A SYSTEM THEORY CRITERION FOR POSITIVE REAL MATRICES

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1. Introduction. The concept of a positive real function is now an old one of network theory, and more recently the concept has been usefully employed in other system theoretic investigations, such as in the development of the Popov criterion for the stability of a feedback system containing a single memoryless nonlinearity [1]. In view of this and other connections between network and control theory, it seems possible that the concept of a positive real matrix could be employed fruitfully in control systems investigations; to assist in such investigations this paper discusses positive real matrices from a system theory viewpoint.

An \( n \times n \) matrix \( Z(\cdot) \) of functions of a complex variable is called positive real if [2]

(i) \( Z(s) \) has elements which are analytic for \( \text{Re} \ s > 0 \);
(ii) \( Z^*(s) = Z(s^*) \) for \( \text{Re} \ s > 0 \);
(iii) \( Z'(s^*) \) is nonnegative definite for \( \text{Re} \ s > 0 \).

Here the superscript asterisk denotes complex conjugation; the prime denotes matrix transposition.

This paper is concerned with developing a criterion for a matrix of rational functions to be positive real. The criterion is a systems theoretic one, in the sense that it is formulated in terms of the parameters of a control system realization of the matrix. Such a result has been developed elsewhere for the scalar case and applied to the development of the Popov criterion [1]. A generalization of the criterion for a scalar function to be positive real has been stated for a subclass of positive real matrices, namely, those which are zero at \( s = \infty \), [3]. In this reference, no proof was given; an outline proof was suggested, but the filling in of the details presented more difficulties than suspected, and a correct proof has not been available hitherto. Similar ideas are discussed in a paper of Popov [4].

After a review of some concepts associated with system realizations and some mathematical preliminaries in §2, we present in §3 the statement and proof of the positive real criterion. Initially, we consider matrices whose elements have no poles on the imaginary axis and which are zero at \( s = \infty \);

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then matrices with poles only on the imaginary axis; and, finally, general positive real matrices (i.e., those which are not necessarily zero at \( s = \infty \), and which have imaginary axis poles permitted). This result appears as Theorem 3, and includes all preceding results as special cases.

Of course Theorem 3 alone could be proved, but the motivation for its proof would then be obscure. As it stands, it is a natural outgrowth of the earlier, more motivated, theorems.

Notation in the paper is straightforward: capital letters will be used for matrices, small letters for vectors. Other symbols will be explained as required.

2. Mathematical and system theoretic preliminaries. In this section we review some of the concepts associated with linear time-invariant dynamical systems; these concepts will appear in the discussion of positive real matrices. We shall also state some mathematical results to be used in the sequel.

We assume that \( M(s) \) is an \( m \times n \) matrix of rational functions, with \( M(\infty) = 0 \). Then a triple \( \{P, G, H\} \) is termed a realization of \( M \) (see [5], [6]) if

\[
M(s) = H'(sI - F)^{-1}G.
\]

This is because \( M(s) \) is the transfer function matrix relating an input vector \( u \) to an output vector \( y \) in the following state-space representation of \( M \):

\[
\begin{align*}
\dot{x} &= Fx + Gu, \\
y &= H'x.
\end{align*}
\]

Here \( x \) is a \( p \)-vector, the state; \( u \) is an \( n \)-vector, the input; \( y \) is an \( m \)-vector, the output; \( F \) is \( p \times p \), \( G \) is \( p \times n \), and \( H \) is \( p \times m \).

For a given \( M(s) \), there exist infinitely many sets \( \{F, G, H\} \) constituting a realization [5]. Clearly there must be a minimal dimension which \( F \) may have; any realization incorporating \( F \) of minimal dimension is termed a minimal realization.

