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# $L_1$ impulse response error bound for balanced truncation

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*Abstract:* This paper presents an upper bound in $L_1$ for the impulse response error between a system and its balanced truncation. It is an *a priori* bound and can be computed easily. Numerical examples are used to illustrate its applications and to compare with other available error bounds.

*Keywords:* Balanced truncation; $L_1$-norm; model reduction; impulse response; Hankel singular values.

## 1. Introduction

The *balanced truncation* method has been an important technique for model reduction of stable systems. It was first introduced to the control engineering field by Moore [4]. The underlying ideas involve Principal Component Analysis which has been widely used in statistical analysis and Kalman's controllable/observable canonical realization of a system. The method was further analyzed by Pernebo and Silverman [5]. Glover [2] and Enns [1] later proved that the reduced model obtained by truncating a balanced realization enjoys an attractive frequency domain $L_\infty$ norm error bound. Such an error bound can be calculated *a priori* based on the Hankel singular values of the full order model. Error bounds for the impulse response error in the $L_1$ and $L_2$ sense of the reduced order model are also given in [3]. The results of [3] were proved for the possibly infinite dimensional case with error expressions involving a minimization. In this paper, we present an upper bound for the impulse response error in the $L_1$ norm for the finite dimensional case. The formula obtained has a simpler form which does not involve a minimization procedure. It is therefore easier to compute than the currently available result given by Glover et al. [3]. Moreover, it can give better a estimation of the achievable error.

We shall provide some definitions and a summary of preliminary results in Section 2. The main results will be presented in Section 3. Before concluding the work in Section 5, numerical examples are given in Section 4 to illustrate the applications of the error bound formula.

## 2. Definitions and preliminary results

Let $G(s)$ be a stable $p \times q$ real rational transfer function matrix of McMillan degree $n$ and admits a state-space realization given by

$$G(s) = C(sI - A)^{-1}B + D$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times q}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times q}$. Since $G(s)$ is stable, we have the eigenvalues of $A$ to $\mathbb{C}^-$. Taking the inverse Laplace transform of $G(s)$ gives an impulse response function $H(t)$ where

$$H(t) = C\,e^{At}B + D\delta(t)$$

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with $\delta(t)$ denoting the Dirac delta function. This impulse response matrix induces an Hankel operator $\Gamma_H : L_2^q(-\infty, 0] \to L_2^p[0, \infty)$ defined for all $t \geq 0$ by

$$(\Gamma_H u)(t) = \int_{-\infty}^{0} C \, e^{A(t-\tau)} B u(\tau) \, d\tau.$$

The singular values of the Hankel operator $\Gamma_H$ denoted by

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0 \tag{1}$$

are referred to as the *Hankel singular values* of $G(s)$ and can be obtained from (see [2])

$$\sigma_i = \sqrt{\lambda_i(PQ)}, \quad i = 1, \ldots, n, \tag{2}$$

with $\lambda_i(\cdot)$ denoting the $i$-th eigenvalue of $(\cdot)$ and $P$, $Q$ are respectively the controllability and observability Gramians of $G(s)$ and they are determined by solving the following Lyapunov equations:

$$AP + PA^* + BB^* = 0, \tag{3}$$

$$A^*Q + QA + C^*C = 0. \tag{4}$$

We sometimes use the notation $\sigma_i(G)$ to denote the $i$-th Hankel singular value of the transfer function $G(s)$ and define the *Hankel norm* of $G(s)$ as $\sigma_1(G)$ denoted by $\|G\|_H$.

A similarity transformation $T$ exists (see [2]) such that

$$\bar{A} = T^{-1}AT, \tag{5}$$

$$\bar{B} = T^{-1}B, \tag{6}$$

$$\bar{C} = CT, \tag{7}$$

with $\bar{P} := T^{-1}PT^{-*}$ and $\bar{Q} := T^*QT$ both equal to

$$\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n) \tag{8}$$

and $\bar{P}$, $\bar{Q}$ satisfy

$$\bar{A}\Sigma + \Sigma\bar{A}^* + \bar{B}\bar{B}^* = 0, \tag{9}$$

$$\bar{A}^*\Sigma + \Sigma\bar{A} + \bar{C}^*\bar{C} = 0. \tag{10}$$

The transformation matrix $T$ is often called the *balancing transformation* of the realization $(A, B, C)$.

