Consider the regression model (2). If $l = p$ then $v_t = u_t$ and so $v_t$ behaves like a moving average order $q$ process. This implies $E(v_t v_{t-a}) = 0$ for $a > q$ but, in general, $E(v_t v_{t-a}) \neq 0$ for $a \leq q$. Also, note that if $l < p$ then $v_t = u_t + \sum_{s=l+1}^{p} \theta_s y_{t-s}$ and so $v_t$ is an ARMA process which again implies $E(v_t v_{t-a}) \neq 0$ in general. Furthermore, note that if either $l < p$ or $k \leq q$ then $\hat{\theta}_{lk} \xrightarrow{P} \theta_l^* \neq \theta_l$ in general. These observations suggest the following test statistics may be used to deduce $p$ and $q$

$$T_{lk} = T^{-1/2} \sum_{t=1}^{T} \hat{w}_t' \hat{V}_{lk}^{-1} T^{-1/2} \sum_{t=1}^{T} \hat{w}_t$$

where $\hat{w}_t = (\hat{v}_t \hat{v}_{t-k}, \hat{v}_t \hat{v}_{t-k-1}, \cdots, \hat{v}_t \hat{v} t_{t-k-b})'$ for some finite constant $b$, $\hat{v}_t = y_t - X_{lt} \hat{\theta}_{lk}$ and $\hat{V}_{lk}$ is a consistent estimator of

$$V_{lk} = \lim_{T \to \infty} E\left[\left(T^{-1/2} \sum \hat{w}_t\right)\left(T^{-1/2} \sum \hat{w}_t\right)'\right].$$

*Proposition 1:* If $y_t$ is generated by (1) and $C1$, $C2$, $C3$ hold then 1) if $l = p$, $k \geq q + 1$ or $l \geq p$, $k = q + 1$ then $T_{lk} \xrightarrow{d} \chi_b^2$ and $\hat{V}_{lk} = T^{-1} \sum_{t=1}^{T} \hat{f}_t \hat{f}_{t'} + 2T^{-1} \sum_{i=1}^{m_T} \omega_{im_T} \sum_{t=1}^{T} \hat{f}_t \hat{f}_{t-j}'$ where $m_T$ is $0(T^{1/4}) \omega_{im_T} = 1 - i/(m_T + 1)$ and $\hat{f}_t = \hat{w}_t - \hat{N}(\hat{M}_{pq+1})^{-1} Z_{pq+1t}' \hat{v}_t$; where $\hat{N}$ is the $b \times p$ matrix with $i$th row $T^{-1} \sum_{t=1}^{T} \hat{v}_{t-q-i} X_{pt}$, $\hat{M}_{pq+t1} = T^{-1} \hat{Z}_{pq+1}' X_p$.

2) if either $l < p$ or $k \leq q$ then if $M_{lk}$ is nonsingular

$$\begin{aligned} T_{lk} &\to \infty \qquad \text{if } m_w \neq M_{lk}^{-1} \mu_{lk} \\ &\xrightarrow{d} \chi_b^2 \qquad \text{else} \end{aligned}$$

$$\text{where } m_w = p\lim T^{-1} \sum_{t=1}^{T} w_t.$$

$D$ is the $b \times l$ matrix with $j$th row $p\lim T^{-1} \sum_{t=1}^{T} v_{t-k-1-j} X_{lt}$

$$\mu_{lk} = p\lim T^{-1} \sum Z_{lkt}' v_t$$

3) if $l > p$, $k > q$ then the limiting distribution of $T_{lk}$ is undefined.

*Proof:*

1)

$$T^{-1/2} \sum \hat{w}_t = T^{-1/2} \sum w_t - N\left(M_{pq+1}\right)^{-1} T^{-1/2} \cdot \sum_{t=1}^{T} Z_{pq+1t}' u_t + op(1), \quad (4)$$

where $N$ is the $b \times p$ matrix with $i$th row $p \lim T^{-1} \sum_{t=1}^{T} u_{t-q-i} X_{pt}$. Under our assumptions $(T^{-1/2} \sum w_t', T^{-1/2} \sum Z_{pq+1t} u_t)$ converge in distribution to a mean zero normal random vector from which the result follows. Our conditions $C1$–$C3$ are special cases of those under which Newey and West [6] show $\hat{V}_{lk} \xrightarrow{P} V_{lk}$.