**Lemma 1** [7]. Let \( \{F_1, G_1, H_1\} \) and \( \{F_2, G_2, H_2\} \) be two minimal realizations of \( M(s) \). Then there exists a nonsingular \( T \) such that

\[
\begin{align*}
F_2 &= TF_1T^{-1}, \\
G_2 &= TG_1, \\
H_2 &= (T')^{-1}H_1.
\end{align*}
\]

Conversely, if \( \{F_1, G_1, H_1\} \) is minimal and \( T \) is nonsingular, \( \{F_2, G_2, H_2\} \) as given by (3a), (3b) and (3c) is minimal.
The dimension of a minimal realization is the dimension of a minimal $F$ matrix and is termed the degree of $M(s)$; it is, naturally, a positive integer number uniquely determined by $M(s)$, written $\delta(M(s))$ or $\delta[M]$. Actually, numerous definitions of degree have appeared over the last fifteen years [8], [9], [10], but recently these have been reconciled with one another [11].

We shall have occasion to use several properties as set out below.

**Lemma 2** [8], [11]. If the elements of $M_1(\cdot)$ and $M_2(\cdot)$ have no poles in common,

$$\delta[M_1 + M_2] = \delta[M_2].$$

**Lemma 3** [8], [11].

$$\delta[M_1M_2] \leq \delta[M_1] + \delta[M_2].$$

**Lemma 4.** If $M_1(\cdot)$ is $n \times r$, $M_2(\cdot)$ is $r \times n$, $r \leq n$, the elements of $M_1$ and $M_2$ have no common poles, rank $M_1(s_0) = r$ at any pole $s_0$ of an element of $M_1$, and rank $M_2(s_0) = r$ at any pole $s_0$ of an element of $M_2$, then

$$\delta[M_1M_2] = \delta[M_1] + \delta[M_2].$$

**Proof.** We use the Smith-McMillan decomposition [8], [11] for $M_1$ and $M_2$. We have $M_1 = A_1T_1B_1$ and $M_2 = A_2T_2B_2$, where $A_1, A_2$ are $n \times n$ polynomial matrices with constant determinant; $B_1, B_2$ are $r \times r$ polynomial matrices with constant determinant; $T_1$ has its first $r$ rows given by $\text{diag} \{e_1/\psi_1, \ldots, e_r/\psi_r\}$ and its last $n-r$ rows zero; $T_2$ has its first $r$ columns given by $\text{diag} \{\mu_1/e_1, \ldots, \mu_r/e_r\}$ and its last $n-r$ columns zero. The $e_i$, etc., are polynomials. Denote by $\tilde{T}_1$ the first $r$ rows of $T_1$ and by $\tilde{T}_2$ the first $r$ columns of $T_2$. Then at poles of the elements of $\tilde{T}_1$, $\tilde{T}_2$ is a non-singular matrix, and vice versa.

This implies (see [8, §§5.26 and 5.4]) that $\delta[\tilde{T}_1B_1A_2\tilde{T}_2] = \delta[\tilde{T}_1] + \delta[\tilde{T}_2]$. The matrix $T_1B_1A_2T_2$ is simply $\tilde{T}_1B_1A_2\tilde{T}_2$ with rows and columns of zeros added, and thus [8, §5.45], $\delta[T_1B_1A_2T_2] = \delta[\tilde{T}_1B_1A_2\tilde{T}_2]$. Finally, by [8, §5.16], we see that $\delta[M_1M_2] = \delta[T_1B_1A_2\tilde{T}_2]$, and $\delta[M_1] = \delta[T_1], \delta[M_2] = \delta[\tilde{T}_2]$, from which the result immediately follows.

The next lemma, though purely algebraic in character, is of great use in studying the stability of linear systems, being originally due to Lyapunov. For a discussion and proof of the first part of the result, see [12], and for the second part, see [1].

**Lemma 5.** Let $F$ be a $p \times p$ matrix with eigenvalues all possessing negative real parts. Then to each $p \times q$ matrix $L$ (arbitrary) there corresponds a unique symmetric nonnegative definite solution $P$ to the equation

$$PF + FP' = -LL'.$$

Moreover if $[F, L']$ is completely observable [5], or equivalently, if
$L' \exp (Ft)x = 0$ for all $t$ implies $x = 0$, $P$ is nonsingular, being given by

$$P = \int_0^\infty \exp(F't)L'L' \exp(Ft) \, dt.$$ 

**COROLLARY.** The only matrices which commute with

$$\begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}$$

are of the form

$$\begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix},$$

where $T_1, T_2'$ commute with $F$.

