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Design problems for sensitivity and complementary sensitivity

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Abstract: Two design problems involving sensitivity and complementary sensitivity are examined in this paper. First, we characterize all pairs of constraints on the sensitivity and the complementary sensitivity which can be simultaneously satisfied in a closed-loop system for a given scalar plant in the context of time-invariant control. Second, we present a simple one-to-one correspondence between solutions of the sensitivity problem and of the complementary sensitivity problem for a multivariable plant.

Keywords: Sensitivity; complementary sensitivity; internal stability; compensators; interpolation.

1. Introduction

It is well known that in the feedback system of Figure 1, sensitivity (the H^∞-norm of the sensitivity function, i.e. the transfer function from the disturbance d to the output y) reflects the measure of output disturbance rejection and tracking, and sensitivity to small additive parameter variations. In parallel to sensitivity, complementary sensitivity (the H^∞-norm of the complementary sensitivity function, i.e. the transfer function from the sensor noise n to the output y) reflects the capacity to suppress sensor noise and is also used as a measure of stability margin. Their physical significance suggests the need to achieve as small a sensitivity and complementary sensitivity as possible in practice. The standard problem of H^∞-norm optimization including the problem of (complementary) sensitivity minimization as a special case has been fully studied in recent years, see e.g. [3] and references therein. However, it should be noted that the standard problem fails to embrace the simultaneous design problem for sensitivity and complementary sensitivity to be considered in this paper.

Intuitively, H^∞-optimization of the sensitivity function may not imply that of the complementary sensitivity function since the sum of these two functions is identical to 1 no matter how a compensator may be designed. Moreover, there are often simultaneous requirements for sensitivity and for complementary sensitivity in practice. This motivates us to consider the following problem: Given a pair of requirements for sensitivity and for complementary sensitivity respectively, does there exist a single internally stabilizing controller which achieves the two requirements simultaneously? Clearly, the solvabil-

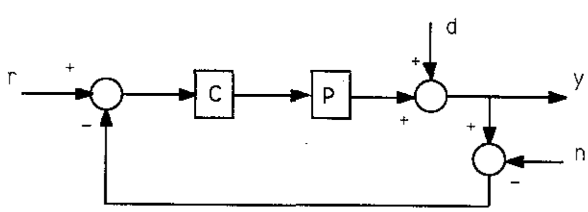

Fig. 1.

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ity of such a problem depends on the nature of the plant and the given pair of requirements, and the problem may not be solvable even if each requirement can be achieved individually. The main aim of the paper is to characterize all the constraints on sensitivity and complementary sensitivity for which the above problem is solvable. Another issue this paper addresses is to establish connections between the sensitivity problem and the complementary sensitivity problem so that the problem of one kind can be reduced to that of the other kind.

Consider a proper scalar linear time-invariant (LTI) continuous-time plant $P(s)$ and a proper real-rational compensator $C(s)$ which internally stabilizes $P(s)$. As usual, the associated sensitivity function and the associated complementary sensitivity function are respectively defined as

$$S(s) \triangleq [1+P(s)C(s)]^{-1} \quad \text{and} \quad T(s) \triangleq P(s)C(s)[1+P(s)C(s)]^{-1}. \tag{1.1}$$

For simplicity, it is assumed that $P(s)$ only has simple poles in $\tilde{H}$ – the closed right half plane (RHP) including the point at infinity, and simple zeros in $\overline{H}$. Suppose the set of its closed RHP poles and the set of its closed RHP zeros are respectively given by

$$\mathbf{P} = \{p_1, \ldots, p_n\} \quad \text{and} \quad \mathbf{Z} = \{z_1, \ldots, z_m\}. \tag{1.2}$$

If both $\mathbf{P}$ and $\mathbf{Z}$ are nonempty, we define

$$\delta \triangleq \sup\Big\{\gamma > 0; \text{ there exists an analytic function } F(s)\colon \overline{H} \to \mathscr{D} \text{ satisfying } F(p_i) = 0 \ (i = 1, \ldots, n) \text{ and } F(z_j) = \gamma \ (j = 1, \ldots, m)\Big\}, \tag{1.3}$$

where $\mathscr{D}$ denotes the open unit disk.

The quantity δ plays an important role in many robust control problems for scalar LTI systems such as the gain margin problem, the phase margin problem, the sensitivity problem, etc. For more details of its relevance and its calculation, we refer the reader to references [5] and [8].

The combined sensitivity and complementary sensitivity (CSCS) problem is that of determining the existence of and then finding a proper stabilizing compensator $C(s)$ such that the following constraints are satisfied simultaneously:

$$\| S(s) \| < r_1 \quad \text{and} \quad \| T(s) \| < r_2, \tag{1.4}$$

where r_1, r_2 are two positive constants given in advance and $\| \cdot \|$ denotes the H^∞-norm of $(\cdot)$.