Partition the realization $(\bar{A}, \bar{B}, \bar{C})$ such that

$$\bar{A} = \begin{pmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \end{pmatrix}, \quad \bar{C} = \begin{pmatrix} \bar{C}_1 & \bar{C}_2 \end{pmatrix}, \tag{11}$$

where the partitioning is done conformally with

$$\Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix} \tag{12}$$

with $\Sigma_1 = \mathrm{diag}(\sigma_1, \ldots, \sigma_k)$, $\Sigma_2 = \mathrm{diag}(\sigma_{k+1}, \ldots, \sigma_n)$ and $\sigma_k > \sigma_{k+1}$. A $k$-th order balanced truncation $\hat{G}_k(s)$ is then given by

$$\hat{G}_k(s) = \bar{C}_1 (sI - \bar{A}_{11})^{-1} \bar{B}_1 + D. \tag{13}$$

In fact, $\hat{G}_k(s)$ defines a stable, minimal, balanced system with controllability and observability Gramians both equal to $\Sigma_1$ [5]. If the multiplicity of $\sigma_n$ is $m$, then a *single-step balanced truncation* given by $\hat{G}_{n-m}(s)$ is referred to the case with $\Sigma_2 = \sigma_n I_m$ where $I_m$ denotes an $m \times m$ unit matrix.

We use $\tilde{G}_k(s)$ to denote a $k$-th order optimal Hankel norm approximation of $G(s)$ and it is an optimal solution for the following minimization problem (see [2]):

$$\min_{G_k \in \mathscr{K}} \| G - G_k \|_{\mathrm{H}}$$

where $\mathscr{K}$ denote the class of stable real rational transfer function matrices with order no greater than $k$. In fact, the minimum value of the above minimization is given by $\| G - \tilde{G}_k \|_{\mathrm{H}} = \sigma_{k+1}(G)$. Glover [2] gave the construction of $\tilde{G}_k(s)$ in state-space form based on a balanced realization of $G(s)$. Suppose the Hankel singular values of $G(s)$ are

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > \sigma_{k+1} = \cdots = \sigma_{k+r} > \sigma_{k+r+1} \geq \cdots \geq \sigma_n > 0.$$

Then form a balanced realization $(\tilde{A}, \tilde{B}, \tilde{C}, D)$ of $G(s)$ such that

$$\tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix}, \qquad \tilde{B} = \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix}, \qquad \tilde{C} = \begin{pmatrix} \tilde{C}_1 & \tilde{C}_2 \end{pmatrix}$$

are partitioned conformally with the corresponding controllability/observability Gramian

$$\tilde{\Sigma} = \begin{pmatrix} \tilde{\Sigma}_1 & O \\ O & \sigma_{k+1} I_r \end{pmatrix}$$

where $\tilde{\Sigma}_1 = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_k, \sigma_{k+r+1}, \ldots, \sigma_n)$. Then a $k$-th order optimal Hankel norm approximation $\tilde{G}_k(s)$ of $G(s)$ is given by the stable part of $(A_h, B_h, C_h, D_h)$ (see [2], Corollary 7.3) with

$$A_h := \Gamma^{-1}\left(\sigma_{k+1}^2 \tilde{A}_{11}^* + \tilde{\Sigma}_1 \tilde{A}_{11} \tilde{\Sigma}_1 - \sigma_{k+1} \tilde{C}_1^* U \tilde{B}_1^*\right), \tag{14}$$

$$B_h := \Gamma^{-1}\left(\tilde{\Sigma}_1 \tilde{B}_1 + \sigma_{k+1} \tilde{C}_1^* U\right), \tag{15}$$

