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Input Conditions for Continuous-Time Adaptive Systems Problems

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Abstract—**Persistence of excitation conditions are derived for the exponential convergence of continuous-time adaptive algorithms. The conditions are independent of the system output and include possibly unstable plants.**

I. Introduction

This note develops persistence of excitation (p.e.) conditions for the exponential convergence of continuous-time adaptive algorithms. Exponential convergence is important for robustness. Adaptive algorithms without such convergence can behave unacceptably in the presence of modeling inadequacies [1]–[5]. Conditions for convergence are usually framed [4]–[10] as spanning conditions on a regressor vector involving the output of the unknown system. In this note we translate these conditions into ones involving the system input only.

An unknown plant, possibly unstable, with a rational transfer function $G(s)$, and imbedded within a feedback loop is considered, with $u(t)$ the control input, $r(t)$ the input to the overall closed-loop system, and $y(t)$ the output. Section II derives conditions on $u(t)$ and $r(t)$ for satisfying the spanning condition when the feedback is time invariant but not necessarily stabilizing. Section III extends these results to stabilizing feedback. Of the two theorems in this section one assumes that the feedback is time invariant while the other allows for time variations. An application to the direct adaptive controller of [11] is given in [12], [13].

Since the completion of the first version [12] of this note, [14], [15] have also derived p.e. conditions using frequency domain ideas. They assume that the plant is minimum phase and asymptotically stable, while the results here are for unstable plants. Similar results for stable plants can be found in [17].

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A. Notations

Throughout this note, s will denote the differential operator and systems like $v(t) = s^i y(t)/(s+\alpha)^n$, unless otherwise mentioned, will be assumed at initial rest.

II. Persistence of Excitation Under Time-Invariant Feedback

The unknown plant $G(s)$ is given by $y(t) = G(s)u(t)$ with

$$G(s) = \frac{B(s)}{A(s)} = \frac{\sum_{j=0}^{m} b_j s^j}{s^n + \sum_{i=0}^{n-1} a_i s^i} \qquad n \ge m \tag{2.1}$$

where a_i, b_i are the unknown constants to be estimated. Defining $V(t)$ as the vector

$$V^T(t) = \left[\frac{y(t)}{(s+\alpha)^n}, \frac{sy(t)}{(s+\alpha)^n}, \cdots, \frac{s^{n-1}y(t)}{(s+\alpha)^n}, \frac{u(t)}{(s+\alpha)^n}, \cdots, \frac{s^m u(t)}{(s+\alpha)^n}\right], \qquad \alpha > 0,$$

a well-known identification scheme [6], [7] reduces to an error model of the form

$$\Delta\dot{\theta}(t) = -\left\{\int_0^t e^{-\gamma(t-\tau)} V(\tau)V^T(\tau)d\tau\right\}\Delta\theta(t). \tag{2.2}$$

Here γ is a positive scalar and $\Delta\theta(t)$ is the parameter estimate error vector. Before stating the conditions for the convergence of (2.2), we make the following definition.

Definition 2.1: A vector function $X(\cdot) = R^+ \to R_p$ is p.e. if there exist $\alpha_1 > 0$ and $\epsilon > 0$ such that

$$\int_\sigma^{\sigma+\epsilon} X(t)X^T(d)\,dt \ge \alpha_1 I \qquad \forall \sigma \ge 0. \tag{2.3}$$

▽▽▽

Then (2.2) is exponentially convergent if $V(t)$ is p.e.

The conditions on $r(t)$ and $u(t)$ involve vectors $W_N(t)$ and $R_N(t)$

$$W_N(t) \triangleq \left[u(t), \frac{u(t)}{s+\beta}, \cdots, \frac{u(t)}{(s+\beta)^{N-1}}\right]^T; \qquad \beta > 0$$

$$R_N(t) \triangleq \left[r(t), \frac{r(t)}{s+\beta}, \cdots, \frac{r(t)}{(s+\beta)^{N-1}}\right]^T; \qquad \beta > 0.$$

All results have the form W_N or R_N p.e. $\leftrightarrow$ V p.e. for a suitable N. Certain signals are required to lie in a set $\Omega_\Delta[0,\infty)$, defined as follows: $v(t)\epsilon\Omega_\Delta[0,\infty)$ if there exists a countable, possibly empty set $C_\Delta = \{t_i\}$, with $t_{i+1} - t_i \ge \Delta > 0$, such that v and $\dot{v}$ are continuous and bounded on $R_+ - C_\Delta$, having finite one-sided limits at each t_i. Notice if $W_N(t)$ is p.e. then so also is $W_l(t)\forall l \le N$.