Note that the (complementary) sensitivity problem is in fact a particular form of the CSCS problem corresponding to $r_2 = \infty$ ($r_1 = \infty$), which has been considered in [5] (respectively [9]). The following result is needed in understanding what follows.

Lemma 1.1. *For the given plant $P(s)$, let $\mathscr{S}$ and $\mathscr{T}$ denote the minimal sensitivity and the minimal complementary sensitivity, respectively. Then the following hold.*

(i) *If $\mathbf{P} \neq \emptyset$ and $\mathbf{Z} \neq \emptyset$, then $\mathscr{S} = \mathscr{T} = 1/\delta$, where δ is defined as* (1.3);
(ii) *if $\mathbf{P} = \mathbf{Z} = \emptyset$, then $\mathscr{S} = \mathscr{T} = 0$;*
(iii) *if $\mathbf{P} \neq \emptyset$ and $\mathbf{Z} = \emptyset$, then $\mathscr{S} = 0$ and $\mathscr{T} = 1$;*
(iv) *if $\mathbf{P} = \emptyset$ and $\mathbf{Z} \neq \emptyset$, then $\mathscr{S} = 1$ and $\mathscr{T} = 0$.*

Remark 1.1. The above result implies that not for all plants, the minimal sensitivity and the minimal complementary sensitivity are equal. This shows that Remark 3.1 in [6] is incomplete.

2. Solvability of the CSCS problem

The purpose of this section is to discuss the solvability of the CSCS problem for a given SISO LTI plant $P(s)$ for r_1, $r_2 > 0$. For this purpose, three important observations can be made immediately. First, the

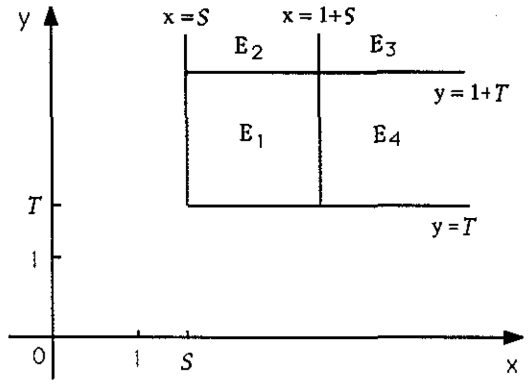

Fig. 2.

CSCS problem is solvable only if $r_1 > \mathscr{S}$ and $r_2 > \mathscr{T}$ since $\mathscr{S}$ and $\mathscr{T}$ are defined to be the minimal achievable sensitivity and the minimal achievable complementary sensitivity, respectively. Second, the CSCS problem is solvable for $r_1 > \mathscr{S}$ and $r_2 \geq \mathscr{S} + 1$, and for $r_1 \geq \mathscr{T} + 1$ and $r_2 > \mathscr{T}$. Third, the CSCS problem is solvable for all $r_1 \geq x$ and $r_2 \geq y$ if it is solvable for $r_1 = x$ and $r_2 = y$. To sum up, the CSCS problem is always solvable for the point (r_1, r_2) in the domain $E_2 \cup E_3 \cup E_4$ and is unsolvable for (r_1, r_2) not in the domain $\cup_{i=1}^{4} E_i$, where E_i, $i = 1, \ldots, 4$, are shown in Figure 2 and defined as

$$E_1 \triangleq \{(x, y) \colon \mathscr{S} < x < 1 + \mathscr{S} \text{ and } \mathscr{T} < y < 1 + \mathscr{T}\}, \tag{2.1}$$

$$E_2 \triangleq \{(x, y) \colon \mathscr{S} < x < 1 + \mathscr{S} \text{ and } y \geq 1 + \mathscr{S}\}, \tag{2.2}$$

$$E_3 \triangleq \{(x, y) \colon x \geq 1 + \mathscr{S} \text{ and } y \geq 1 + \mathscr{T}\}, \tag{2.3}$$

$$E_4 \triangleq \{(x, y) \colon x \geq 1 + \mathscr{T} \text{ and } \mathscr{T} < y < 1 + \mathscr{T}\}. \tag{2.4}$$

Thus, the only unclear domain for the solvability of the CSCS problem is E_1, in which (r_1, r_2) is obviously desired to be.

