, we can now conclude that $\| C(sI - A)^{-1}B \|_\infty < 1$. Thus, we have now established condition (i).

In order to complete the proof of the theorem, we now suppose that statements (i)–(iii) hold. We must show that $P < \tilde{P}$. However, we have already proved that $P \leqslant \tilde{P}$. Now suppose that there exists a vector z such that $z'\tilde{P}z = z'Pz$. That is, $Sz = 0$ where $S = \tilde{P} - P \geqslant 0$ is defined as above. Applying this fact to equation (17), it follows that $z'\tilde{R}z = 0$. However, this contradicts the fact that $\tilde{R} > 0$. Thus we must have $P < \tilde{P}$. This completes the proof of the theorem. □

3. THE MAIN RESULT

The following result is the main result of this paper; see also Reference 14.

Theorem 3.1

Consider the system (1) and suppose assumptions A1–A4 are satisfied. Then the following statements are equivalent:

(i) There exists a dynamic compensator

$$\begin{aligned} \dot{\eta} &= A_K\eta + B_Ky \\ u &= C_K\eta + D_Ky \end{aligned} \tag{19}$$

such that the resulting closed loop system satisfies

$$\begin{bmatrix} A + B_2D_KC_2 & B_2C_K \\ B_KC_2 & A_K \end{bmatrix} \text{ stable} \tag{20}$$

and

$$\Bigg\| [C_1 + D_{12}D_KC_2 \quad D_{12}C_K] \times \left\{ sI - \begin{bmatrix} A + B_2D_KC_2 & B_2C_K \\ B_KC_2 & A_K \end{bmatrix} \right\}^{-1} \begin{bmatrix} B_1 + B_2D_KD_{21} \\ B_KD_{21} \end{bmatrix} + D_{12}D_KD_{12} \Bigg\|_\infty < 1 \tag{21}$$

(ii) The Riccati equations (2) and (3) have stabilizing solutions $X \geqslant 0$ and $Y \geqslant 0$ such that $\rho(XY) < 1$.

Furthermore, if conditions (ii) holds, a compensator meeting the requirements of condition (i) is described by equations (4) and (5).

3.1. An outline of our approach

The necessity proof proceeds as follows. If there exists a compensator which solves the H^∞ control problem, we show that there also exists a strictly proper compensator which solves this H^∞ control problem. We then show that corresponding state feedback and output injection H^∞ control problems will also have solutions. Finally, we show that if a state feedback H^∞ control problem has a solution then the corresponding game Riccati equation will have a stabilizing solution $X \geqslant 0$. The existence of a solution to the other Riccati equation follows by duality. Finally, the fact that $\rho(XY) < 1$ follows from the relationship between the solutions of the primal and dual Riccati equations.

The sufficiency proof is quite elementary. Using the solution X and Y to Riccati equations (2) and (3), the required compensator is constructed as in (4) and (5). Also, it is shown that a related matrix $W \geqslant 0$ is the stabilizing solution to a third Riccati equation. The Bounded Real Lemma is then used to prove the required H^∞ norm bound and internal stability for the closed loop system. This is done by showing that the matrix $\Sigma = [\begin{smallmatrix} X & 0 \\ 0 & W \end{smallmatrix}]$ satisfies a bounded real Riccati equation corresponding to the closed loop system.

3.2. The necessity proof

3.2.1. A preliminary result. The following preliminary result shows that if an H^∞ control problem has a solution using a proper compensator, it necessarily has a solution using a strictly proper compensator.