2)

$$T^{-1/2} \sum \hat{w}_t = T^{-1/2} \sum w_t - D M_{lk}^{-1} T^{-1/2} \sum_{t=1}^{T} Z_{lkt}' v_t + op(1).$$

Under our assumptions $[T^{-1/2} \sum (w_t - m_w)', T^{-1/2} \sum (Z_{lkt}' v_t - \mu_{lk})']$ converges in distribution to a mean zero normal random vector. The result then follows directly.

3) This follows from Stoica [8]. If $l > p$ and $k > q$ then $M_{lk}$ is singular and $\hat{\theta}_{lk}$ is asymptotically undefined.

Several comments are in order. First, notice that we can use $T_{lk}$ as a basis for a model selection procedure. An iterative updating mechanism for $(l, k)$ must be derived and for each $(l, k)$ $T_{lk}$ is calculated. If $(l^*, k^*)$ are the first values in the updating sequence for which $T_{lk} < \chi_b^2(1 - \alpha)$, where $\chi_b^2(1 - \alpha)$ is the $100(1 - \alpha)$ percentile of $\chi_b^2$, then $p$ and $q$ are chosen as $\hat{p} = l^*$, $\hat{q} = k^* - 1$. One such updating mechanism might be $(l, k) = (0, 1), (1, 1), (0, 2), (1, 2) \cdots$. Given Proposition 1, 2), one would expect $T_{lk} > \chi_b^2(1 - \alpha)$ for conventional choices of $\alpha$ (e.g., $\alpha = .05$) when $l < p$ or $k \leq q$, because, in general, $m_w \neq D M_{lk}^{-1} \mu_{lk}$. Second, note that in a finite sample there is no guarantee that $T_{lk} < \chi_b^2(1 - \alpha)$ for some finite $l$, $k$. A similar problem exists for the overfitting tests based on nonlinear squares estimators discussed in the introduction. It is not possible to show that the probability of type I error for a sequence of $T_{lk}$ tests tends to zero as the number of tests in the sequence tends to infinity, because in the overfit case the limiting distribution is undefined. Therefore, there is no guarantee that the model selection procedure converges even asymptotically. Hall [4] has extended Proposition 1 to cover the case where $y_t$ is ARIMA $(p, 1, q)$. In that context, he reports simulation evidence which suggests that in practice $T_{lk} < \chi_b^2(1 - \alpha)$ for some "reasonable" choice of $l$ and $k$, and so this may not be a problem. He also found that the equivalent of $\hat{\theta}_{lk}$ always existed even in large samples and so Proposition 1, 3) may not be of practical importance. Of course, if a singularity is encountered the natural solution given Stoica's result is to reduce the order of the model. Finally, note that $T_{lk}$ can be viewed as a transformation of Stoica's [8] statistic as $T^{-1/2} \sum \hat{w}_t$ is asymptotically a linear function of $M_{lk}$.

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# New Relations Between the Schur–Cohn–Fujiwara and Nour Eldin Stability Criteria

Brian D. O. Anderson and Mahomed Mansour

*Abstract*—**Root counting and stability criteria involving polynomial roots inside the unit circle are available using the Schur–Cohn–Fujiwara**

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B. D. O. Anderson is with the Department of Systems Engineering, Australian National University, Canberra, ACT, 2601, Australia.

M. Mansour was with the Department of Systems Engineering, Australian National University, Canberra, ACT, 2601, Australia. He is now with the Institute for Automatic Control, Swiss Federal Institute of Technology, Zurich, Switzerland.

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**matrix and certain Cauchy index calculations. This note illustrates a close connection between these criteria, including reduced versions of them.**

## I. Introduction

Let $\phi(z)$ be a real polynomial of degree $n$. In this note, we examine some connections between certain criteria for $\phi(z)$ to have the discrete-time stability property, *viz.* all zeros of $\phi(\cdot)$ lie in $|z| < 1$. In particular, we look on the one hand at the Schur–Cohn–Fujiwara criteria [1]–[5] and reduced versions thereof [6]–[7] and on the other hand at the Nour Eldin–Markov criterion [8], and its reduced form [9]. Important background is provided by [10]; this reference shows how to connect a Bezout matrix (of which the Schur–Cohn–Fujiwara matrix is an example), and a Hankel matrix formed from Markov coefficients of a certain transfer function.

For the sake of simplicity, and as is frequently done, we shall assume that the degree $n$ of $\phi(z)$ is even: $n = 2m$. Otherwise, we replace $\phi(z)$ by $z\phi(z)$. We shall say $\phi(z)$ is *stable* when all zeros lie in $|z| < 1$.