Theorems 2.1 and 2.2, respectively, derive sufficient and necessary conditions on $u(t)$ so that $V(t)$ is p.e. They both require that $u(t)\epsilon\Omega_\Delta[0,\infty)$, and are useful when $G(s)$ operates in open loop. All proofs of this section are in Appendix A.

Theorem 2.1: For the system of (2.1) with $u(t)\epsilon\Omega_\Delta[0,\infty)$, suppose $A(s)$ and $B(s)$ are coprime. Then $W_N(t)$ p.e. implies $V(t)$ is p.e., with $N = m + n + 1$. ▽▽▽

This means that if $u(t)$ is a linear combination of sinusoids, it must have at least $(n+m+1)/2$ distinct frequency components. This is a sufficient condition. If $A(s)$ has zeros in the closed right-half plane, then these can contribute nondecaying excitation components to the output. Thus, for a necessary condition, the excitation condition may be relaxed as in Theorem 2.2.

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Theorem 2.2: Suppose the assumptions of Theorem 2.1 hold and $A(s)$ has μ zeros in the closed right-half plane. Then V p.e. implies $W_{n+m+1-\mu}$ is p.e. $\nabla\nabla\nabla$

The next theorem relates $r(t)$ to the p.e. conditions on $V(t)$. Consider a reconfigured closed loop, with $u(t) = (c(s)/d(s))r(t)$, and $c(s)/d(s)$ a possibly nonminimal transfer function. Then each imaginary zero of $c(s)$ could block one sinusoidal component and $r(t)$ must be accordingly enriched. Notice such cancellations do not arise in relating $W(t)$ to $V(t)$, as components of the latter involving $y(t)$ are essentially the states of the system and are unaffected by transfer function zeros.

Theorem 2.3: Suppose $r(t)\epsilon\Omega_\Delta[0,\infty)$, $A(s)$ and $B(s)$ are coprime and $c(s)$ has p imaginary zeros. Then $R_{n+m+1+p}(t)$ p.e. implies that $V(t)$ is p.e.

The precise form of transfer function used to construct a particular entry of V, W_N or R_N, e.g., $s^j(s+\alpha)^{-n}$ is not important. Only stability and linear independence over the reals of the first n and last $m+1$ transfer functions are required.

III. Persistence of Excitation with Stabilizing Feedback

In Section II no stability assumptions were made on the closed loop. In this section, the closed loop is assumed to be stable, although $G(s)$ can still be unstable. While Theorem 2.1 still holds, Theorem 2.2 needs adjustment. Closed-loop stability results in the suppression of the unstable modes. Thus, only imaginary axis poles of the plant can generate additional nondecaying excitations. We thus have Theorem 3.1, proved in Appendix B. The theorem holds regardless of whether the feedback is time varying or not.

Theorem 3.1: Suppose the closed loop is stable and all assumptions of Theorem 2.2 hold. Then Theorem 2.2 holds with p [imaginary zeros of $A(s)$] replacing μ. $\nabla\nabla\nabla$

Theorem 2.3 must also be adjusted. With the closed-loop reconfigured, as $y(t) = G(s)c(s)/d(s)r(t)$ all the imaginary poles of $G(s)$ must now appear in $c(s)$. Notice $c(s)/d(s)$ is nonminimal. The excitations cancelled by $G(s)$ reappear due to initial condition effects in the plant. We thus have Theorem 3.2, whose proof, being similar to those of Theorems 2.3 and 3.1, is omitted.

Theorem 3.2: Suppose all assumptions of Theorem 3.1 hold and the imaginary zeros of $C(s)$ are the same as those of $A(s)$. Then $R_{n+m+1}(t)$ p.e. implies that $V(t)$ is p.e. $\nabla\nabla\nabla$

IV. Conclusions

We have developed a general set of tools which help to establish p.e. conditions on system inputs. These techniques have been applied elsewhere [14] to establish the exponential convergence of the model reference adaptive control algorithm of [11] with known high frequency gain.

Appendix A

Proof of Theorems in Section II

The basic strategy in all proofs is the following. To show that Z_1 p.e. $\rightarrow Z_2$ p.e., we show that Z_2 not p.e. $\rightarrow$ for some nonzero vector θ, $|\theta^T Z_2|$ is small $\rightarrow |\bar{\theta}^T Z_1|$ is small $(\bar{\theta} \neq 0) \rightarrow Z_1$ not p.e. We need several lemmas which have the form $v(t)$ small $\rightarrow$ some functional of $v(t)$ is also small. The first of these lemmas has been obtained from [16].