**Proof of Corollary.** Suppose that

$$\begin{bmatrix} T_1 & S \\ R & T_2 \end{bmatrix} \begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix} = \begin{bmatrix} F' & 0 \\ 0 & -F' \end{bmatrix} \begin{bmatrix} T_1 & S \\ R & T_2 \end{bmatrix}.$$ 

Then

(10a) \hspace{1cm} FT_1 = T_1F,

(10b) \hspace{1cm} F'T_2 = T_2F',

(10c) \hspace{1cm} FS + SF' = 0,

(10d) \hspace{1cm} RF + F'R = 0.

Clearly $S = 0$ and $R = 0$ satisfy (10c) and (10d). Lemma 5 guarantees these solutions are unique.

To conclude this section we state a lemma on spectral factorization, due to Youla [13]. Let $Z(s)$ be positive real. Let $Y(s) = Z(s) + Z'(-s)$; $Y(s)$ is termed "parahermitian", i.e., $Y(s) = Y'(-s)$. If $s = jo$, $Y(jo)$ is non-negative definite hermitian as a consequence of the positive real character of $Z(s)$. Then [12, Theorem 2] yields a factorization of $Y(s)$ as follows.

**Lemma 6.** Let the $n \times n$ matrix $Z(s)$ be positive real, and suppose that $Z(s) + Z'(-s)$ has rank $r$ almost everywhere. Then there exists an $r \times n$ matrix $W(s)$ such that

$$Y(s) = Z(s) + Z'(-s) = W'(-s)W(s),$$

and

(i) $W$ has elements which are analytic for $\text{Re } s > 0$, and for $\text{Re } s \geq 0$ if $Z(s)$ has elements which are analytic for $\text{Re } s \geq 0$;

(ii) rank $W = r$ for $\text{Re } s > 0$;

(iii) $W$ is unique save for multiplication on the left by an arbitrary orthogonal matrix.
3. Principal results. We shall assume until further notice that \( Z(s) \) is positive real, with \( Z(\infty) = 0 \), and that it possesses no imaginary axis poles, i.e., all poles lie in the half-plane \( \text{Re}\, s < 0 \). If \( [F, G, H] \) is a minimal realization for \( Z \), then \( F \) will have eigenvalues with negative real parts.

**Lemma 7.** Let \( [F, G, H] \) be a minimal realization for \( Z(s) \). With \( Z \) and \( W \) related as in Lemma 6, suppose \( W \) has a minimal realization \([A, K, L]\). Then the matrices \( A \) and \( F \) are similar.

**Proof.** Because \([F, G, H]\) is a realization for \( Z(s) \), direct calculation shows that

\[
(F_1, G_1, H_1) = \begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}, \begin{bmatrix} G' \\ H \end{bmatrix}, \begin{bmatrix} H \\ -G' \end{bmatrix}
\]

is a realization for \( Z'(\cdot-s) + Z(s) \). Because \( Z(s) \) and \( Z'(\cdot-s) \) can have no poles in common (those of \( Z(s) \) being in \( \text{Re}\, s < 0 \) and those of \( Z'(\cdot-s) \) in \( \text{Re}\, s > 0 \)), by Lemma 1,

\[
\delta[Z(s) + Z'(\cdot-s)] = 2\delta[Z(s)].
\]

Hence \([F_1, G_1, H_1]\) is minimal.

By direct calculation

\[
W'(\cdot-s)W(s) = K'(-sI - A')^{-1}LL'(sI - A)^{-1}K
\]

\[
= H_2(sI - F_2)^{-1}G_2,
\]

where

\[
(F_2, G_2, H_2) = \begin{bmatrix} A \\ LL' \\ 0 \\ -A' \end{bmatrix}, \begin{bmatrix} K \\ 0 \\ -K \end{bmatrix}, \begin{bmatrix} 0 \\ -K \end{bmatrix}.
\]

By Lemma 4 and (i) and (ii) in Lemma 6, \( \delta[W'(\cdot-s)W(s)] = 2\delta[W] \), and thus \([F_2, G_2, H_2]\) is minimal.