$$C_h := \tilde{C}_1 \tilde{\Sigma}_1 + \sigma_{k+1} U \tilde{B}_1^*, \tag{16}$$

$$D_h := D - \sigma_{k+1} U, \tag{17}$$

$$\Gamma := \tilde{\Sigma}_1^2 - \sigma_{k+1}^2 I_r, \tag{18}$$

where $U$ such that $UU^* \leq I$ satisfies

$$\tilde{B}_2 + \tilde{C}_2^* U = 0. \tag{19}$$

The above condition on $U$ stems from the fact that one has $\tilde{B}_2 \tilde{B}_2^* = \tilde{C}_2^* \tilde{C}_2$ in the realization $(\tilde{A}, \tilde{B}, \tilde{C}, D)$ of $G(s)$. Notice that a choice of $UU^* = I$ (resp. $U^*U = I$) is always possible when $p \leq q$ (resp. $p \geq q$). For ease of reference, we shall refer to $U$ as the *U-matrix* of a Hankel norm approximation. In the case with $k = n - m$ where $m$ is the multiplicity of $\sigma_n$, then we refer to $\tilde{G}_{n-m}(s)$ as a *single-step Hankel norm approximation* of $G(s)$.

For the transfer function $G(s)$ above, we define the $L_\infty$ norm of $G(s)$ as

$$\| G \|_\infty = \sup_{0 \leq \omega < \infty} \bar{\sigma}(G(\mathrm{j}\omega))$$

where $\bar{\sigma}(\cdot)$ denotes the maximum singular value of $(\cdot)$. The *nuclear norm* of $G(s)$ is

$$\| G \|_{\mathrm{N}} := \sum_{i=1}^{n} \sigma_i$$

and the $L_1$ norm of the impulse response $H(t)$ is

$$\| H \|_1 := \int_0^\infty \bar{\sigma}(H(t))\, \mathrm{d}t.$$

The following lemma is true for any stable strictly proper transfer function.

**Lemma 2.1** (Glover et al. [3]). *For any strictly proper transfer function $G(s)$, we have*

$$2\| G \|_{\mathrm{N}} \geq \| H \|_1 \geq \| G \|_\infty \geq \sigma_1(G). \qquad \square$$

Furthermore, the following result was proved in [3].

**Lemma 2.2.** *If $\hat{G}_k(s)$ (with impulse response $\hat{H}_k(t)$) is a k-th order balanced truncation of $G(s)$ (with impulse response $H(t)$), then*

$$\| G - \hat{G}_k \|_{\mathrm{N}} \leq E_k := \min_{0 \leq i \leq k} \left\{ (4\sqrt{2}i - 1) \sum_{j=k+1}^{n} \sigma_j + 2 \sum_{j=i+1}^{n} \sigma_j \right\} \tag{20}$$

*and*

$$\| H - \hat{H}_k \|_1 \leq 2E_k. \qquad \square \tag{21}$$

The currently available $L_1$ error bound on the impulse response for a balanced truncation as seen above involves a process of minimization, the computation of which is complicated especially when $k$ and $n$ are large. Apart from this, it fails to provide simple and direct insight into its behaviour from a given set of Hankel singular values. Thus, it would be convenient if a simpler upper bound expression were available.

The following lemma gives a similar result for an optimal Hankel norm approximation which can also be found in [3].

**Lemma 2.3.** *If $\tilde{G}_k(s)$ is a k-th order optimal Hankel norm approximation of $G(s)$, then*

$$\| G - \tilde{G}_k \|_{\mathrm{N}} \leq 2k\sigma_{k+1} + \sum_{j=k+1}^{n} \sigma_j. \qquad \square \tag{22}$$

## 3. Main results

We first give the following result which relates the nuclear norm of the error between $\hat{G}_{n-m}(s)$ and $\tilde{G}_{n-m}(s)$.