Theorem 3.2

(See Reference 21 for proof): Consider the system (1) and suppose there exists a compensator of the form (19) such that when this is applied to (1), the resulting closed loop system satisfies conditions (20) and (21). Then there exists a strictly proper compensator

$$\begin{aligned} \dot{\xi} &= \tilde{A}_K\xi + \tilde{B}_Ky \\ u &= \tilde{C}_K\xi \end{aligned} \tag{22}$$

such that when this is applied to (1), the resulting closed loop system satisfies

$$\begin{bmatrix} A & B_2\tilde{C}_K \\ \tilde{B}_K C_2 & \tilde{A}_K \end{bmatrix} \text{ stable} \tag{23}$$

and

$$\left\| [C_1 \quad D_{12}\tilde{C}_K]\left\{sI - \begin{bmatrix} A & B_2\tilde{C}_K \\ \tilde{B}_K C_2 & \tilde{A}_K \end{bmatrix}\right\}^{-1}\begin{bmatrix} B_1 \\ \tilde{B}_K D_{21} \end{bmatrix}\right\|_\infty < 1 \tag{24}$$

3.2.2. Reduction to state feedback and output injection problems. The following theorem shows that if there is a strictly proper compensator solving the H^∞ control problem, then there is a state feedback law and an output injection law, each of which solve a related H^∞ control problem. This result is taken from Reference 21. However, the key ideas behind the proof were previously published in Reference 9.

Theorem 3.3

(See Reference 21 for proof): Consider the system (1) and suppose there exists a compensator of the form (22) such that when this is applied to (1), the resulting closed loop system satisfies conditions (23) and (24). Then the following conditions hold:

(i) There exists a state feedback matrix K and a matrix $P > 0$ such that

$$(A + B_2K)'P + P(A + B_2K) + PB_1B_1'P + (C_1 + D_{12}K)'(C_1 + D_{12}K) < 0 \tag{25}$$

Hence

$$A + B_2K \text{ is stable and } \|(C_1 + D_{12}K)\{sI - A - B_2K\}^{-1}B_1\|_\infty < 1 \tag{26}$$

(Note that $(C_1 + D_{12}K)\{sI - A - B_2K\}^{-1}B_1$ is the transfer function from w to z which will result with the state feedback law $u = Kx$.)

(ii) There exists an output injection matrix L and a matrix $\Sigma > 0$ such that

$$(A + LC_2)\Sigma + \Sigma(A + LC_2)' + \Sigma C_1'C_1\Sigma + (B_1 + LD_{21})(B_1 + LD_{21})' < 0 \tag{27}$$

Hence

$$(A + LC_2) \text{ is stable and } \| C_1(sI - A - LC_2)^{-1}(B_1 + LD_{21}) \|_\infty < 1 \tag{28}$$

(Note that $C_1(sI - A - LC_2)^{-1}(B_1 + LD_{21})$ is the transfer function from w to z which will result with the output injection $\dot{x} = Ax + B_1w + Ly$.)

Furthermore,these solutions satisfy $\rho(P\Sigma) \leqslant 1$.

3.2.3. Necessity for the state feedback H$^\infty$ *control problem.* The next step is to relate the existence of a state feedback law solving an H^∞ control problem to the existence of a stabilizing solution to a Riccati equation of the same type. At this point conditions A1 and A3 enter the picture.

Theorem 3.4

Consider the system (1) and suppose assumptions A1 and A3 are satisfied. Furthermore, suppose there exists a state feedback matrix K such that condition (26) is satisfied. Also suppose $P > 0$ satisfies inequality (25). (The existence of such a $P > 0$ follows from Theorem 2.1.) Then the Riccati equation (2) will have a stabilizing solution $X \geqslant 0$ such that $X < P$.

In order to establish this theorem, we first prove the following preliminary lemma.

Lemma 3.1

Consider the system (1) and suppose assumptions A1 and A3 are satisfied. Then there exists a state transformation which when applied to the system (1) allows the following decomposition of the matrices $A - B_2E_1^{-1}D_{12}'C_1, B_1, B_2$, and $(I - D_{12}E_1^{-1}D_{12}')C_1$:

$$A - B_2E_1^{-1}D_{12}'C_1 = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \quad B_1 = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}, \quad B_2 = \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix},$$
$$(I - D_{12}E_1^{-1}D_{12}')C_1 = [C_{11} \quad 0] \quad (29)$$

where $(-A_{11}', C_{11}')$ is a stabilizable pair and A_{22} is stable.