It is a well-known result [10] that a proper compensator internally stabilizes the plant $P(s)$ if and only if the associated sensitivity function is analytic in $\overline{H}$ and includes the plant unstable poles $\{p_1, \ldots, p_n\}$ as its zeros, and the associated complementary sensitivity function includes the plant closed RHP zeros $\{z_1, \ldots, z_m\}$ as its zeros. Using this fact, one can easily establish that the CSCS problem is equivalent to finding a real rational analytic function

$$S(s) \colon \overline{H} \to G_0 \triangleq \{s \in \mathbb{C} \colon |s| < r_1 \text{ and } |s - 1| < r_2\} \tag{2.5}$$

satisfying the interpolation conditions

$$S(p_i) = 0, \quad i = 1, \ldots, n, \tag{2.6}$$

$$S(z_i) = 1, \quad i = 1, \ldots, m. \tag{2.7}$$

Let us first deal with trivial cases. To do this, note that the function $S(s) \equiv 0$ is a mapping from $\overline{H}$ to G_0 and satisfies (2.6) for all pairs (r_1, r_2) with $r_1 > 0$, $r_2 > 1$ and the function $S(s) \equiv 1$ is a mapping from $\overline{H}$ to G_0 and satisfies (2.7) for all pairs (r_1, r_2) with $r_1 > 1$, $r_2 > 0$. Also, there always exists a real positive number a such that $S(s) \equiv a$ is a mapping from $\overline{H}$ to G_0 for all pairs (r_1, r_2) with $r_1 + r_2 > 1$. In this way, one can readily conclude the following result.

Theorem 2.1. *For the given plant $P(s)$, there hold*
(i) *if $\mathbf{P} \neq \emptyset$ and $\mathbf{Z} = \emptyset$, the CSCS problem is solvable for (r_1, r_2) iff $r_1 > 0$ and $r_2 > 1$;*
(ii) *if $\mathbf{P} = \emptyset$ and $\mathbf{Z} \neq \emptyset$, the CSCS problem is solvable for (r_1, r_2) iff $r_1 > 1$ and $r_2 > 0$;*
(iii) *if $\mathbf{P} = \mathbf{Z} = \emptyset$, the CSCS problem is solvable for (r_1, r_2) iff $r_1 + r_2 > 1$.*

In view of the above theorem, we need only restrict our attention to the case where the plant $P(s)$ has both a closed RHP pole and a closed RHP zero in the sequel. Moreover, we might as well assume that $(r_1, r_2) \in E_1$. From (i) of Lemma 1.1, we have

$$E_1 \triangleq \{(x, y): 1/\delta < x < 1 + 1/\delta \text{ and } 1/\delta < y < 1 + 1/\delta\}.$$

It is clear that $\delta \leq 1$, and G_0 defined as in (2.5) is a region containing 0 and 1 and a strict subset in each of the two disks $|s| < r_1$ and $|s-1| < r_2$. The following auxiliary result is crucial for solving the CSCS problem.

Lemma 2.1 [8]. *Let $G \subsetneq \mathbb{C}$ be a given simply connected region containing* 0 *and* 1. *Suppose that there exists a conformal equivalence $\phi(s): G \to \mathscr{D}$ with*

$$\overline{\phi(\bar{s})} = \phi(s) \quad \text{and} \quad \phi(0) = 0. \tag{2.8}$$

Let δ be defined as in (1.3). *Then there exists a real rational analytic function $S(s)$ mapping from $\overline{H}$ to G and satisfying the interpolation conditions* (2.6)–(2.7) *if and only if $|\phi(1)| < \delta$.*

Remark 2.1. The Riemann Conformal Mapping Theorem [1] guarantees that if the region G contains at least two boundary points, then there always exists a unique conformal mapping $\phi(s): G \to \mathscr{D}$ with the properties $\phi(0) = 0$ and $\phi'(0) > 0$. If the region G is further assumed to be symmetric with respect to the real axis, it is easily seen that the unique conformal mapping $\phi(s)$ must satisfy the conditions (2.8).

From the above remark, we can see that there is no problem with the existence of a conformal equivalence $\phi(s): G_0 \to \mathscr{D}$ with the required properties (2.8). The construction of such a $\phi(s)$ will proceed as follows. In doing this, it is convenient to refer to Figure 3. Here

$$s_0 = \tfrac{1}{2}\left\{r_1^2 - r_2^2 + 1 + \mathrm{j}\sqrt{\left[(r_1+1)^2 - r_2^2\right]\left[r_2^2 - (r_1-1)^2\right]}\right\} \triangleq x_0 + \mathrm{j}y_0,$$

$$\alpha = \tan^{-1}\left(\frac{y_0}{1-x_0}\right), \qquad \beta = \tan^{-1}\left(\frac{y_0}{x_0}\right),$$

$$\alpha + \beta = \tan^{-1}\left(\frac{y_0}{x_0 - r_1^2}\right) \quad \left(\tfrac{1}{2}\pi < \alpha + \beta < \pi\right).$$