The exposing of these connections is made here for the first time. It is the understanding of such connections which often proves important in making developments in areas such as robust stability, 2-D stability, and so on.

## II. Review of Background Results

We shall review here the Schur–Cohn–Fujiwara matrix test for stability, the connection between Bezout and Hankel matrices and the Cauchy index, and the Nour Eldin–Markov test for stability.

The entries of the Schur–Cohn–Fujiwara matrix $S = [s_{ij}]$ are defined by (see [3]–[5])

$$\frac{\phi(z)w^n\phi(w^{-1}) - \phi(w)z^n\phi(z^{-1})}{z - w} = \sum_{i,j=1}^{n} z^{i-1}s_{ij}w^{n-j}. \tag{2.1}$$

The matrix is symmetric and centrosymmetric

$$s_{ij} = s_{ji} = s_{n+1-i,n+1-j}. \tag{2.2}$$

The half-size reduced-order matrices [6], [7] are defined by

$$A = (a_{ij}) = (s_{ij} + s_{i,n+1-j}) \qquad 1 \le i, j \le m \tag{2.3a}$$

$$B = (b_{ij}) = (s_{ij} - s_{i,n+1-j}) \qquad 1 \le i, j \le m. \tag{2.3b}$$

The matrix $S$ is congruent to $A \dotplus B$. The key result is that

$$S > 0 \Leftrightarrow A \dotplus B > 0 \Leftrightarrow \phi(z) \text{ is stable}. \tag{2.4}$$

(Here, the symbol $\dotplus$ denotes direct sum.) The reduced-order criterion [6], [7] states that

$$\left.\begin{array}{l}\text{Either } A > 0 \text{ or } B > 0, \text{ and}\\ \text{either one of two sets of linear}\\ \text{inequalities on coefficients of}\\ \phi(z) \text{ are fulfilled}\end{array}\right. \Leftrightarrow \phi(z) \text{ is stable}. \tag{2.5}$$

Next, consider a rational proper function $p(x)/q(x)$, with $q(x)$ of degree $m$ and $p$, $q$ coprime. One can associate a Bezout form and a symmetric $m \times m$ Bezout matrix $C = (c_{ij})$ with this function, as follows:

$$\sum c_{ij}x^{i-1}y^{j-1} = \frac{p(y)q(x) - q(y)p(x)}{x - y}. \tag{2.6}$$

When the Markov coefficients $s_i$ associated with $p(x)/q(x)$ are defined by

$$\frac{p(x)}{q(x)} = s_{-1} + \frac{s_0}{x} + \frac{s_1}{x^2} + \frac{s_2}{x^3} + \cdots + \tag{2.7}$$

the right side of (2.6) can also be rewritten, see [10], so that

$$\sum c_{ij}x^{i-1}y^{j-1} = q(x)[x^{-1}x^{-2}\cdots]H[y^{-1}y^{-2}\cdots]'q(y) \tag{2.8}$$

where $H$ is the infinite Hankel matrix

$$H = \begin{bmatrix} s_0 & s_1 & s_2 & \cdot \\ s_1 & s_2 & s_3 & \cdot \\ s_2 & s_3 & s_4 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}. \tag{2.9}$$

Moreover, if

$$q(x) = \alpha_0 x^m + \alpha_1 x^{m-1} + \cdots + \alpha_m \tag{2.10}$$

then there holds, see [10]

$$C = \begin{bmatrix} \alpha_{m-1} & \alpha_{m-2} & \cdots & \alpha_1 & \alpha_0 \\ \alpha_{m-2} & \alpha_{m-3} & \cdots & \alpha_0 & 0 \\ \vdots & & & & \\ \alpha_0 & 0 & \cdots & 0 & 0 \end{bmatrix} \cdot H_{mm} \begin{bmatrix} \alpha_{m-1} & \alpha_{m-2} & \cdots & \alpha_1 & \alpha_0 \\ \alpha_{m-2} & \alpha_{m-3} & \cdots & \alpha_0 & 0 \\ \vdots & & & & \\ \alpha_0 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{2.11}$$

where $H_{mm}$ denotes the first $m$ rows and columns of $H$. Also

$$\mathrm{I}_{-\infty}^{+\infty} \frac{p(x)}{q(x)} = \text{signature } C = \text{signature } H_{mm} = \text{signature } H. \tag{2.12}$$

(Here, $\mathrm{I}_a^b \dfrac{p(x)}{q(x)}$, the Cauchy index over $(a, b)$ of $p/q$, is the number of jumps in $p/q$ from $-\infty$ to $+\infty$ less the number of jumps from $+\infty$ to $-\infty$ as $x$ moves from $a$ to $b$, with jumps at $a$ or $b$ being excluded.)