Proof. We first apply a state transformation to the system (1) which decomposes the pair $(A - B_2E_1^{-1}D_{12}'C_1, (I - D_{12}E_1^{-1}D_{12}')C_1)$ into observable and unobservable parts. Thus,

$$A - B_2E_1^{-1}D_{12}'C_1 = \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \quad B_1 = \begin{bmatrix} \tilde{B}_{11} \\ \tilde{B}_{12} \end{bmatrix}, \quad B_2 = \begin{bmatrix} \tilde{B}_{21} \\ \tilde{B}_{22} \end{bmatrix},$$
$$(I - D_{12}E_1^{-1}D_{12}')C_1 = [\tilde{C}_{11} \quad 0]$$

where $(\tilde{A}_{11}, \tilde{C}_{11})$ is an observable pair.

We now show that the matrix $\tilde{A}_{22}$ has no purely imaginary eigenvalues. Indeed, suppose $s = j\omega$ is an eigenvalue of $\tilde{A}_{22}$ with corresponding eigenvector x_2. Letting $\bar{x} = [0 \quad x_2']'$, it follows that $(A - B_2E_1^{-1}D_{12}'C_1)\bar{x} = j\omega\bar{x}$ and $(I - D_{12}E_1^{-1}D_{12}')C_1\bar{x} = 0$. We now let $\bar{y} = -E_1^{-1}D_{12}'C_1\bar{x}$ which implies that $(j\omega I - A)\bar{x} = B_2\bar{y}$ and $C_1\bar{x} + D_{12}\bar{y} = (C_1 + D_{12}E_1^{-1}D_{12}'C_1)\bar{x} = 0$. That is,

$$\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = 0$$

However, this contradicts assumption A3 and thus, we can conclude that $\tilde{A}_{22}$ has no purely imaginary eigenvalues.

Using the fact that $\tilde{A}_{22}$ has no purely imaginary eigenvalues, we now apply a further state transformation to the system (1) which decomposes $\tilde{A}_{22}$ into anti-stable and stable parts. Thus, we obtain

$$A - B_2E_1^{-1}D_{12}'C_1 = \begin{bmatrix} \tilde{A}_{11} & 0 & 0 \\ \tilde{A}_{21a} & \tilde{A}_a & 0 \\ \tilde{A}_{21b} & 0 & \tilde{A}_b \end{bmatrix}, \quad B_1 = \begin{bmatrix} \tilde{B}_{11} \\ \tilde{B}_{12a} \\ \tilde{B}_{12b} \end{bmatrix}, \quad B_2 = \begin{bmatrix} \tilde{B}_{21} \\ \tilde{B}_{22a} \\ \tilde{B}_{22b} \end{bmatrix},$$
$$(I - D_{12}E_1^{-1}D_{12}')C_1 = [\tilde{C}_{11} \quad 0 \quad 0]$$

where $\tilde{A}_a$ is antistable and $\tilde{A}_b$ is stable. Using the fact that the pair $(\tilde{A}_{11}, \tilde{C}_{11})$ is observable, it now follows that the pair

$$\left\{ -\begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21a} & \tilde{A}_a \end{bmatrix}, \quad [\tilde{C}_{11} \quad 0] \right\}$$

is detectable. Hence

$$A_{11}=\begin{bmatrix}\tilde{A}_{11} & 0\\ \tilde{A}_{21a} & \tilde{A}_{a}\end{bmatrix},\qquad A_{21}=[\tilde{A}_{21b}\quad 0],\qquad A_{22}=\tilde{A}_{b}$$

$$B_{11}=\begin{bmatrix}\tilde{B}_{11}\\ \tilde{B}_{12a}\end{bmatrix},\qquad B_{12}=\tilde{B}_{12b},\qquad B_{21}=\begin{bmatrix}\tilde{B}_{21}\\ \tilde{B}_{22a}\end{bmatrix},\qquad B_{22}=\tilde{B}_{22b},\qquad C_{11}=[\tilde{C}_{11}\quad 0]$$

gives the required decomposition in (29). □

Proof of Theorem 3.4

Suppose $P>0$ satisfies (25) and let $S=P^{-1}$. Define $V=E_1^{1/2}KS+E_1^{-1/2}B_2'+E_1^{-1/2}D_{12}'C_1S$. It follows from (25) that S satisfies the inequality