Lemma A.1: If $f(\cdot)$ is an n times differentiable function on an interval I of length Δ and if $|f(x)| \leq M_0$ and $|f^{(n)}(x)| \leq M_n$, then for $x \in I$ and for $0 < k < n$,

$$|f^{(k)}(x)| \leq 4e^{2k}\{^nC_k\}^k M_0^{(1-k/n)} M_n'^{(k/n)} \tag{A.1}$$

where $M_n' = \max(M_n, M_0 n!\Delta^{-n})$ and $^nC_k = (n!)/(k!)(n-k)!$

Lemma A.2: For any asymptotically stable system with a proper transfer function $W(s)$, if the input $u(t)$ is such that

$$|u(t)| \leq M \qquad \text{on } [0, T]$$

$$|u(t)| \leq \epsilon \qquad \text{for all } t > T \tag{A.2}$$

and if the initial state lies in some fixed ball B of radius R, then there exists a $\delta_1(\epsilon, M, R)$, independent of T, such that for $t > \delta_1 + T$, the output $y(t)$ satisfies $|y(t)| < 0(\epsilon)$. We designate a function as being $0(\epsilon)$ if the function goes to zero as $\epsilon \rightarrow 0$.

Proof: For any minimal realization $\{F, G, H, J\}$ of $W(s)$, asymptotic stability and (A.2) imply the existence of a $K(M, R)$ such that $\|x(t)\| < K$, $\forall t \leq T$, with $x(t)$ the state vector. Then for $t > T$

$$|y(t)| \leq K\|e^{F(t-T)}\| \, \|H\| + 0(\epsilon).$$

If δ_1 is selected to make $K\|e^{F\delta_1}\| \, \|H\| < \epsilon$, it follows that for $t \geq \delta_1 + T$, $|y(t)| < 0(\epsilon)$.

Lemma A.3: If $u(t)\epsilon\Omega_\Delta[0,\infty)$, then under the assumption of arbitrary finite initial conditions, for any Hurwitz polynomial $D(s)$ and polynomials $N_1(s)$ and $N_2(s)$, such that $\delta[N_1(s)] \leq \delta[D(s)]$ and $\delta[N_2(s)] = 1 + \delta[D(s)]$ (δ refers to the degree), the following two properties hold:

i) $\{N_1(s)/D(s)\}u\epsilon\Omega_\Delta[0,\infty)$;

ii) $\{N_2(s)/D(s)\}u$ is continuous and bounded on $\{[0,\infty) - C_\Delta\}$, and has finite limits as $t \downarrow t_i$ and $t \uparrow t_i$, $t_i \epsilon C_\Delta$.

Proof: The proof follows similarly to that of Lemma A.2. $\nabla\nabla\nabla$

Now we come to a somewhat more technical result, but again one of the form "if something is small, then something else related is also small."

Lemma A.4: Consider the proper systems

$$v_2(t) = \left[\frac{\theta_2(s)}{A_2(s)} + \frac{\theta_1(s)}{A_2(s)}\frac{B(s)}{A(s)}\right]\frac{P(s)}{Q(s)}v_1(t) \tag{A.3a}$$

$$v_3(t) = \frac{Q(s)A(s)A_2(s)v_2(t)}{A_1(s)} \tag{A.3b}$$

with $v_1(t)\epsilon\Omega_\Delta(0,\infty)$, $|v_1(t)| < M$, and $A_1(s)$ and $A_2(s)$ Hurwitz. All initial conditions lie in some ball B of radius R. Then $v_3(t)$ is bounded. Further, given arbitrary $\epsilon > o$, if $\delta > \delta_1(\epsilon, M, R)$, where the procedure for computing δ_1 is given in the course of proving this lemma

$$\int_\sigma^{\sigma+\delta} |v_2(t)|\,dt < \epsilon \tag{A.4}$$

for any σ implies

$$\int_{\sigma+\delta_1(\epsilon,R,M)}^{\sigma+\delta} |v_3(t)|\,dt < 0(\epsilon). \tag{A.5}$$

Proof: The minimal transfer function from $v_1(t)$ to $v_3(t)$ has a Hurwitz denominator, so the zero-state component of $v_3(t)$ is bounded. Any unstable modes must be associated with unstable zeros of $A(s)$ in (A.3a). They will then appear in $v_2(t)$ but be blocked by the numerator zeros of the transfer function in (A.3b). Put another way, all unstable states in a realization of (A.3) obtained by cascading two minimal realizations are unobservable. Hence, the zero-input component of $v_3(t)$ and $v_3(t)$ are bounded.