Last, we consider the stability test of Nour Eldin [8] and modification of [9]. Define polynomials $P_1(\cdot)$, $P_2(\cdot)$ of degree $m$ and $m - 1$ by

$$P_1\left(\frac{z + z^{-1}}{2}\right) = \frac{1}{2}[z^{-m}\phi(z) + z^m\phi(z^{-1})] \tag{2.13a}$$

$$P_2\left(\frac{z + z^{-1}}{2}\right) = \frac{2}{z - z^{-1}}[z^{-m}\phi(z) - z^m\phi(z^{-1})]. \tag{2.13b}$$

Then

$$\mathrm{I}_{-1}^{1} \frac{P_2(x)}{P_1(x)} = m \Leftrightarrow \mathrm{I}_{-\infty}^{\infty} \frac{(x+1)P_2(x)}{P_1(x)} = m \text{ and } \mathrm{I}_{-\infty}^{\infty} \frac{(1-x)P_2(x)}{P_1(x)} = m \Leftrightarrow \phi(z) \text{ is stable}. \tag{2.14}$$

In [9], it is asserted (without proof in the journal version of the note) that if either of the two equalities in the second member of (2.14) holds, *and* a number of linear inequalities on the coefficients of $\phi(z)$ are fulfilled, then this is enough to guarantee stability of $\phi(z)$.

The forms of the reduced Schur–Cohn Fujiwara criterion (2.5) and reduced Nour Eldin criterion in [9] suggest that the conditions $A > 0$, $B > 0$ may be separately equivalent to the two Cauchy index conditions appearing in the second member of (2.14). Indeed, this is the case, and we develop a relationship in some detail, which then allows the derivation also of conditions for root counting, see [5], [9].

## III. Relation Between Reduced Schur–Cohn Fujiwara Matrices and Certain Bezout Forms

The key to this section is the following proposition.

*Proposition 3.1:* Let $\phi(z)$ be a polynomial of degree $n = 2m$, and let the Schur–Cohn–Fujiwara matrix and the reduced-order matrices be defined by (2.1), (2.3) and let polynomials of degree $m$ and $m-1$ be defined by (2.13). Set

$$x = \frac{z + z^{-1}}{2} \quad y = \frac{w + w^{-1}}{2}. \tag{3.1}$$

Then the following identities are valid

$$\frac{(x-1)P_2(x)P_1(y) - (y-1)P_2(y)P_1(x)}{x-y} = 2\frac{z^{-(m-1)}w^{-(m-1)}}{(z+1)(w+1)} \sum_{i,j=1}^{m} (z^{i-1} + z^{2m-i}) \cdot a_{ij}(w^{j-1} + w^{2m-j}) \tag{3.2}$$

$$\frac{(x+1)P_2(x)P_1(y) - (y+1)P_2(y)P_1(x)}{x-y} = 2\frac{z^{-(m-1)}w^{-(m-1)}}{(w-1)(z-1)} \sum_{i,j=1}^{m} (z^{i-1} - z^{2m-i}) \cdot b_{ij}(w^{2m-j} - w^{j-1}). \tag{3.3}$$

*Proof:* We outline how (3.2) can be proved. Using (3.1) and (2.13), tedious manipulation yields

$$\frac{(x-1)P_2(x)P_1(y) - (y-1)P_2(y)P_1(x)}{x-y} = \frac{2z^{-(m-1)}w^{-(m-1)}}{(1+w)(1+z)} \times \left[\frac{\phi(w)\phi(z) - w^n z^n \phi(w^{-1})\phi(z^{-1})}{zw - 1} + \frac{w^n\phi(w^{-1})\phi(z) - \phi(w)z^n\phi(z^{-1})}{z-w}\right]. \tag{3.4}$$