$$\begin{aligned}0 &> (A+B_2K)S+S(A+B_2K)'+S(C_1+D_{12}K)'(C_1+D_{12}K)S+B_1B_1'\\ &=(A-B_2E_1^{-1}D_{12}'C_1)S+S(A-B_2E_1^{-1}D_{12}'C_1)'+SC_1'(I-D_{12}E_1^{-1}D_{12}')C_1S\\ &\quad +B_1B_1'-B_2E_1^{-1}B_2'+V'V\\ &\geqslant (A-B_2E_1^{-1}D_{12}'C_1)S+S(A-B_2E_1^{-1}D_{12}'C_1)'+SC_1'(I-D_{12}E_1^{-1}D_{12}')C_1S\\ &\quad +B_1B_1'-B_2E_1^{-1}B_2.\end{aligned}\tag{30}$$

We now consider the decomposition defined in Lemma 3.1 and let

$$S=\begin{bmatrix}S_{11} & S_{12}\\ S_{12}' & S_{22}\end{bmatrix}$$

Substituting this into inequality (30) and taking the (1,1) block, it follows that $S_{11}>0$ satisfies the inequality

$$A_{11}S_{11}+S_{11}A_{11}'+S_{11}C_{11}'C_{11}S_{11}+B_{11}B_{11}'-B_{21}E_1^{-1}B_{12}'<0$$

That is, there exists a matrix $Q>0$ such that

$$-A_{11}S_{11}-S_{11}A_{11}'-S_{11}C_{11}'C_{11}S_{11}-B_{11}B_{11}'+B_{21}E_1^{-1}B_{21}'-Q=0\tag{31}$$

Using the fact that the pair $(-A_{11}', C_{11}')$ is stabilizable, it now follows from a standard result on the monotonicity of Riccati equation solutions (see Theorem 2.2 of Reference 27) that the Riccati equation

$$A_{11}T+TA_{11}'+TC_{11}'C_{11}T+B_{11}B_{11}'-B_{21}E_1^{-1}B_{21}'=0\tag{32}$$

has a unique symmetric solution T such that the matrix $A_{11}+TC_{11}'C_{11}$ has all of its eigenvalues in the closed right half plane. Furthermore, $T\geqslant S_{11}>0$. Also, it follows from (32) that T^{-1} satisfies the Riccati equation

$$A_{11}'T^{-1}+T^{-1}A_{11}+T^{-1}(B_{11}B_{11}'-B_{21}E_1^{-1}B_{21}')T^{-1}+C_{11}'C_{11}=0\tag{33}$$

We will now show that $A_{11}+TC_{11}'C_{11}$ is in fact an anti-stable matrix and $T>S_{11}$. In order to establish these facts, we first add equations (31) and (32) to obtain

$$-(A_{11}+TC_{11}'C_{11})Z-Z(A_{11}+TC_{11}'C_{11})'+\tilde{Q}=0$$

where $Z=T-S_{11}\geqslant 0$ and $\tilde{Q}=ZC_{11}'C_{11}Z+Q>0$. A standard Lyapunov argument now implies that $-(A_{11}+TC_{11}'C_{11})$ is stable and $Z>0$ as required; see Lemma 12.2 of Reference 28.

We will show that the matrix

$$X = \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} \geqslant 0$$

is the required stabilizing solution to Riccati equation (2). Indeed using (33), direct substitution shows that this matrix satisfies (2). Furthermore, the matrix $A - B_2E_1^{-1}D_{12}'C_1 + (B_1B_1' - B_2E_1^{-1}B_2')X$ is given by

$$A - B_2E_1^{-1}D_{12}'C_1 + (B_1B_1' - B_2E_1^{-1}B_2')X = \begin{bmatrix} A_{11} + (B_{11}B_{11}' - B_{21}E_1^{-1}B_{21}')T^{-1} & 0 \\ A_{21} + (B_{12}B_{11}' - B_{22}E_1^{-1}B_{21}')T^{-1} & A_{22} \end{bmatrix}$$

However, it is straightforward to establish using Riccati equation (32) that the matrix $A_{11} + (B_{11}B_{11}' - B_{21}E_1^{-1}B_{21}')T^{-1}$ is similar to the matrix $-(A_{11} + TC_{11}'C_{11})$ and hence stable. Thus, the fact that A_{22} is a stability matrix, we can conclude that the matrix X is a stabilizing solution to Riccati equation (2).