**Lemma 3.1.** *Let $\hat{G}_{n-m}(s)$ and $\tilde{G}_{n-m}(s)$ be respectively a single-step balanced truncation and a single-step Hankel-norm approximation of $G(s)$ with McMillan degree n, then*

$$\| \hat{G}_{n-m} - \tilde{G}_{n-m} \|_{\mathrm{N}} \leq 2(n-m)\sigma_n$$

*where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{n-m} > \sigma_{n-m+1} = \cdots = \sigma_n > 0$.*

**Proof.** We first construct a single-step Hankel norm approximation $\tilde{G}_{n-m}(s)$ of $G(s)$ with the formulae provided in Section 2 based on [2] (Corollary 7.3). In particular, we have $U$ such that $UU^* \leq I$ satisfying

$$\tilde{B}_2 + \tilde{C}_2^* U = 0$$

as defined in (19). Consider an augmented system $G_a(s)$ given by

$$G_a(s)=\begin{pmatrix} G(s) & O \\ O & O \end{pmatrix}$$

where $G_a(s)$ is $(p+q)\times(p+q)$ and the McMillan degrees of $G(s)$ and $G_a(s)$ are equal. Moreover, the Hankel singular values of $G(s)$ and $G_a(s)$ are the same. A balanced realization of $G_a(s)$ can be obtained from a balanced realization of $G(s)=\tilde{C}(sI-\tilde{A})^{-1}\tilde{B}+D$ by augmenting $p$ columns of zeros to the right of $\tilde{B}$ and $q$ rows of zeros at the bottom of $\tilde{C}$. A single-step Hankel-norm approximation of $G_a(s)$ can be obtained using the same construction as for $\tilde{G}_{n-m}(s)$ with the corresponding $U$-matrix, say $V$, satisfying

$$\begin{pmatrix}\tilde{B}_2 & O\end{pmatrix}+\begin{pmatrix}\tilde{C}_2^* & O\end{pmatrix}V=0. \tag{23}$$

A particular choice of $V$ such that $V$ is unitary is given by

$$V=\begin{pmatrix} U & (I-UU^*)^{1/2} \\ (I-U^*U)^{1/2} & -U^* \end{pmatrix}$$

and the property that $\tilde{C}_2^*(I-UU^*)\tilde{C}_2=\tilde{C}_2^*\tilde{C}_2-\tilde{B}_2\tilde{B}_2^*=0$ is employed in order that $V$ satisfies constraint (23). It is then easily seen that a single-step Hankel norm approximation of $G_a(s)$ is given by $\tilde{G}_a(s)$ where

$$\tilde{G}_a(s)=\begin{pmatrix} \tilde{G}_{n-m} & x \\ x & x \end{pmatrix}$$

with $x$ denote some transfer function matrix of no immediate significance. If $\hat{G}_a(s)$ is a single-step balanced truncation of $G_a(s)$, then from [2] (Lemma 9.5) we have that

$$\left(\hat{G}_a-\tilde{G}_a\right)/\sigma_n$$

is all-pass (so that all Hankel singular values are 1) and by applying Lemma 7.4 in [2], we obtain

$$\sigma_i\left(\hat{G}_{n-m}-\tilde{G}_{n-m}\right)\le\sigma_n,\quad i=1,\dots,l,$$

where $l$ is the McMillan degree of $\hat{G}_{n-m}-\tilde{G}_{n-m}$. The result then follows since $l$ is not greater than $2(n-m)$. □

Now, the result on the $L_1$ bound for impulse response error is given in the following theorem.