Define \( P \) to be the unique positive definite symmetric solution of

\[
PA + A'P = -LL'.
\]

The existence of \( P \) follows by Lemma 5, (i) in Lemma 6 and the minimality of \([A, K, L]\). We may now apply Lemma 1, taking

\[
T = \begin{bmatrix} I & 0 \\ P & I \end{bmatrix}
\]

to obtain the following alternative minimal realization of \( W'(\cdot-s)W(s) \):

\[
(F_3, G_3, H_3) = \begin{bmatrix} A \\ 0 \\ -A' \end{bmatrix}, \begin{bmatrix} K \\ PK \end{bmatrix}, \begin{bmatrix} -K \\ -K \end{bmatrix}.
\]

Since (12) and (15) are minimal realizations of the same matrix, viz., \( Y(s) \), and since \( F \) has strictly negative eigenvalues, as also has \( A \), from (i) in Lemma 6, the result of the lemma follows by (3a) in Lemma 1.
COROLLARY A. Let $Z(s)$ have a minimal realization $(F, G, H)$ and let $Z$ and $W$ be related as in Lemma 6. Then there exist matrices $K, L$ such that $W$ has a minimal realization $(F, K, L)$.

COROLLARY B. With the notation of Corollary A, two minimal realizations of $Y(s) = Z(s) + Z'(-s) = W'(-s)W(s)$ are given by

$$\begin{align*}
(F_1, G_1, H_1) &= \left\{\begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}, \begin{bmatrix} G' \\ H' \end{bmatrix}, \begin{bmatrix} H' \\ -G' \end{bmatrix}\right\},
\end{align*}$$

and

$$\begin{align*}
(F_2, G_2, H_2) &= \left\{\begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}, \begin{bmatrix} P \\ PK \end{bmatrix}, \begin{bmatrix} PK \\ -K \end{bmatrix}\right\},
\end{align*}$$

where $P$ is uniquely defined (see Lemma 5) by

$$\begin{align*}
PF + F'P = -LL',
\end{align*}$$

and $K, L$ are now as defined in Corollary A.

The proofs of these two corollaries follow from Lemma 1.

The next lemma is concerned with making a further identification between minimal realizations of $Z$ and $W$.

LEMMA 8. Let $Z(s)$ have a minimal realization $(F, G, H)$ and let $Z$ and $W$ be related as in Lemma 6. Then there exists a matrix $L$ such that $(F, G, L)$ is a minimal realization for $W$.

Proof. By Corollary A of the previous lemma, $W$ has a minimal realization $(F, K, L)$ for some $K$ and $L$. By Corollary B and Lemma 1 there exists a nonsingular matrix $T$ which must commute with

$$\begin{align*}
\begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}
\end{align*}$$

such that

$$\begin{align*}
T \begin{bmatrix} G \\ H \end{bmatrix} = \begin{bmatrix} K \\ PK \end{bmatrix}
\end{align*}$$

By Lemma 5, there exists $T_1$ commuting with $F$ such that

$$\begin{align*}
T_1G = K.
\end{align*}$$

Moreover, since $T$ is nonsingular, so is $T_1$. Then by Lemma 1 again, since $(F, K, L)$ is a minimal realization for $W$, so is $(T_1^{-1}FT_1, T_1^{-1}K, (T_1)^'L) = (F, G, L)$.

We can now state the following theorem.

THEOREM 1. Let $Z(\cdot)$ be a matrix of rational functions such that $Z(\infty) = 0$ and $Z$ has poles only in $Re \ s < 0$. Let $(F, G, H)$ be a minimal realization of $Z$. Then $Z(\cdot)$ is positive real if and only if there exist a symmetric positive defi-
nite matrix $P$ and a matrix $L$ such that

$$PP + F'P = -LL'$$

and

$$PG = H.$$ 

Proof of sufficiency. Of the three conditions listed in §1 which $Z$ must satisfy, the only one which needs to be verified is the third, viz., $Z'(s^*) + Z(s)$ is nonnegative definite for Re $s > 0$. We have