In order to show that $P > X$, we first write

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}' & P_{22} \end{bmatrix}$$

Hence, using the fact that $S = P^{-1}$, it follows that $S_{11}^{-1} = P_{11} - P_{12}P_{22}^{-1}P_{12}'$. However, $T > S_{11} > 0$ which implies $T^{-1} < S_{11}^{-1} = P_{11} - P_{12}P_{22}^{-1}P_{12}'$. Hence, the matrix

$$P - X = \begin{bmatrix} P_{11} - T^{-1} & P_{12} \\ P_{12}' & P_{22} \end{bmatrix}$$

is positive definite since $P_{22} > 0$ and $P_{11} - T^{-1} - P_{12}P_{22}^{-1}P_{12}' > 0$. This completes the proof. □

3.2.4. The necessity proof for the main result

Proof of the necessity part of Theorem 3.1

Suppose that condition (i) holds. It follows from Theorem 3.2 there exists a strictly proper controller of the form (22) such that conditions (23) and (24) hold. Hence, using Theorem 3.3, we can conclude the following:

(a) There exists a matrices $P > 0$ and K such that conditions (25) and (26) hold.
(b) There exist a matrices $\Sigma > 0$ and L such that conditions (27) and (28) hold.

Moreover, $\rho(P\Sigma) \leqslant 1$.

Using Theorem 3.4, (a) now implies that Riccati equation (2) will have a stabilizing solution $X \geqslant 0$ such that $X < P$. Also, if we consider the system dual to system (1):

$$\begin{aligned} \dot{x} &= A'x + C_1'z + C_2'y \\ w &= B_1'x \qquad\quad + D_{21}'y \\ u &= B_2'x + D_{12}'z \end{aligned}$$

This system will also satisfy assumptions A1–A4. Hence if we apply Theorem 3.4 with the matrices A, B_2, C_1, D_{12} replaced by the matrices A', C_2', B_1', D_{21}' respectively, (b) implies that Riccati equation (3) has a stabilizing solution $Y \geqslant 0$ such that $Y < \Sigma$. Moreover, $\rho(XY) = \rho(Y^{1/2}XY^{1/2}) \leqslant \rho(Y^{1/2}PY^{1/2}) = \rho(P^{1/2}YP^{1/2}) < \rho(P^{1/2}\Sigma P^{1/2}) = \rho(P\Sigma) \leqslant 1$. This completes the proof of the necessity part of Theorem 3.1. □

3.3 The sufficiency proof

We first present a preliminary result required in our proof of the sufficiency part of Theorem 3.1 (see also Theorem 3.7 of Reference 20 for a related result).

Lemma 3.2

Suppose condition (ii) of Theorem 3.1 holds. Then the matrix $Z := (I - YX)^{-1}Y = Y(I - XY)^{-1} \geqslant 0$ is a stabilizing solution to the Riccati equation

$$A_* Z + Z A_*' - Z M_* Z + N_* = 0 \tag{34}$$

where

$$A_* := A - B_1 D_{21}' E_2^{-1} C_2 + B_1 (I - D_{21}' E_2^{-1} D_{21}) B_1' X$$
$$M_* := (C_2 + D_{21} B_1' X)' E_2^{-1} (C_2 + D_{21} B_1' X) - (B_2' X + D_{12}' C_1)' E_1^{-1} (B_2' X + D_{12}' C_1)$$
$$N_* := B_1 (I - D_{21}' E_2^{-1} D_{21}) B_1'$$