**Theorem 3.1.** *Suppose the distinct Hankel singular values of a stable $G(s)$ of degree $N$ are given by*

$$\sigma_1>\sigma_2>\cdots>\sigma_n>0$$

*where each $\sigma_i$ has multiplicity $m_i$ $(m_1+\cdots+m_n=N)$. Let $\hat{H}_K(t)$ $(K=m_1+\cdots+m_k)$ be the impulse response of the $K$-th order balanced truncation $\hat{G}_K(s)$ of $G(s)$ with impulse response $H(t)$. Then we have*

$$\|H(t)-\hat{H}_K(t)\|_1\le2\left(4N\sigma_n+4\sum_{i=k+1}^{n-1}\left(N-\sum_{j=i+1}^{n}m_j\right)\sigma_i-3\sum_{i=k+1}^{n}m_i\sigma_i\right) \tag{24}$$

*and the middle term in the right-hand-side disappears when $k=n-1$.*

**Proof.** First consider a single-step balanced truncation of $G(s)$ and notice that the smallest Hankel singular value is $\sigma_n$ with multiplicity $m_n$. From Lemma 2.1, we have

$$\begin{aligned}\|H-\hat{H}_{N-m_n}\|_1&\le2\|G-\hat{G}_{N-m_n}\|_{\mathrm{N}}\\&\le2\left(\|G-\tilde{G}_{N-m_n}\|_{\mathrm{N}}+\|\tilde{G}_{N-m_n}-\hat{G}_{N-m_n}\|_{\mathrm{N}}\right)\\&\le2(2(N-m_n)\sigma_n+m_n\sigma_n+2(N-m_n)\sigma_n)\end{aligned}$$

where Lemma 2.3 and Lemma 3.1 are employed. Hence,

$$\|H - \hat{H}_{N-m_n}\|_1 \le 2(4N - 3m_n)\sigma_n.$$

Similarly, $\hat{H}_{N-m_n}(s)$ can be approximated by a single-step balanced truncation $\hat{H}_{m-m_n-m_{n-1}}(s)$ and we have

$$\|\hat{H}_{N-m_n} - \hat{H}_{N-m_n-m_{n-1}}\|_1 \le 2(4(N-m_n) - 3m_{n-1})\sigma_{n-1}$$

by noticing the property of the preservation of Hankel singular values in a balanced truncation. By arguing inductively for more truncation steps until $\hat{G}_K(s)$ (with impulse response $\hat{H}_K(t)$) is reached which we have

$$\|\hat{H}_{N-m_n-\cdots-m_{k+2}} - \hat{H}_K\|_1 \le 2(4(N-m_n-\cdots-m_{k+2}) - 3m_{k+1})\sigma_{k+1}$$

since $N - m_n - m_{n-1} - \cdots - m_{k+1} = K$ and the overall approximation error is then given by

$$\|H(t) - \hat{H}_K(t)\|_1 \le 2\left(4N\sigma_n + 4\sum_{i=k+1}^{n-1}\left(N - \sum_{j=i+1}^{n} m_j\right)\sigma_i - 3\sum_{i=k+1}^{n} m_i\sigma_i\right)$$

and the first summation term in the right-hand-side of the above inequality disappears for $k = n-1$. □

The result in the above theorem can be much simplified when the Hankel singular values of $G(s)$ are distinct. In this case, we summarize the result in the following corollary.

**Corollary 3.1.** *With the notation in Theorem 3.1, if the Hankel singular values are distinct, then*

$$\|H(t) - \hat{H}_k(t)\|_1 \le 2\sum_{i=k+1}^{n}(4i-3)\sigma_i. \tag{25}$$

**Proof.** For distinct Hankel singular values, we have $m_i = 1$, $i = 1, \ldots, n$, $N = n$ and $K = k$. Hence, (24) becomes

$$\begin{aligned}\|H - \hat{H}_k\|_1 &\le 2\left(4n\sigma_n + 4\sum_{i=k+1}^{n-1}\left(n - \sum_{j=i+1}^{n} 1\right)\sigma_i - 3\sum_{i=k+1}^{n}\sigma_i\right)\\ &= 2\left(4n\sigma_n + 4\sum_{i=k+1}^{n-1} i\sigma_i - 3\sum_{i=k+1}^{n}\sigma_i\right) = 2\left(4\sum_{i=k+1}^{n} i\sigma_i - 3\sum_{i=k+1}^{n}\sigma_i\right)\\ &= 2\sum_{i=k+1}^{n}(4i-3)\sigma_i. \quad \square\end{aligned}$$

**Remark 3.1.** Although the upper bound in Theorem 3.1 is slightly more complicated than (25) in the case of distinct Hankel singular values, it is nevertheless easily computable. Result (25) is also very important in its own right since the Hankel singular values are distinct for most practical systems.