Next suppose the transfer function from $v_2(t)$ to $v_3(t)$ has a minimal state-variable realization defined by a quadruple $\{F_3, G_3, H_3, J_3\}$, so that

$$v_3(t) = \int_\sigma^t [H_3 \exp F_3(t-\tau)G_3 + J_3\delta(t-\tau)]v_2\,d\tau + H_3 \exp F_3(t-\sigma)x_3(\sigma). \tag{A.6}$$

Call the first summand on the right $v_4(t)$. Under (A.4), it follows that

$$\int_\sigma^{\sigma+\delta} |v_4(t)|\,dt < \epsilon.$$

Because $v_3(\cdot)$ is bounded, it follows that

$$\left|\int_\sigma^{\sigma+\delta} H_3 \exp F_3(t-\sigma)\,dt\,x_3(\sigma)\right|$$

is bounded. $x(\sigma)$ is bounded due to observability of $[F_3, H_3]$ (else a contradiction can be established). This bound depends on R, M. Now the stability of F_3 implies that there exists a $\delta_1 = \delta_1(\epsilon, M, R)$ such that for all $v_1(\cdot)$, initial states at $t = 0$, and $v_3(\cdot)$ satisfying (A.4)

$$\|H_3 \exp F_3(t-\sigma)x_3(\sigma)\| = 0(\epsilon) \qquad t \geq \delta_1. \tag{A.7}$$

Use of (A.4) again in (A.6) shows that (A.5) holds.

Proof of Theorem 2.1: Suppose $v(t)$ is not p.e. Then by definition of p.e. [see (2.3)], for arbitrary $\epsilon > 0$ and $\delta > 0$, there exist σ and $\xi = [\gamma_0, \cdots, \gamma_{n-1}, \theta_0, \cdots, \theta_m]^T$, of unit magnitude, such that

$$\int_\sigma^{\sigma+\delta} \left|\xi^T V(t)\right|^2 dt < \epsilon^4 \tag{A.8}$$

$$\int_\sigma^{\sigma+\delta} \left|\xi^T V(t)\right| dt \leq \delta^{1/2} \left(\int_\sigma^{\sigma+\delta} \left|\xi^T V(t)\right|^2 dt\right)^{1/2} \leq \delta^{1/2}\epsilon^2. \tag{A.9}$$

Below, we shall require that $\delta \geq \delta_i$, $(i = 1, 2)$, δ_1 is a constant associated with the decay of initial conditions and ϵ, and δ_2 depends on ϵ, n, m, and ultimately on the bounds of $u(t)$. Assume $\delta - \delta_1 > \Delta$. Now, from (A.9)

$$\int_\sigma^{\sigma+\delta} \left| \frac{\sum_{i=0}^{n-1} \gamma_i s^i y(t)}{(s+\alpha)^n} + \frac{\sum_{j=0}^{m} \theta_j s^j u(t)}{(s+\alpha)^n} \right| dt \leq \delta^{1/2}\epsilon^2. \tag{A.10}$$

Let the signal within the modulus sign be $v_2(t)$. Defined $v_3(t)$ as

$$v_3(t) = \frac{(s+\alpha)^n}{(s+\beta)^{2n}} A(s) v_2(t). \tag{A.11}$$

Then noting that $y(t) = [B(s)/A(s)]u(t)$, with $u(t)$ as $v_1(t)$, $P(s) = Q(s) = 1$ the various conditions set out in Lemma A.4 are accordingly satisfied with obvious identifications of $\theta_i(s)$ and $A_i(s)$. Hence, provided $\delta > \delta_1$,

$$\int_{\sigma+\delta_1}^{\sigma+\delta} |v_3(t)|\, dt < 0(\delta^{1/2}\epsilon^2) = g(\epsilon). \tag{A.12}$$

Rewriting $v_3(t)$ as

$$v_3(t) = \frac{\left(\sum_{i=0}^{n-1} \gamma_i s^i\right) B(s) + \left(\sum_{j=0}^{m} \theta_j s^j\right) A(s)}{(s+\beta)^{2n}} u(t) \tag{A.13}$$

by Lemma A.3, $v_3(t) \in \Omega_\Delta(0, \infty)$, $|v_3(t)| < C$ on $[\sigma + \delta_1, \sigma + \delta] - C_\Delta$ for some positive constant C. If ϵ^2 is chosen to be small enough to ensure that $M_2' = \max(C, g(\epsilon)2!\Delta^{-2}) = C$ holds, Lemma A.1 shows that on $[\sigma + \delta_1, \sigma + \delta] - C_\Delta$

$$|v_3(t)| \leq 0(\delta^{1/2}\epsilon^2)^{1/2} = 0(\delta^{1/4}\epsilon). \tag{A.14}$$