Proof. Given that $\rho(XY) < 1$, it follows that Z is well defined and furthermore, $Y(I - XY) \geqslant 0$. Hence $Z = (I - YX)^{-1} Y \geqslant 0$. Moreover, it follows from the definition of Z that $(I + ZX) = (I - YX)^{-1}$ and $Y = (I + ZX)^{-1} Z = Z(I + XZ)^{-1}$. Substituting these identities into equation (3) and using (2), it is straightforward to verify that $Z \geqslant 0$ satisfies (34). Furthermore, using (2) and (3), it is straightforward to show that

$$(I - YX)^{-1} \{A - B_1 D_{21}' E_2^{-1} C_2 - Y(C_2' E_2^{-1} C_2 - C_1' C_1)\} (I - YX) = A_* - Z M_*$$

Hence, using the fact that Y is a stabilizing solution to (3), it now follows that Z is a stabilizing solution to (34). □

Proof of the sufficiency part of Theorem 3.1

We will suppose that condition (ii) holds and prove that the compensator given by (4) and (5) satisfies the requirements of condition (i). In order to establish this fact, note that Lemma 3.2 implies that matrix $Z = (I - YX)^{-1} Y \geqslant 0$ is a stabilizing solution to Riccati equation (34). Substituting $(I - YX)^{-1} Y = Z$ and $(I - YX)^{-1} = (I + ZX)$ into (5), it follows that the compensator input matrix B_K can be written as

$$B_K = B_1 D_{21}' E_2^{-1} + Z(C_2' + X B_1 D_{21}') E_2^{-1} \tag{35}$$

We now form the closed loop system associated with system (1) and compensator matrices A_K, B_K, C_K. This system is described by the state equations

$$\begin{aligned} \dot{\eta} &= \bar{A}\eta + \bar{B}w \\ z &= \bar{C}\eta \end{aligned} \tag{36}$$

where $e := x - \hat{x}$,

$$\eta := \begin{bmatrix} x \\ e \end{bmatrix} \qquad \bar{A} := \begin{bmatrix} A + B_2 C_K & -B_2 C_K \\ A - A_K + B_2 C_K - B_K C_2 & A_K - B_2 C_K \end{bmatrix} \qquad \bar{B} := \begin{bmatrix} B_1 \\ B_1 - B_K D_{21} \end{bmatrix}$$

and

$$\bar{C} := [C_1 + D_{12} C_K \quad -D_{12} C_K] \tag{37}$$

In order to verify that this system satisfies conditions (20) and (21), we first recall that $Z \geqslant 0$ is a stabilizing solution to Riccati equation (34). This implies that $Z \geqslant 0$ will also be a stabilizing solution to the Riccati equation

$$A_0 Z + ZA_0' + ZC_0'C_0 Z + B_0 B_0' = 0 \tag{38}$$

where

$$A_0 := A - B_1 D_{21}' E_2^{-1} C_2 + B_1(I - D_{21}' E_2^{-1} D_{21}) B_1' X - Z(C_2 + D_{21} B_1' X)' E_1^{-1}(C_2 + D_{21} B_1' X)$$
$$B_0 := B_1(I - D_{21}' E_2^{-1} D_{21}) - Z(C_2 + D_{21} B_1' X)' E_2^{-1} D_{21}$$
$$C_0 := E_1^{-1/2}(B_2' X + D_{12}' C_1)$$

It now follows from Theorem 2.1 that A_0 is stable and $\| C_0(sI - A_0)^{-1} B_0 \|_\infty = \| B_0'(sI - A_0')^{-1} C_0' \|_\infty < 1$. Again using Theorem 2.1, it follows that the Riccati equation

$$A_0' W + WA_0 + WB_0 B_0' W + C_0' C_0 = 0 \tag{39}$$

has a stabilizing solution $W \geqslant 0$. Using equations (2), (5), (35), (37) and (39), it is now straightforward but tedious to verify that

$$\Sigma := \begin{bmatrix} X & 0 \\ 0 & W \end{bmatrix} \geqslant 0$$

satisfies the Riccati equation

$$\bar{A}'\Sigma + \Sigma\bar{A} + \Sigma\bar{B}\bar{B}'\Sigma + \bar{C}'\bar{C} = 0 \tag{40}$$