Also,

$$\begin{aligned}
&\frac{\phi(w)\phi(z) - w^n z^n \phi(w^{-1})\phi(z^{-1})}{zw-1} + \frac{w^n\phi(w^{-1})\phi(z) - \phi(w)z^n\phi(z^{-1})}{z-w} \\
&= \sum_{i,j=1}^{n} z^{i-1}s_{ij}w^{j-1} + \sum_{i,j=1}^{n} z^{i-1}s_{ij}w^{n-j} \qquad \text{by (2.1)} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} z^{i-1}(s_{ij}w^{j-1} + s_{i,2m+1-j}w^{2m-j}) + \sum_{i=1}^{n}\sum_{j=1}^{m} z^{i-1}(s_{ij}w^{2m-j} + s_{i,2m+1-j}w^{j-1}) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} z^{i-1}(s_{ij} + s_{i,2m+1-j})(w^{j-1} + w^{2m-j}) \\
&= \sum_{i=1}^{m}\sum_{j=1}^{m} \left[z^{i-1}(s_{ij} + s_{i,2m+1-j}) + z^{2m-i}(s_{2m+1-i,j} + s_{2m+1-i,2m+1-j})\right] \times (w^{j-1} + w^{2m-j}) \\
&= \sum_{i=1}^{m}\sum_{j=1}^{m} \left[z^{i-1}(s_{ij} + s_{i,2m+1-j}) + z^{2m-i}(s_{i,2m+1-j} + s_{ij})\right] \times (w^{j-1} + w^{2m-j}) \\
&= \sum_{i=1}^{m}\sum_{j=1}^{m} (z^{i-1} + z^{2m-i})a_{ij}(w^{j-1} + w^{2m-j}) \qquad \text{by (2.3a)}
\end{aligned} \tag{3.5}$$

The identity (3.2) is an immediate consequence of (3.4) and (3.5). Identity (3.3) is proved in a similar way. The identities of Proposition (3.1) strongly suggest the following conjecture:

$$\mathrm{I}_{-\infty}^{+\infty} \frac{(1-x)P_2(x)}{P_1(x)} = m \Leftrightarrow A > 0 \tag{3.6}$$

$$\mathrm{I}_{-\infty}^{+\infty} \frac{(1+x)P_2(x)}{P_1(x)} = m \Leftrightarrow B > 0. \tag{3.7}$$

In fact, a little more again is true.

*Proposition 3.2:* With notation as above define $D = (d_{ij})$ by

$$\sum_{i,j=1}^{m} d_{ij}x^{m-i}y^{m-j} = \frac{(1-y)P_2(y)P_1(x) - (1-x)P_2(x)P_1(y)}{x-y}. \tag{3.8}$$

(Thus, $D$ is a Bezout matrix with the ordering of rows and column reversed.) Let

$$\frac{P_2(x)}{P_1(x)} = \frac{s_0}{x} + \frac{s_1}{x^2} + \frac{s_2}{x^3} + \cdots + \tag{3.9}$$

so that

$$\frac{(1-x)P_2(x)}{P_1(x)} = -s_0 + \frac{s_0 - s_1}{x} + \frac{s_1 - s_2}{x^2} + \cdots + \tag{3.10}$$

Then there exist lower triangular matrices $U$, $W$, with

$$D = 2UAU' = W\begin{bmatrix} s_0 - s_1 & s_1 - s_2 & \cdots & s_{m-1} - s_m \\ s_1 - s_2 & s_2 - s_3 & \cdots & s_m - s_{m+1} \\ \vdots & \vdots & & \vdots \\ s_{m-1} - s_m & s_m - s_{m+1} & \cdots & s_{2m-2} - s_{2m-1} \end{bmatrix} W'. \tag{3.11}$$

The matrix $U$ has diagonal entries $2^{m-1}, 2^{m-2}, \ldots, 1$ and with

$$P_1(x) = \alpha_0 x^m + \alpha_1 x^{m-1} + \cdots \alpha_m \tag{3.12}$$

there holds

$$W = \begin{bmatrix} \alpha_0 & 0 & \cdots & 0 \\ \alpha_1 & \alpha_0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \alpha_{m-1} & \alpha_{m-2} & \cdots & \alpha_0 \end{bmatrix}. \tag{3.13}$$

*Proof:* Recall the definition of the Chebyshev polynomial of the first kind: if $x = \cos\phi$, $T_k(x) = \cos k\phi$ and $T_0(x) = 1$. Notice then that with $x = \frac{1}{2}(z + z^{-1})$, there holds

$$\frac{z^{-(m-1)}}{z+1}(z^{i-1} + z^{2m-i}) = \frac{z^{m-i+1} + z^{i-m}}{z+1}$$

$$= z^{m-i} - z^{m-i-1} + \cdots + z^{-(m-i)}$$
$$= 2\left[T_{m-i}(x) - T_{m-i-1}(x) + \cdots + (-1)^{m-i}T_0(X)\right]$$
$$\triangleq S_{m-i}(x) \tag{3.14}$$