Since $v_3(t)$ has one-sided limits everywhere, we have that

$$|v_3(t)| < 0(\delta^{1/4}\epsilon) \text{ on } [\sigma + \delta_1, \sigma + \delta]. \tag{A.15}$$

Thus,

$$\left| \frac{\left(\sum_{i=1}^{n-1} \gamma_i s^i\right) B(s) + \left(\sum_{j=0}^{m} \theta_j s^j\right) A(s)}{(s+\beta)^{2n}} u(t) \right| < 0(\delta^{1/4}\epsilon). \tag{A.16}$$

From Lemma A.3, the first $n + m + 1$ derivatives of $v_3(t)$ are bounded in magnitude by some constant K_1; K_1 independent of ξ as $\|\xi\| = 1$. Thus, from Lemma A.1 and the one-sided limits of u and $\dot{u}$ for small enough ϵ, $|(s+\beta)^{n-m} v_3(t)|$

$< 0((\delta^{1/4}\epsilon)^{1/(n-m+1)})$, whence on $[\sigma + \delta_1, \sigma + \delta]$

$$\left| \frac{\left(\sum_{i=0}^{n-1} \gamma_i s^i\right) B(s) + \left(\sum_{j=0}^{m} \theta_j s^j\right) A(s)}{(s+\beta)^{n+m}} u(t) \right| < 0((\delta^{1/4}\epsilon)^{1/(n-m+1)}). \tag{A.17}$$

Defining $\bar{R}$ as

$$\bar{R} = \begin{bmatrix} R & 0 \\ a^T & 1 \end{bmatrix}$$

where $a^T = [0 \cdots 0\, a_0 \quad a_1 \cdots a_{n-1}] \in R^{n+m}$ and R is the $(n+m) \times (n+m)$ resultant matrix corresponding to the polynomials $B(s)$ and $A(s)$, we find that the numerator on the left-hand side of (A.17) can be expressed as $\xi^T \bar{R}[1 \quad s \cdots s^{n+m}]^T$. Define $\bar{\xi} = \bar{R}^T \xi$. The coprimeness of $A(s)$ and $B(s)$ assures nonsingularity of R, and hence of $\bar{R}$. Thus, as $\|\xi\| = 1$ by hypothesis $\|\bar{\xi}\| > K_3 > 0$, where K_3 depends on the a_i and b_i only. Hence, (A.17) implies after minor manipulations that

$$\int_{\sigma+\delta_1}^{\sigma+\delta} W_{n+m+1}(\tau) W_{n+m+1}^T(\tau) d\tau$$

cannot be uniformly positive definite, since ϵ is arbitrary, whence $W_{n+m+1}(t)$ is not p.e. Thus, the result follows. ∇∇∇

Theorem 2.2 needs the following additional lemma.

Lemma A.5: Consider the following proper systems with arbitrary, but finite, initial conditions

$$\nu_2(t) = W(s)\nu_1(t) \tag{A.18}$$

$$\nu_3(t) = W(s)A_+(s)\frac{1}{A_+(s)A_-(s)}\nu_1(t). \tag{A.19}$$

Here $\nu_1(t) \epsilon \Omega_\Delta[0, \infty]$, $W(s)$ is asymptotically stable, $A_+(s)$ has roots in the closed right-half plane, and $A_-(s)$ is Hurwitz. Suppose for some δ and $\sigma > 0$, there exists ϵ such that

$$|\nu_2(t)| \leq \epsilon \qquad \forall t \epsilon]\sigma, \sigma + \delta]. \tag{A.20}$$

Then, for large enough δ, there exists $\delta_1 < \delta$ such that

$$|\nu_3(t)| \leq 0(\epsilon) \qquad \forall t \epsilon [\sigma + \delta_1, \sigma + \delta].$$

Proof: Define

$$\nu_4(t) = A_+(s)\frac{1}{A_+(s)A_-(s)}\nu_1(t). \tag{A.21}$$

Then, a similar argument to the one used in Lemma A.4 shows that $\nu_4(t)$ is bounded and all nonasymptotically stable modes of this system are unobservable. Hence, by [18] there exists the following realization of (A.21) with A_1 having eigenvalues in the open left-half plane.