Furthermore, it is straightforward to verity that

$$\bar{A} + \bar{B}\bar{B}'\Sigma = \begin{bmatrix} A - B_2 E_1^{-1} D_{12}' C_1 - (B_2 E_1^{-1} B_2' - B_1 B_1') X & B_2 E_1^{-1} B_2' + B_2 E_1^{-1} D_{12}' C_1 + B_1(I - D_{21}' E_2^{-1} D_{21}) B_1' W - B_1 D_{21}' E_2^{-1}(C_2 + D_{21}' X) ZW \\ 0 & A_0 + B_0' BW \end{bmatrix}$$

Hence using the fact that X is a stabilizing solution to (2) and W is a stabilizing solution to (39), it follows that $\Sigma \geqslant 0$ is a stabilizing solution to (40). Therefore using Theorem 2.1, we can conclude that $\bar{A}$ is stable and $\| \bar{C}(sI - \bar{A})^{-1} \bar{B} \|_\infty < 1$. This completes the proof of Theorem 3.1. □

We now present a corollary to the proof of the above theorem which gives the solution to a corresponding state feedback H^∞ control problem. This state feedback H^∞ control problem relates to the system (1). However, it is assumed that the full state x is available for measurement. The case in which the full state is available for measurement is of interest in its own right. Furthermore, by referring to Theorem 3.1, it can be seen that the solution to the state feedback H^∞ control problem in fact plays an important role in the general solution.

Corollary 3.1

Consider the system (1) and assume that assumptions A1 and A3 are satisfied. Then the following statements are equivalent.

(i) There exists a dynamic state feedback compensator of the form

$$\dot{\eta} = A_K \eta + B_K x$$
$$u = C_K \eta + D_K x$$

such that the resulting closed loop system satisfies

$$\begin{bmatrix} A + B_2D_K & B_2C_K \\ B_K & A_K \end{bmatrix} \tag{41}$$

and

$$\left\| [C_1 + D_{12}D_K \quad D_{12}C_K] \left\{ sI - \begin{bmatrix} A + B_2D_K & B_2C_K \\ B_K & A_K \end{bmatrix} \right\}^{-1} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \right\|_\infty < 1 \tag{42}$$

(ii) The Riccati equation (2) has a stabilizing solution $X \geqslant 0$.

Furthermore, if condition (ii) holds, then condition (i) can be satisfied using the static state feedback control law

$$u = -E_1^{-1}(B_2'X + D_{12}'C_1)x \tag{43}$$

Proof. The necessity part of this corollary is a direct corollary to the *proof* of Theorem 3.1.

In order to establish sufficiency, suppose that Riccati equation (2) has a stabilizing solution $X \geqslant 0$. We will apply a state feedback control of the form (43). That is, the state feedback gain matrix K is given by $K := -E_1^{-1}(B_2'X + D_{12}'C_1)$. Using this notation, it follows that $X \geqslant 0$ is a stabilizing solution to the Riccati equation $A_0'X + XA_0 + XB_1B_1'X + C_0'C_0 = 0$ where

$$A_0 := A - B_2E_1^{-1}B_2'X - B_2E_1^{-1}D_{12}'C_1 = A + B_2K$$

and

$$C_0 := (I - D_{12}E_1^{-1}D_{12}')C_1 - D_{12}E_1^{-1}B_2'X = C_1 + D_{12}K$$

Hence, it follows from Theorem 2.1 that $A + B_2K$ is stable and

$$\| (C_1 + D_{12}K)\{sI - A - B_2K\}^{-1}B_1 \|_\infty < 1.$$

This completes the proof of the corollary. □

Remark. A similar result to the above can also be found in Reference 30. However, in contrast the above results, the main result of Reference 30 considers a state feedback H^∞ control problem in which the closed loop H^∞ norm is required to be *less than or equal to* one rather than *strictly less than* one. In this case, the solution is obtained in terms on a strong solution to Riccati equation (2) rather than the stabilizing solution required in our case.

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