**Remark 3.2.** It can be easily verified that the error bound provided in Corollary 3.1 for distinct $\sigma_i$ has a form which is also applicable to the non-distinct case (that is, by allowing the $\sigma_i$ in (25) to have multiplicity greater than unity), although the bound is not as tight as that of Theorem 3.1. We can see this from a single-step balanced truncation of $G(s)$ with the Hankel singular values given by

$$\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_{n-m} > \sigma_{n-m+1} = \cdots = \sigma_n > 0$$

where $1 \le m \le n-1$. In this case, Theorem 3.1 gives

$$\| H - \hat{H}_{n-m} \|_1 \le 2(4n - 3m)\sigma_n \tag{26}$$

while (25) in Corollary 3.1 gives (by applying the formula disregarding the distinctness condition on $\sigma_i$)

$$\| H - \hat{H}_{n-m} \|_1 \le 2\left(4\frac{(2n-m+1)m}{2} - 3m\right)\sigma_n = 2(4mn - 2m^2 - m)\sigma_n. \tag{27}$$

Comparing the right-hand-side of (26) and (27), it is easy to see that $4mn - 2m^2 - m \ge 4n - 3m$ for $1 \le m \le 2n$. The same argument can be applied to further truncation steps. Thus, the upper bound formula in Corollary 3.1 is always applicable. However, it may be much less tight than that of Theorem 3.1 if $m$ is close to $n$ with $n$ large.

**Remark 3.3.** For a single-step balanced truncation of $G(s)$ with unit multiplicity on $\sigma_n$, Corollary 3.1 gives $\| H - \hat{H}_{n-1} \|_1 \le 2(4n-3)\sigma_n$. In arriving at this result, we have employed inequalities due to Lemma 2.1, Lemma 2.3, Lemma 3.1 and the triangle inequality $\| G - \hat{G}_{n-1} \|_N \le \| G - \tilde{G}_{n-1} \|_N + \| \tilde{G}_{n-1} - \hat{G}_{n-1} \|_N$. Thus, because of the number of inequalities and the fact that they involve $L_1$ bounds (which will often be conservative), the given bound will often be conservative.

## 4. Numerical examples

In this section, we shall illustrate the results in Section 3 by means of examples. Comparisons will be made with Glover's bound in Lemma 2.2.

**Example 1.** Consider the following balanced realization of a 6-th order SISO system $G(s)$ with impulse response $H(t)$:

$$G(s) = \left(\begin{array}{cccccc|c} -1.3816 & 0.6754 & -0.4485 & 0.0676 & 0.0494 & -0.0071 & 14.3292 \\ 0.6754 & -4.8209 & 4.8936 & -0.9776 & -0.6907 & 0.1000 & -3.5645 \\ -0.4485 & 4.8936 & -7.0946 & 2.3789 & 1.5409 & -0.2244 & 2.3381 \\ -0.0676 & 0.9776 & -2.3789 & -1.4901 & -2.3552 & 0.3127 & 0.3501 \\ 0.0494 & -0.6907 & 1.5409 & 2.3552 & -10.6043 & 3.3202 & -0.2559 \\ 0.0071 & -0.1000 & 0.2244 & 0.3127 & -3.3202 & -2.8409 & -0.0370 \\ \hline -14.3292 & 3.5645 & -2.3381 & 0.3501 & 0.2559 & -0.0370 & 0 \end{array}\right),$$

where

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) := C(sI - A)^{-1}B + D.$$

The poles of $G(s)$ are at $-11.8979$, $-5.6044 \pm 1.8874j$, $-0.9996$, $-1.8731$, $-2.2530$ and the Hankel singular values are

$$\sigma_1 = 74.309, \quad \sigma_2 = 1.3178, \quad \sigma_3 = 0.38526,$$
$$\sigma_4 = 0.041138, \quad \sigma_5 = 0.0030886, \quad \sigma_6 = 0.00024029.$$

In this example, we have $\| H \|_1 \approx 151.95$. The $k$-th order balanced truncation $\hat{G}_k(s)$ of $G(s)$ with impulse response $\hat{H}_k(t)$ has error properties summarized in Table 1.