$$\dot{x}(t) \triangleq \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \nu_1(t)$$

$$\nu_4(t) = [C_1^T \quad 0] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

Now, $x_1(\sigma)$ is bounded as

$$x_1(\sigma) = e^{A_1\sigma}x_1(0) + \int_0^\sigma e^{A_1(\sigma-\tau)} b_1 \nu_1(\tau) d\tau.$$

Thus, from (A.19) and the fact that $t \geq \sigma$

$$v_3(t) = W(s)\left\{C_1 e^{A_1(t-\sigma)}x_1(\sigma) + \int_\sigma^t C_1(\tau)e^{A_1(t-\tau)}b_1 v_1(\tau)d\tau\right\}$$

the result follows. Here δ_1 is chosen to make initial conditions, observable in $v_3(t)$, decay to $0(\epsilon)$ as $t = \sigma + \delta_1$.

Proof of Theorem 2.2: We shall sketch the outline of the proof only, as it is similar to that of Theorem 2.1. Suppose $W_{n-m+1-\mu}$ is not p.e. As $W_{n+m+1-\mu}(\cdot) \in \Omega_\Delta[0, \infty)$, by Lemma A.1, we have for arbitrary $\epsilon > 0$ a σ, a suitably large δ, and a $\theta \triangleq [\theta_0, \cdots, \theta_{n+m-\mu}]^T$ of unit magnitude such that

$$\left|\sum_{i=0}^{n+m-\mu} \frac{\theta_i (s+\beta)^{n+m-\mu-i}}{(s+\beta)^{n+m-\mu}} u(t)\right| < 0(\epsilon) \text{ on } [\sigma, \sigma+\delta]. \quad \text{(A.22)}$$

Factorize $A(s)$ as $A(s) = A_-(s)A_+(s)$, where $A_-(s)$ has all and only the asymptotically stable zeros of $A(s)$. Thus, from Lemma A.5, identifying $W(s)$ as the operator in (A.22), we have for some δ_1,

$$\left|\frac{\left(\sum_{i=0}^{n+m-\mu} \theta_i (s+\beta)^{n+m-\mu-i}\right) A_+(s)u(t)}{(s+\beta)^{n+m-\mu}A(s)}\right| < 0(\epsilon) \quad \text{on } [\sigma+\delta_1, \sigma+\delta].$$

By hypothesis, $\delta[A_+(s)] = \mu$, whence the numerator in the above operator has degree $n + m$. As $A(s)$ and $B(s)$ are coprime, there exists $\gamma = [\gamma_0, \cdots, \gamma_{n+m}]^T$, $\|\gamma\|$ bounded away from zero, such that

$$\sum_{i=0}^{n+m-\mu} (s+\beta)^{n+m-\mu-i} A_+(s)\theta_i = \sum_{i=0}^{n-1} \gamma_i s^i B(s) + \sum_{j=0}^{m} \gamma_{n+j} s^j A(s)$$

and then for arbitrary finite initial conditions

$$\left|\frac{\left(\sum_{i=0}^{n-1}\gamma_i s^i\right) B(s)}{(s+\beta)^{n+m-\mu}A(s)} u(t) + \frac{\left(\sum_{j=0}^{m}\gamma_{n+j}s^j\right)}{(s+\beta)^{n+m-\mu}} u(t)\right| < 0(\epsilon).$$

Then from (2.1), after minor manipulations the result follows.

∇∇∇

Theorem 2.3 requires the following lemma.

Lemma A.6: Consider the systems

$$v_2(t) = \frac{\bar{C}(s)}{A_1(s)} v_1(t) \qquad v_3(t) = \frac{1}{A_1(s)} v_1(t)$$

where $A_1(s)$ and $\bar{C}(-s)$ are Hurwitz and $v_1(t) \in \Omega_\Delta(0, \infty)$. Suppose for some δ, $\sigma > 0$

$$|v_2(t)| \leq \epsilon \qquad \forall t\epsilon[\sigma, \sigma+\delta].$$

Then for large enough δ, there exists $\delta_1 < \delta$ for which

$$|v_3(t)| \leq \epsilon \qquad \forall t\epsilon[\sigma, \sigma+\delta-\delta_1].$$

Proof: Clearly, $v_i(t)$ are bounded and

$$\bar{C}(s)v_3(t) = v_2(t). \quad \text{(A.23)}$$

Consider a minimal realization $\{F, g, h\}$ of (A.23). Then for $t \leq \sigma + \delta$

$$|v_3(t)| \leq \|h\| \, \|e^{-F(t-\sigma-\delta)}\| \, \|x(\sigma+\delta)\| + \left\|\int_{\sigma+\delta}^{t} he^{F(t-\tau)}gv_2(\tau)d\tau\right\|.$$

As before the boundedness of $v_3(t)$ and the observability of $[F, h]$ imply that $\|x(\sigma+\delta)\|$ is bounded. Then noting that $-F$ has eigenvalues in the open left-half plane one can find δ_1 such that $\|e^{-F\delta_1}\| = \epsilon$, whence the result follows.