with $S_0(x) = 1$. The highest power term in $T_k(x)$ is $2^{k-1}x^k$. Hence the highest power term in $S_k(x)$ is $2^k x^k$. From (3.2), it follows that

$$\frac{(x-1)P_2(x)P_1(y) - (y-1)P_2(y)P_1(x)}{x-y}$$
$$= 2\sum_{i,j=1}^{m} S_{m-i}(x)a_{ij}S_{m-j}(y)$$
$$= 2\left[x^{m-1}x^{m-2}\cdots 1\right]UAU'\left[y^{m-1}y^{m-2}\cdots 1\right]'$$

where $U$ is a lower triangular matrix, the $j$th column containing the coefficients of $S_{m-j}(x)$. The diagonal entries of $U$ are, in order

$$2^{m-1}, 2^{m-2}, \ldots, 1.$$

The first equality in (3.11) is then clear. The second follows from the connection between Bezout and Hankel matrices as set out in Section II, noting that the columns and rows of $D$ are in reversed order to the usual Bezout matrix.

This proposition shows that the signature of the truncated Hankel matrix [which is the Cauchy index over $(-\infty, \infty)$ of $(1-x)P_2(x)/P_1(x)$] is identical with the signature of $A$. In fact, the leading principal minors of the two matrices are very simply related. Denote the $i \times i$ leading principal minors by $|S_{-1}|$ and $|A_i|$. Then (3.11) shows that

$$2^i(2^{m-1}\cdots 2^{m-i})^2|A_i| = \alpha_0^{2i}|S_{-i}|,$$

i.e.,

$$|S_{-i}| = \frac{2^{i(2m-i)}}{\alpha_0^{2i}}|A_i|. \tag{3.15}$$

Of course, a similar result to Proposition 3.2 can be obtained involving the other reduced-order matrices.

*Proposition 3.3:* With notation as above, define $E = (e_{ij})$ by

$$\sum_{i,j=1}^{m} e_{ij}x^{m-i}y^{m-j}$$
$$= \frac{(1+y)P_2(y)P_1(x) - (1+x)P_2(x)P_1(y)}{x-y}. \tag{3.16}$$

Then there exists a lower triangular matrix $V$ with

$$E = 2VBV'$$
$$= W\begin{bmatrix} s_0+s_1 & s_1+s_2 & \cdots & s_{m-1}+s_m \\ s_1+s_2 & s_2+s_3 & \cdots & s_m+s_{m+1} \\ \vdots & \vdots & & \vdots \\ s_{m-1}+s_m & s_m+s_{m+1} & \cdots & s_{2m-2}+s_{2m-1}\end{bmatrix}W'. \tag{3.17}$$

The matrix $V$ satisfies

$$\left[x^{m-1}\ x^{m-2}\ \cdots\ 1\right]V$$
$$= \left[R_{m-1}(x)\ R_{m-2}(x)\ \cdots\ R_1(x)\ 1\right] \tag{3.18}$$

where

$$R_k(x) = 2\left[T_k(x) + T_{k-1}(x) + \cdots T_0(x)\right] \tag{3.19}$$

and is lower triangular, with the same diagonal entries as $U$.

The two preceding propositions have shown that

$$\mathrm{I}_{-\infty}^{+\infty}\frac{(1-x)P_2(x)}{P_1(x)} = \text{signature } A = \text{signature } D = \text{signature } S_- \tag{3.20}$$

and

$$\mathrm{I}_{-\infty}^{+\infty}\frac{(1+x)P_2(x)}{P_1(x)} = \text{signature } B = \text{signature } E = \text{signature } S_+ \tag{3.21}$$

where

$$S_\pm = \begin{bmatrix} s_0 \pm s_1 & s_1 \pm s_2 & \cdots & s_{m-1} \pm s_m \\ s_1 \pm s_2 & s_2 \pm s_3 & \cdots & s_m \pm s_{m+1} \\ \vdots & & & \\ s_{m-1} \pm s_m & s_m \pm s_{m+1} & \cdots & s_{2m-2} \pm s_{2m-1}\end{bmatrix}. \tag{3.22}$$

The relation between the reduced-order stability criteria of [6], [7], [9] is thereby exposed.

## IV. Conclusions

We have demonstrated relationships between two different stability criteria, including reduced-order versions of these criteria. Modifications of the criteria can also be used for root counting. The various matrices whose signatures are the key to obtaining root distributions are related by congruency transformations with triangular structure, implying that the sign patterns of the leading principal minors are identical.

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