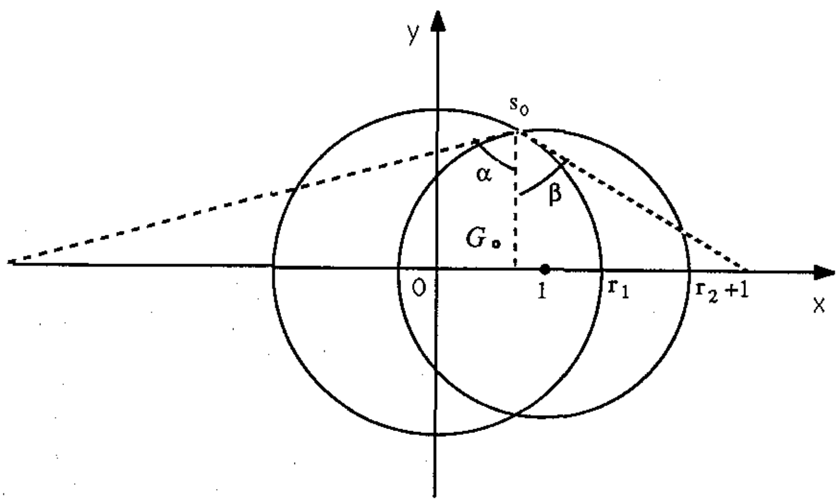

Fig. 3.

From [4], the conformal mapping from G_0 to $\{u;\ 0 < \arg(u) < \alpha+\beta\}$ can be derived as follows:

$$u = \phi_1(s) = \rho\frac{s_0 - s}{s - \bar{s}_0}, \qquad \rho = \sqrt{\frac{r_1^3}{2y_0^2(r_1 - x_0)}}\ \mathrm{e}^{\mathrm{j}\alpha}.$$

Meanwhile, it is trivial to find the following conformal mappings from $\{u\colon 0 < \arg(u) < \alpha+\beta\}$ to $\{v\colon 0 < \arg(v) < \pi\}$ and from $\{v\colon 0 < \arg(v) < \pi\}$ to $\mathscr{D}$, respectively,

$$v = \phi_2(u) = u^{\pi/(\alpha+\beta)} \quad \text{and} \quad z = \phi_3(v) = k\frac{v - v_0}{v - \bar{v}_0},$$

where $v_0 = \phi_2(\phi_1(0))$ and k is any complex number of unit magnitude.

Now, the conformal mapping ϕ from G_0 to $\mathscr{D}$ can be constructed as a composition of the above three conformal mappings:

$$z = \phi(s) = \phi_3\{\phi_2[\phi_1(s)]\} = k\frac{\left(\rho\frac{s_0 - s}{s - \bar{s}_0}\right)^{\pi/(\alpha+\beta)} - \left(\rho\frac{s_0}{-\bar{s}_0}\right)^{\pi/(\alpha+\beta)}}{\left(\rho\frac{s_0 - s}{s - \bar{s}_0}\right)^{\pi/(\alpha+\beta)} - \overline{\left[\left(\rho\frac{s_0}{-\bar{s}_0}\right)^{\pi/(\alpha+\beta)}\right]}}.$$

With the relations

$$\left(\rho\frac{s_0 - s}{s - \bar{s}_0}\right)^{\pi/(\alpha+\beta)} = \rho^{\pi/(\alpha+\beta)}\frac{(s_0 - s)^{\pi/(\alpha+\beta)}}{(s - \bar{s}_0)^{\pi/(\alpha+\beta)}} = \frac{[\rho(s_0 - s)]^{\pi/(\alpha+\beta)}}{(s - \bar{s}_0)^{\pi/(\alpha+\beta)}}, \quad \forall s \in G_0, \tag{2.9}$$

$$\overline{\left(\rho\frac{s_0}{-\bar{s}_0}\right)^{\pi/(\alpha+\beta)}} = \frac{(\bar{\rho}\bar{s}_0)^{\pi/(\alpha+\beta)}}{(-s_0)^{\pi/(\alpha+\beta)}}, \tag{2.10}$$

$$(s_0 - s)^{\pi/(\alpha+\beta)}(-\bar{s}_0)^{\pi/(\alpha+\beta)} = [\bar{s}_0(s - s_0)]^{\pi/(\alpha+\beta)}, \tag{2.11}$$

$$(s - \bar{s}_0)^{\pi/(\alpha+\beta)}(s_0)^{\pi/(\alpha+\beta)} = [s_0(s - \bar{s}_0)]^{\pi/(\alpha+\beta)}, \tag{2.12}$$

$$[\rho(s_0 - s)]^{\pi/(\alpha+\beta)}(-s_0)^{\pi/(\alpha+\beta)} = [\rho s_0(s - s_0)]^{\pi/(\alpha+\beta)}\mathrm{e}^{\mathrm{j}2\pi^2/(\alpha+\beta)}, \tag{2.13}$$

$$(s - \bar{s}_0)^{\pi/(\alpha+\beta)}(\bar{\rho}\bar{s}_0)^{\pi/(\alpha+\beta)} = [\bar{\rho}\bar{s}_0(s - \bar{s}_0)]^{\pi/(\alpha+\beta)}, \quad \forall s \in G_0, \tag{2.14}$$