Proof of Theorem 2.3: Again an outline is given. If V is p.e., then as in Theorem 2.1 for some $\|\xi\| = 1$, σ and any δ

$$\int_\sigma^{\sigma+\delta}\left|\left(\frac{\sum_{i=0}^{n-1} y_i s^i}{(s+\alpha)^n}\frac{B(s)}{A(s)} + \frac{\sum_{j=0}^{m}\theta_j s^j}{(s+\alpha)^n}\right)\frac{c(s)}{d(s)}r(t)\right| dt \leq 0(\epsilon)^2. \quad \text{(A.24)}$$

Then applying Lemma A.5 with $d(s)(s+\alpha)^n A(s)/(s+\beta)^{2n+\nu}$ (where $\nu = \delta[d(s)]$) as the last operator block in the lemma, and using the coprimeness of $A(s)$ and $B(s)$, we have a nonzero vector Ψ and $\delta_2 > 0$

$$\int_\sigma^{\sigma+\delta}\left|\frac{\sum_{i=0}^{n+m}\psi_i s^i}{(s+\beta)^{\mu+2n}} c(s)r(t)\right| dt < 0(\epsilon)^2. \quad \text{(A.25)}$$

Since the signal inside the modulus sign is in $\Omega_\Delta[0, \infty)$, Lemma A.1 gives for $t\epsilon[\sigma+\delta_2, \sigma+\delta]$

$$\left|\frac{\left(\sum_{i=0}^{n+m}\psi_i s^i\right) c(s)}{(s+\beta)^{\mu+2n}} r(t)\right| < 0(\epsilon). \quad \text{(A.26)}$$

Factorize $c(s) = \bar{c}(s)c_0(s)$, where $c_0(s)$ has all, and only, the imaginary zeros of $c(s)$. Then applying Lemmas A.1, A.2, and A.6 we have, for some δ_3, δ_4 and $\forall t\epsilon[\sigma+\delta_3, \sigma+\delta-\delta_4]$

$$\left|\frac{\left(\sum_{i=0}^{n+m}\psi_i s^i\right) c_0(s)}{(s+\beta)^{\mu+2n}} r(t)\right| < 0(\epsilon). \quad \text{(A.27)}$$

Note $\delta[c_0(s)] = p$. Thus, with some manipulation the result follows.

Appendix B

Theorem 3.1 can be proved in the same way as Theorem 2.2 by replacing μ by p and Lemma A.1 by Lemma B.1, below.

Lemma B.1: Consider the following proper systems, having arbitrary but finite initial conditions.

$$v(t) = W(s)u(t) \quad \text{(B.1)}$$

$$y(t) = W(s)A_0(s)\frac{1}{\bar{A}(s)A_0(s)}u(t) \quad \text{(B.2)}$$

where $A_0(s)$ has only imaginary zeros and $\bar{A}(s)$ has none; $u(t) \in \Omega_\Delta[0, \infty)$, $W(s)$ is asymptotically stable and proper and all signals are bounded. Suppose for some δ and $\sigma > 0$ there exists ϵ such that

$$|v(t)| \leq \epsilon \, \forall t\epsilon[\sigma, \sigma+\delta]. \quad \text{(B.3)}$$

Then for large enough δ, δ_1, $\delta_2 > 0$, and $\delta_1 + \delta_2 < \delta$ such that

$$|y(t)| \leq 0(\epsilon) \qquad \forall t\epsilon[\sigma+\delta_1, \sigma+\delta-\delta_2].$$

Proof: The proof is in the same vein as that of Lemma A.6. Since the imaginary axis poles are unobservable, the following representation of (B.2) with

$$F = \begin{bmatrix} F_{11} & 0 & 0 \\ 0 & F_{22} & 0 \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \quad C = \begin{bmatrix} C_1 \\ C_2 \\ 0 \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

is possible. Here F_{11} and F_{22} have all the asymptotically stable and unstable poles of the system, respectively, and Re $\lambda_i(F_{33}) = 0$. The zero input response of $y(t)$ now has only the unstable and asymptotically stable components. The latter decay to $0(\epsilon)$ by $\sigma + \delta_1$. Since all signals are bounded, as in the proof of Lemma A.6 the former must be $0(\epsilon)$ on the interval $[\sigma, \sigma + \delta - \delta_2]$. The rest of the proof is straightforward.