In Table 1, $\sigma_{k+1}$ gives a lower bound for the achieved error $\| H - \hat{H}_k \|_1$. The best upper bound is given by $2 \| G - \hat{G}_k \|_N$. However, such bound cannot be calculated *a priori*. The *a priori* bound given by Glover is consistently higher than our bound using Corollary 3.1 in this example. The upper bound in Corollary 3.1 is around $3 \sim 5$ times higher than the achieved error for all cases. In fact, if we are allowed to have an *a posterori* estimation of the error, the upper bound $2 \| G - \hat{G}_k \|_N$ should be used and it

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ummary of $L_1$ errors and their bounds

| $\sigma_{k+1}$ | $\Vert H-\hat{H}_k \Vert_1$ | $2\Vert G-\hat{G}_k \Vert_N$ | Glover's [a] | Ours [b] |
|---|---|---|---|---|
| 1.32 | 6.81 | 11.7 | 23.3 | 21.3 |
| 0.385 | 2.08 | 4.68 | 10.6 | 8.12 |
| 0.0411 | 0.232 | 0.677 | 1.60 | 1.185 |
| 0.00309 | 0.0292 | 0.0877 | 0.157 | 0.115 |
| 0.000240 | 0.00181 | 0.00659 | 0.0141 | 0.0101 |

.emma 2.2.
Corollary 3.1.

ıpears to be reasonably close to the achieved values. For most situations, this quantity is usually much ısier and cheaper to compute than the exact error.

**xample 2.** Consider the 2-input-3-output system $G(s)$ with a balanced realization given by

$$G(s)=\left(\begin{array}{cccc|ccc} -\frac{1}{6} & 0 & \frac{1}{8} & 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{6} & -\frac{5}{8} & 0 & 0 & 1 & 0 \\ -\frac{3}{8} & -\frac{1}{8} & -2 & -1 & 1 & 1 & \sqrt{2} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 \end{array}\right)$$

ith Hankel singular values $\sigma_1=\sigma_2=3$, $\sigma_3=\sigma_4=1$. The $L_1$ norm of the impulse response is $\Vert H \Vert_1 \approx$ .00. A single-step balanced truncation results in a second order system $\hat{G}_2(s)$ (with response $\hat{H}_2(t)$). For ıis system, we have

$$\sigma_3 = 1 \leq \Vert H-\hat{H}_2 \Vert_1 \approx 4.03 \leq 2\Vert G-\hat{G}_2 \Vert_N \approx 6.22.$$

Glover's upper bound in Lemma 2.2 gives 28 while our bound in (24) gives 20 which is smaller. Our new ıound is about 5 times higher than the achieved error. It is also important to point out that Glover's and ıur bound both exceed $\Vert H \Vert_1$ with only the *a posterori* bound $2\Vert G-\hat{G}_2 \Vert_N$ less than $\Vert H \Vert_1$.

The above two examples showed that (24) of Theorem 3.1 can provide a better estimate of the $L_1$ error than that of Lemma 2.2 due to Glover. However, we remark that there are cases where Glover's ɔound is tighter than ours. Since neither Glover's bound nor our bound gives a better result always, it seems prudent that the two bounds should be considered whenever an estimation of $L_1$ impulse response error is required.

## 5. Conclusion

We have derived an $L_1$ error bound in the impulse response domain for a balanced truncation approximation. The bound is *a priori* and is a function of the Hankel singular values of the full order system. Numerical results have shown that the bound obtained is reasonably good as an *a priori* estimate of the achieved error. Furthermore, it can provide a better estimate than the currently available error bound.

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