the expression for the conformal mapping ϕ can be further simplified as

$$\begin{aligned} z = \phi(s) &= k\rho^{\pi/(\alpha+\beta)}\frac{(-s_0)^{\pi/(\alpha+\beta)}}{(-\bar{s}_0)^{\pi/(\alpha+\beta)}}\,\frac{(s_0 - s)^{\pi/(\alpha+\beta)}(-\bar{s}_0)^{\pi/(\alpha+\beta)} - (s - \bar{s}_0)^{\pi/(\alpha+\beta)}(s_0)^{\pi/(\alpha+\beta)}}{[\rho(s_0 - s)]^{\pi/(\alpha+\beta)}(-s_0)^{\pi/(\alpha+\beta)} - (s - \bar{s}_0)^{\pi/(\alpha+\beta)}(\bar{\rho}\bar{s}_0)^{\pi/(\alpha+\beta)}} \\ &= k\rho^{\pi/(\alpha+\beta)}\frac{(-s_0)^{\pi/(\alpha+\beta)}}{(-\bar{s}_0)^{\pi/(\alpha+\beta)}}\,\frac{[\bar{s}_0(s - s_0)]^{\pi/(\alpha+\beta)} - [s_0(s - \bar{s}_0)]^{\pi/(\alpha+\beta)}}{[\rho s_0(s - s_0)]^{\pi/(\alpha+\beta)}\,\mathrm{e}^{\mathrm{j}2\pi^2/(\alpha+\beta)} - [\bar{\rho}\bar{s}_0(s - \bar{s}_0)]^{\pi/(\alpha+\beta)}}. \end{aligned}$$

Apparently, the above construction implies that $\phi(0) = 0$. Furthermore, for the requirement $\overline{\phi(\bar{s})} = \phi(s)$ to be satisfied, from Remark 2.1 it suffices to choose the constant k such that $|k| = 1$ and $\phi'(0)$ is real and positive. Since

$$\overline{[\bar{s}_0(1 - s_0)]^{\pi/(\alpha+\beta)}\ \mathrm{e}^{-\mathrm{j}\pi^2/(\alpha+\beta)}} = [s_0(1 - \bar{s}_0)]^{\pi/(\alpha+\beta)}\ \mathrm{e}^{-\mathrm{j}\pi^2/(\alpha+\beta)}, \tag{2.15}$$

$$\overline{[\rho s_0(1 - s_0)]^{\pi/(\alpha+\beta)}} = [\bar{\rho}\bar{s}_0(1 - \bar{s}_0)]^{\pi/(\alpha+\beta)}\ \mathrm{e}^{-\mathrm{j}2\pi^2/(\alpha+\beta)}, \tag{2.16}$$

it turns out that

$$|\phi(1)| = |\rho|^{\pi/(\alpha+\beta)} \left| \frac{[\bar{s}_0(1-s_0)]^{\pi/(\alpha+\beta)} - [s_0(1-\bar{s}_0)]^{\pi/(\alpha+\beta)}}{[\rho s_0(1-s_0)]^{\pi/(\alpha+\beta)} e^{j2\pi^2/(\alpha+\beta)} - [\bar{\rho}\bar{s}_0(1-\bar{s}_0)]^{\pi/(\alpha+\beta)}} \right| \tag{2.17}$$

$$= |\rho|^{\pi/(\alpha+\beta)} \left| \frac{[\bar{s}_0(1-s_0)]^{\pi/(\alpha+\beta)} e^{-j\pi^2/(\alpha+\beta)} - [s_0(1-\bar{s}_0)]^{\pi/(\alpha+\beta)} e^{-j\pi^2/(\alpha+\beta)}}{[\rho s_0(1-s_0)]^{\pi/(\alpha+\beta)} - [\bar{\rho}\bar{s}_0(1-\bar{s}_0)]^{\pi/(\alpha+\beta)} e^{-j2\pi^2/(\alpha+\beta)}} \right| \tag{2.18}$$

$$= |\rho|^{\pi/(\alpha+\beta)} \left| \frac{\mathrm{Im}\{[\bar{s}_0(1-s_0)]^{\pi/(\alpha+\beta)} e^{-j\pi^2/(\alpha+\beta)}\}}{\mathrm{Im}\{[\rho s_0(1-s_0)]^{\pi/(\alpha+\beta)}\}} \right| \tag{2.19}$$

$$= |\rho|^{\pi/(\alpha+\beta)} \left| \frac{\mathrm{Im}\{[r_1 r_2 e^{j(\pi-\alpha-\beta)}]^{\pi/(\alpha+\beta)}\}}{\mathrm{Im}\{[|\rho| r_1 r_2 e^{j\beta}]^{\pi/(\alpha+\beta)}\}} \right| \tag{2.20}$$

$$= \left| \frac{\sin[\pi^2/(\alpha+\beta)]}{\sin[\beta\pi/(\alpha+\beta)]} \right|. \tag{2.21}$$

A direct application of Lemma 2.1 yields the following result.