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An Indirect Stochastic Adaptive Scheme with On-Line Choice of Weighting Polynomials

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Abstract—**This note presents an indirect adaptive control algorithm combining a quadratic cost measure of the Clarke-Gawthrop type and the classical control strategy of pole placement. It updates weighting polynomials of cost function on line in such a way that the closed-loop poles are located at prespecified positions. The convergence and stability of the adaptive control algorithm is also given without assuming that the system is minimum phase.**

I. Introduction

In recent years a number of authors have focused their attention on the problem of controlling systems having constant but unknown parameters. Complete stability and convergence results have been derived for both the direct [1], [2] and indirect adaptive algorithms [3], [4] based on minimum variance control strategy. However, the above algorithms cannot control nonminimum phase systems. In order to overcome this deficiency, Clarke and Gawthrop developed the self-tuning controller based on the generalized minimum variance control law [5]. Recently, global convergence of such self-tuning controllers has been established for both deterministic [6] and stochastic [7] systems.

The assumption made in [6] and [7] is that the off-line choice of the weighting polynomials P and Q is such that the polynomial $T(z^{-1}) = P(z^{-1})B(z^{-1}) + Q(z^{-1})A(z^{-1})$ is stable. The self-tuning controller [5] uses a "cut and try" procedure for off-line choice of P and Q. For unknown systems, it is difficult to choose weighting polynomials. The technique of choosing P and Q on-line was presented in [8], but the on-line choice of weighting polynomials affects the control-law estimation, so that convergence problems have arisen [8]. A generalized self-tuning controller was presented to avoid the above problems in [9]. However, the convergence analysis of the above self-tuning control algorithms has not been established.

Because the indirect scheme based on the generalized minimum-variance control law not only leads to a reduction in the number of parameters to be estimated, especially for systems with large time delay [3], but also makes the control-law estimation independent of updating the weighting polynomials so that the good convergence properties are obtained, a new indirect stochastic adaptive control algorithm is presented in this note. A convergence and stability proof for this algorithm is also given under certain conditions.

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II. Derivation of the Control Law

Consider a SISO linear time-invariant system described by

$$A(z^{-1})y(t) = B(z^{-1})u(t-k) + C(z^{-1})\xi(t) \tag{2.1}$$

where $\{y(t)\}$, $\{u(t)\}$, and $\{\xi(t)\}$ denote the output, input, and disturbance sequences, respectively. $A(z^{-1})$, $B(z^{-1})$, and $C(z^{-1})$ are polynomials in the unit delay operator z^{-1}. The leading coefficient b_0 of $B(z^{-1})$ is assumed to be different from zero; k is the time delay. The scalar sequence $\xi(t)$ is a real stochastic process defined on a probability space (Ω, A, P) on which we denote the sequence of increasing sigma-algebras $(F_t, t \in N)$ where F_t is generated by the observations up to and including time t. The sequence $\{\xi(t)\}$ is assumed to be white Gaussian noise; thus,

$$E(\xi(t)/F_{t-1}) = 0 \quad \text{a.s.} \tag{2.2}$$

$$E(\xi(t)^2/F_{t-1}) = \sigma^2 \quad \text{a.s.} \tag{2.3}$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \xi(t)^2 < \infty \quad \text{a.s.} \tag{2.4}$$

We make the following assumptions:

A1) k is known;

A2) upper bounds n_a, n_b, and n_c of the degrees of A, B, and C are known;

A3) $C(z^{-1})$ is a stable polynomial;

A4) $A(z^{-1})$ and $B(z^{-1})$ are relatively prime (but having unknown coefficients).

The control objective is to find a control law to minimize the variance of the following generalized tracking error:

$$e(t+k) = P(z^{-1})y(t+k) - R(z^{-1})w(t) + Q(z^{-1})u(t) \tag{2.5}$$

where P, R, and Q are weighting polynomials in z^{-1} such that $P_0 \neq 0$, and $w(t)$ is the known reference signal. The cost function is of the form

$$J = E(e(t+k)^2/F_t). \tag{2.6}$$

Define an auxiliary output $\phi(t+k)$ as

$$\phi(t+k) = P(z^{-1})y(t+k). \tag{2.7}$$