$$Z'(s^*) + Z(s) = G'(s^*I - F')^{-1}H + H'(sI - F)^{-1}G = G'((s^*I - F')^{-1}P + P(sI - F)^{-1}G) = G'((s^*I - F')^{-1}(P(sI - F') + (s^*I - F')P(sI - F)^{-1}G = G'((s^*I - F')^{-1}(Ps + s^*P - PP - F'P)(sI - F)^{-1}G = G'(s^*I - F')^{-1}(PsI - F)^{-1}G(s^* + s^*) + G'(s^*I - F')^{-1}LL'(sI - F)^{-1}G.$$

Since the right-hand side is of the form $[2Re s][A'(s^*)PA(s)] + B'(s^*)B(s)$, it is clearly nonnegative definite in the right half-plane.

Proof of necessity. Let $W(s)$ be as in Lemma 6, and let $[F, G, L]$ be a minimal realization of $W$, where we are using Lemma 8 and dropping the hat notation. The matrix $L$ is used to define through (17) the matrix $P$ which we know from Lemma 5 to be unique and symmetric positive definite. By Corollary B to Lemma 7,

$$\begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}, \begin{bmatrix} G \\ H' \end{bmatrix}, \begin{bmatrix} H \\ -G \end{bmatrix}$$

and

$$\begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}, \begin{bmatrix} G \\ PG \end{bmatrix}, \begin{bmatrix} PG \\ -G \end{bmatrix}$$

are two minimal realizations of $Y(s)$. By Lemma 1, there exists nonsingular $T$ commuting with

$$\begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}$$

such that

$$T \begin{bmatrix} G \\ H \end{bmatrix} = \begin{bmatrix} G \\ PG \end{bmatrix}$$

and $(T^{-1})' \begin{bmatrix} H \\ -G \end{bmatrix} = \begin{bmatrix} PG \\ -G \end{bmatrix}$. 

By Lemma 5, there exists $T_1$ commuting with $F$ such that $T_1G = G$ and $(T_1^{-1})'H = PG$. Now, since $T_1$ commutes with $F$,

$$[G, FG, \cdots] = [T_2G, FT_2G, \cdots] = [T_1G, T_1FG, \cdots] = T_1[G, FG, \cdots].$$
The matrix \( IG, FG, \cdots \) has rank \( p \), where \( p \) is the dimension of \( F \), since \( [F, G, H] \) is a minimal realization of \( Z(s) \) and all minimal realizations are completely controllable [5]. Hence \( T_1 = I \), and thus \( PG = H \).

In preparation for dealing with matrices which may possess imaginary axis poles, we consider now positive real matrices whose only poles are on the imaginary axis.

**Theorem 2.** Let a positive real \( Z(s) \) have all pure imaginary poles, with \( Z(\infty) = 0 \), and let \( [F, G, H] \) be a minimal realization for \( Z \). Then there exists a symmetric positive definite \( P \) such that

\[
PF + F'P = 0, \tag{19}
\]

\[
PG = H. \tag{20}
\]

**Proof.** First note that if \( P \) satisfies the above equations, then \( P^* = (T^')^{-1}PT^{-1} \) satisfies the corresponding equations for the minimal realization \( [TFT^{-1}, TG, (T^')'H] \). Consequently if we exhibit a symmetric positive definite \( P \) for any one minimal realization of \( Z \), it follows that a symmetric positive definite \( P \) exists for all minimal realizations. Our procedure will in fact be to choose a minimal realization \( [F, G, H] \) for which \( P \) has a particularly obvious form.