Theorem 2.2. *Consider the plant $P(s)$ with δ defined as in* (1.3). *Assume that $\mathbf{P} \neq \emptyset$ and $\mathbf{Z} \neq \emptyset$. The the* CSCS *problem is always solvable for all $(r_1, r_2) \in \bigcup_{i=2}^{4} E_i$; the* CSCS *problem for $(r_1, r_2) \in E_1$ is solvable if and only if*

$$\left| \frac{\sin(\pi^2/\mu)}{\sin(\nu\pi/\mu)} \right| < \delta, \tag{2.22}$$

where

$$\mu = \tan^{-1}\left\{ -\frac{\sqrt{[(r_1+1)^2 - r_2^2][r_2^2 - (r_1-1)^2]}}{r_1^2 + r_2^2 - 1} \right\},$$

$$\nu = \tan^{-1}\left\{ \frac{\sqrt{[(r_1+1)^2 - r_2^2][r_2^2 - (r_1-1)^2]}}{r_1^2 - r_2^2 + 1} \right\}.$$

In a special case where $r_1 = r_2$, the solvability condition (2.22) has a quite simple form to be given below.

Corollary 2.1. *With the same assumption as in Theorem* 2.2, *the* CSCS *problem is solvable for $r_1 = r_2 = r$ with $r > 1/\delta$ if and only if*

$$r > \frac{1}{2\cos[\pi^2/2(\pi + \sin^{-1}\delta)]}. \tag{2.23}$$

Proof. It is not hard to verify that with $r_1 = r_2 = r$, the following hold:

$$\mu = 2\nu \quad \text{and} \quad \nu = \cos^{-1}(1/2r). \tag{2.24}$$

Since $r > 1/\delta \geq 1$, one can see that

$$\pi < \frac{\pi^2}{2\nu} < \frac{3\pi}{2}. \tag{2.25}$$

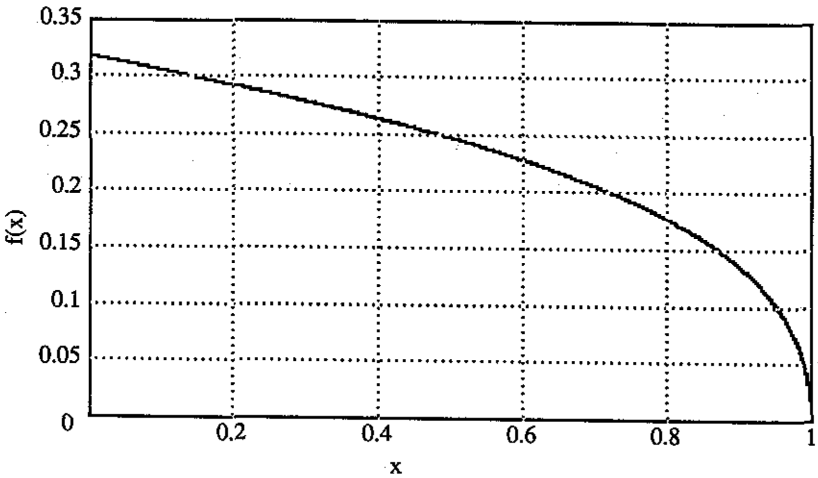

Fig. 4.

In view of (2.24) and (2.25), the condition (2.22) turns out to be $\sin(\pi^2/2\nu - \pi) < \delta$, which is obviously equivalent to (2.23) because $0 < \pi^2/2\nu - \pi < \frac{1}{2}\pi$. □

We close this section with several remarks.

Remark 2.2. As is shown in Figure 4,

$$f(x) \triangleq \frac{1}{2\cos[\pi^2/2(\pi + \sin^{-1}x)]} - \frac{1}{x} > 0, \quad 0 < x < 1;$$

that is, the right-hand expression of (2.23) as a function of δ is always greater than $1/\delta$. This implies that for a plant with $\delta < 1$, it is impossible to achieve simultaneously the sensitivity and the complementary sensitivity both arbitrarily close to their minimums with a single LTI compensator, though they have the same minimum.

Remark 2.3. It is trivial to see that under the additional assumption that $\delta = 1$, the solvability condition of the CSCS problem reduces to that $r_1, r_2 > 1$ since $|\phi(1)| < 1 = \delta$, $\forall (r_1, r_2) \in E_1$.