The form of \( Z(s) \) has been established (see, for example, [2]) as

\[
Z(s) = \sum_{i} \frac{A_i s + B_i}{s^2 + \omega_i^2}, \tag{21}
\]

where the \( \omega_i \) are all different and the matrices \( A_i \) and \( B_i \) satisfy certain requirements. By realizing separately each term \( (A_i s + B_i)/(s^2 + \omega_i^2) \) with minimal \( \{F_i, G_i, H_i\} \) and selecting a \( P \) such that (19) and (20) are satisfied, one can obtain a minimal \( \{F, G, H\} \) and a \( P \) satisfying (19) and (20) with \( F = \bigoplus F_i \) (where \( \bigoplus \) denotes direct sum), \( G' = \{G'_1, G'_2, \cdots\}, H' = \{H'_1, H'_2, \cdots\} \) and \( P = \bigoplus P_i \). Consequently we shall consider the realization of the simpler

\[
Z(s) = \frac{As + B}{s^2 + \omega_i^2}. \tag{22}
\]

In [2, Chap. 6] it is pointed out that if \( 2k \) is the degree of \( Z \) in (22), there exist \( k \) complex vectors \( x_i \) such that

\[
x_i^*x_i = 1, \quad x_i^*x_i = \mu_i, \quad 0 < \mu_i \leq 1, \quad \mu_i \text{ real}, \tag{23}
\]

and

\[
Z(s) = \sum_{i=1}^{k} \left[ \frac{z_i^*z_i^*}{s - j\omega_i} + \frac{x_i^*x_i}{s + j\omega_i} \right]. \tag{24}
\]

Direct sum techniques then allow us to restrict consideration to obtaining
a minimal realization for the degree 2
\begin{equation}
Z(s) = \frac{xx^*'}{s - j\omega_0} + \frac{x^*x'}{s + j\omega_0}.
\end{equation}

It is easy now to verify that if
\begin{equation}
y_1 = \frac{x + x^*}{\sqrt{2}}, \quad y_2 = \frac{x - x^*}{\sqrt{2}},
\end{equation}
we have
\begin{equation}
Z(s) = [y_1, y_2] \frac{1}{s^2 + \omega_0^2} \begin{bmatrix} s & \omega_0 \\ -\omega_0 & s \end{bmatrix} \begin{bmatrix} y_1' \\ y_2' \end{bmatrix},
\end{equation}
and then
\begin{equation}
\begin{bmatrix} F, G, H, P \end{bmatrix} = \begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix} \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}
\end{equation}
defines a minimal realization for (25), with (19) and (20) being satisfied.

All preliminaries are now in hand to give the final theorem, which applies to general positive real matrices. Since any positive real matrix can be written as the sum of \(D_0\) and \(Z(s)\), where \(D\) is nonnegative definite and \(Z(s)\) is also positive real, but with \(Z(\infty)\) finite, we shall lose no generality in restricting the theorem statement to such \(Z(s)\) (see [2, Chap. 5]).

By way of notation, we shall say that \(\begin{bmatrix} F, G, H, Z(\infty) \end{bmatrix}\) is a minimal realization of \(Z(s)\) if \(\begin{bmatrix} F, G, H \end{bmatrix}\) is a minimal realization of \(Z(s) - Z(\infty)\).

**Theorem 3.** Let \(Z(\cdot)\) be a matrix of rational transfer functions such that \(Z(\infty)\) is finite and \(Z\) has poles which lie in \(\text{Re} \ s < 0\) or are simple on \(\text{Re} \ s = 0\). Let \(\begin{bmatrix} F, G, H, Z(\infty) \end{bmatrix}\) be a minimal realization of \(Z\). Then \(Z(\cdot)\) is positive real if and only if there exist is symmetric positive definite \(P\) and matrices \(W_0\) and \(L\) such that
\begin{align}
PF + FP' &= -LL', \\
PG &= H - LW_0, \\
W_0'W_0 &= Z(\infty) + Z'(\infty).
\end{align}

**Proof of sufficiency.** It only remains to verify the positive real behavior of \(Z'(s*) + Z(s)\) in the right half-plane. We have
\begin{align}
Z'(s*) + Z(s) &= Z'(\infty) + Z(\infty) + G'(s^*I - F')^{-1}H + H'(sI - F)^{-1}G \\
&= W_0'W_0 + G'((s^*I - F')^{-1}P + P(sI - F)^{-1})G \\
&\quad + G'(s^*I - F')^{-1}LW_0 + W_0'L'(sI - F)^{-1}G
\end{align}
which is plainly nonnegative definite for \( \text{Re} \ s > 0 \).