Remark 2.4. It should be pointed out that as for the solvability of the CSCS problem, the condition (2.22) is actually applicable to (r_1, r_2) with $r_1, r_2 > 1/\delta$ and $|r_1 - r_2| < 1$ rather than being restricted to $(r_1, r_2) \in E_1$, which can be seen from the above derivation. It is interesting to note that

$$\lim_{r_1 \to (1+r_2)^-} \left| \frac{\sin(\pi^2/\mu)}{\sin(\nu\pi/\mu)} \right| = \frac{1}{r_2} \quad \text{and} \quad \lim_{r_2 \to (1+r_1)^-} \left| \frac{\sin(\pi^2/\mu)}{\sin(\nu\pi/\mu)} \right| = \frac{1}{r_1}.$$

Thus, it follows that for any fixed $r_0 > 1/\delta$, there exists $\eta > 0$ (possibly dependent on r_0) such that the CSCS problem is solvable for all $r_1 \geq 1 + r_0 - \eta$ and $r_2 \geq r_0$, and for all $r_1 \geq r_0$ and $r_2 \geq 1 + r_0 - \eta$. This is evidently consistent with the first part of Theorem 2.2.

3. Connection between the sensitivity problem and its complementary problem

In this section, we turn to establishing a connection between the sensitivity problem and the complementary sensitivity problem for a multivariable plant. This connection will allow us to solve either of the two problems by way of tackling the other one. Recall that the sensitivity problem is to find a stabilizing compensator $C(s)$ such that the H^∞-norm of its associated sensitivity function is less than a given $r>0$ while the complementary sensitivity problem is to find a stabilizing compensator $C(s)$ such that the H^∞-norm of its associated complementary sensitivity function is less than a given $r>0$.

Let RH_+^∞ denote the set of proper rational matrices which have no poles in the closed RHP. Suppose the transfer matrix $P(s)$ is given and has a stable doubly-coprime factorization in RH_+^∞:

$$P(s)=ND^{-1}(s)=\tilde{D}^{-1}\tilde{N}(s)$$

with the Bezout identities

$$\begin{bmatrix} X & Y \\ \tilde{D} & -\tilde{N} \end{bmatrix}\begin{bmatrix} N & \tilde{Y} \\ D & -\tilde{X} \end{bmatrix}=\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \tag{3.1}$$

or equivalently,

$$\begin{bmatrix} N & \tilde{Y} \\ D & -\tilde{X} \end{bmatrix}\begin{bmatrix} X & Y \\ \tilde{D} & -\tilde{N} \end{bmatrix}=\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{3.2}$$

Then, it is a standard result that the set of all internally stabilizing proper compensators for $P(s)$ can be described by

$$S(P)=\left\{(X-Z\tilde{N})^{-1}(X+Z\tilde{D}):\ Z\in RH_+^\infty \text{ and } |Y-Z\tilde{N}|\neq 0\right\} \tag{3.3}$$

$$=\left\{(\tilde{X}+DZ)(\tilde{Y}-NZ)^{-1}:\ Z\in RH_+^\infty \text{ and } |\tilde{Y}-NZ|\neq 0\right\}. \tag{3.4}$$

It follows that with $C(s)=(\tilde{X}+DZ)(\tilde{Y}-NZ)^{-1}$, there holds

$$(I+PC)^{-1}(s)=(\tilde{Y}-NZ)\tilde{D}(s). \tag{3.5}$$

Let us now state the main result of this section.

Theorem 3.1. *Let $P(s)$ denote the $p\times m$ transfer matrix of a multivariable* LTI *plant, and $r>1$ be given. Then, $C(s)$ solves the sensitivity problem for r if and only if $(r^2/(r^2-1))C(s)$ solves the complementary sensitivity problem for r.*

Proof. Assume that $C_0(s)$ solves the sensitivity problem for r with $S_0(s)$ its corresponding sensitivity function. Since $C_0(s)$ is in $S(P)$, there exists $Z_0\in\mathrm{RH}_+^\infty$ such that

$$C_0(s)=(\tilde{X}+DZ_0)(\tilde{Y}-NZ_0)^{-1}. \tag{3.6}$$

Let us show that $C_1(s)\triangleq(r^2/(r^2-1))C_0(s)$ is also in $S(P)$. In other words, we need to find a $Z\in\mathrm{RH}_+^\infty$ with $|Y-Z\tilde{N}|\neq 0$ such that

$$\frac{r^2}{r^2-1}(\tilde{X}+DZ_0)(\tilde{Y}-NZ_0)^{-1}=(Y-Z\tilde{N})^{-1}(X+Z\tilde{D}). \tag{3.7}$$

With the Bezout identities, one can see that under the condition that $|Y-Z\tilde{N}|\neq 0$, (3.7) is equivalent to

$$r^2Z_0+X(\tilde{Y}-NZ_0)=Z\left[r^2I-\tilde{D}(\tilde{Y}-NZ_0)\right]. \tag{3.8}$$

Since

$$\|S_0(s)\|=\|(\tilde{Y}-NZ_0)\tilde{D}(s)\|<r,$$

$r^2I-(\tilde{Y}-NZ_0)\tilde{D}$ is unimodular in RH_+^∞; hence, so is $r^2I-\tilde{D}(\tilde{Y}-NZ_0)$. In this way, it follows from (3.8) that

$$Z_1 \triangleq \left[r^2Z_0+X(\tilde{Y}-NZ_0)\right]\left[r^2I-\tilde{D}(\tilde{Y}-NZ_0)\right]^{-1}\in \mathrm{RH}_+^\infty.$$

Further, we have

$$\begin{aligned}Y-Z_1\tilde{N} &= Y-\left[r^2Z_0+X(\tilde{Y}-NZ_0)\right]\left[r^2I-\tilde{D}(\tilde{Y}-NZ_0)\right]^{-1}\tilde{N}\\ &= Y-\left[r^2Z_0+X(\tilde{Y}-NZ_0)\right]\left[(r^2-1)I+\tilde{N}(\tilde{X}+DZ_0)\right]^{-1}\tilde{N}\\ &= Y-\left[r^2Z_0+X(\tilde{Y}-NZ_0)\right]\tilde{N}\left[(r^2-1)I+(\tilde{X}+DZ_0)\tilde{N}\right]^{-1}\\ &= (r^2-1)(Y-Z_0\tilde{N})\left[(r^2-1)I+(\tilde{X}+DZ_0)\tilde{N}\right]^{-1},\end{aligned}$$

which implies that $|Y-Z_1\tilde{N}|\neq 0$ amounts to $|Y-Z_0\tilde{N}|\neq 0$ because

$$\det\left[(r^2-1)I+(\tilde{X}+DZ_0)\tilde{N}\right]=\det\left[r^2I-\tilde{D}(\tilde{Y}-NZ_0)\right]\neq 0.$$

But,

$$\begin{aligned}|(Y-Z_0\tilde{N})D| &= |I-(X+Z_0\tilde{D})N|\\ &= |I-N(X+Z_0\tilde{D})| = |(\tilde{Y}-NZ_0)\tilde{D}|\neq 0;\end{aligned}$$

so $|Y-Z_1\tilde{N}|\neq 0$. Therefore, $Z=Z_1$ satisfies (3.7). Having shown that $C_1(s)$ is in $S(P)$, we now verify that $C_1(s)$ indeed solves the complementary sensitivity problem for r, i.e.

$$\| I-(I+PC_1)^{-1}\| < r.$$

On noting that

$$I-(I+PC_1)^{-1}=r^2(S_0-I)(r^2I-S_0)^{-1},$$

it suffices to check that

$$\| r(S_0-I)(r^2I-S_0)^{-1}\| <1.$$

By definition, the above inequality is equivalent to

$$r^2\left[r^2I-S_0'(-\mathrm{j}\omega)\right]^{-1}\left[S_0'(-\mathrm{j}\omega)-I\right]\left[S_0(\mathrm{j}\omega)-I\right]\left[r^2I-S_0(\mathrm{j}\omega)\right]^{-1}-I<0,\quad \forall\omega\in\mathbb{R}. \tag{3.9}$$

Since the left-hand term of the above is equal to

$$(r^2-1)\left[r^2I-S_0'(-\mathrm{j}\omega)\right]^{-1}\left[S_0'(-\mathrm{j}\omega)S_0(\mathrm{j}\omega)-r^2I\right]\left[r^2I-S_0(\mathrm{j}\omega)\right]^{-1},$$

$\| S_0\| < r$ implies (3.9). Thus, the 'only if' part of Theorem 3.1 is proved.

The 'if' part can be proved by showing with a similar argument as above that if $C(s)$ solves the complementary sensitivity problem for r, then $((r^2-1)/r^2)C(s)$ solves the sensitivity problem for r. □

As an immediate consequence of Theorem 3.1, the following corollary is obtained.

Corollary 3.1. *For a given multivariable* LTI *plant, the solvability of the sensitivity problem for* $r>1$ *is equivalent to the solvability of the complementary sensitivity problem for* r; *in particular, if the minimal achievable sensitivity or the minimal achievable complementary sensitivity of the plant is greater than* 1, *they must be equal.*

4. Conclusions

The problem of finding a time-invariant controller for a given SISO plant so that the closed-loop system satisfies two tolerances for sensitivity and for complementary sensitivity simultaneously has been considered. The solvability condition of the problem has been derived in terms of an explicit inequality. It has also been discovered that for a multivariable plant, there exists a very simple one-to-one correspondence between the sensitivity problem for $r > 1$ and the complementary sensitivity problem for the same r. Namely, a solution to either of the two problems can be obtained simply by multiplying a solution to the other problem by some certain constant. This exhibits that sensitivity and complementary sensitivity have the same minimum for a plant if either minimum is greater than 1.

There is still some work remaining for further research. For instance, it is currently unclear how to extend the obtained results on the CSCS problem to the multivariable case or the weighted case, though the generalization to the discete-time case is direct.

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