This completes the proof of sufficiency.

Proof of necessity. Initially, suppose \( Z(s) \) has strict left half-plane poles. We shall consider the general case later, with the aid of Theorem 2. Let \( W(s) \) be the matrix defined in Lemma 6. Then the arguments used to establish Lemmas 7 and 8 carry through in essentially the same fashion to establish that there exist matrices \( L \) and \( W \) such that \( W \) has a minimal realization \((F, G, L, W)\), with two minimal realizations for \( Z(s) + Z'(-s) = W'(-s)W(s) \) being given by

\[
[F, G, H, W] = \left\{ \begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}, \begin{bmatrix} G \\ H \end{bmatrix}, \begin{bmatrix} H' \\ -G \end{bmatrix}, W \right\}
\]

and

\[
[F, G, H, W] = \left\{ \begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}, \begin{bmatrix} PG + LW \end{bmatrix}, \begin{bmatrix} PG + LW \end{bmatrix}, W \right\}
\]

Here \( P \) is the unique symmetric positive definite solution of (29).

The arguments of Theorem 1 can now be followed directly to conclude that (30) holds. Equation (31) follows by setting \( s = \infty \) in \( Z(s) + Z'(-s) = W'(-s)W(s) \).

Now let us relax the restriction on the poles of \( Z(s) \). Then \( Z(s) \) may be decomposed as

\[
Z(s) = Z_1(s) + Z_2(s),
\]

where \( Z_1 \) has purely imaginary axis poles, \( Z_2 \) has poles only in \( \text{Re} \ s < 0 \), and \( Z_1 \) and \( Z_2 \) are both positive real [2, §5.2].

Now select for \( Z_1 \) a minimal \([F_1, G_1, H_1]\) and \( P_1 \) using Theorem 2, such that

\[
P_1F_1 + F_1'P_1 = 0,
\]
and for $Z_2$ select $\{F_2, G_2, H_2\}$ and $P_2$, using the material just proved, such that

\[(29') P_2F_2 + F_2'P_2 = -L_2L_2', \]

\[(30') P_2G_2 = H_2 - L_2W_2, \]

\[(31') W_2'W_2 = Z_2(\infty) + Z_2'(\infty). \]

Then it is easily verified that (29), (30) and (31) are satisfied by taking

\[P = P_1 + P_2, \]

\[F = F_1 + F_2, \]

\[G' = [G_1', G_2'], \]

\[H' = [H_1', H_2'], \]

\[L' = [0, L_2'] . \]

Moreover, with $\{F_1, G_1, H_1\}$ and $\{F_2, G_2, H_2\}$ minimal realizations for $Z_1(s)$ and $Z_2(s) - Z_2(\infty)$, $\{F, G, H\}$ is a minimal realization for $Z(s) - Z(\infty)$. This is because, by Lemma 2, the degree of $Z$ is the sum of the degrees of $Z_1$ and $Z_2$, the latter having no common poles, while the dimension of $F$ is the sum of the dimensions of $F_1$ and $F_2$. One should at this stage verify that (29), (30) and (31) are valid under a state space coordinate transformation, as they have merely been established for a particular class of $F$ (i.e., those of the form $F_1 + F_2$). This is easy to do, however, along the lines given in Theorem 2 for a more particular case.

4. Conclusions. The significance of the theorems in their own right is self-evident. They provide a conceptual link between basic concepts of control theory and network theory. Their proofs have a number of interesting features, such as the necessity to use the particular $\bar{W}(s)$ in the factorization of $Z(s)$ and $Z'(\infty)$, the heavy reliance on the concept of degree in the network [8] and control theory [11] sense, and the canonical $[F, G, H]$ representation (believed new) of a "lossless" $Z$. Hopefully the results themselves as well as their proofs will help forge another link in the growing chain [3], [14] between control and network theory.

There are several immediate applications of the theory. The stability of control systems containing multiple nonlinearities is discussed in [15], a new passive network synthesis of positive real functions and matrices in [16], and the properties of multivariable control systems with linear feedback laws in [17].
REFERENCES