J. SIAM CONTROL
Vol. 5, No. 2, 1967
Printed in U.S.A.

A SYSTEM THEORY CRITERION FOR POSITIVE REAL MATRICES*

B. D. O. ANDERSON†

1. Introduction. The concept of a positive real function is now an old one of network theory, and more recently the concept has been usefully employed in other system theoretic investigations, such as in the development of the Popov criterion for the stability of a feedback system containing a single memoryless nonlinearity [1]. In view of this and other connections between network and control theory, it seems possible that the concept of a *positive real matrix* could be employed fruitfully in control systems investigations; to assist in such investigations this paper discusses positive real matrices from a system theory viewpoint.

An $n \times n$ matrix $Z(\cdot)$ of functions of a complex variable is called positive real if [2]

(i) $Z(s)$ has elements which are analytic for $\operatorname{Re} s > 0$;
(ii) $Z^*(s) = Z(s^*)$ for $\operatorname{Re} s > 0$;
(iii) $Z'(s^*) + Z(s)$ is nonnegative definite for $\operatorname{Re} s > 0$.

Here the superscript asterisk denotes complex conjugation; the prime denotes matrix transposition.

This paper is concerned with developing a criterion for a matrix of *rational* functions to be positive real. The criterion is a systems theoretic one, in the sense that it is formulated in terms of the parameters of a control system realization of the matrix. Such a result has been developed elsewhere for the scalar case and applied to the development of the Popov criterion [1]. A generalization of the criterion for a scalar function to be positive real has been stated for a subclass of positive real matrices, namely, those which are zero at $s = \infty$, [3]. In this reference, no proof was given; an outline proof was suggested, but the filling in of the details presented more difficulties than suspected, and a correct proof has not been available hitherto. Similar ideas are discussed in a paper of Popov [4].

After a review of some concepts associated with system realizations and some mathematical preliminaries in §2, we present in §3 the statement and proof of the positive real criterion. Initially, we consider matrices whose elements have no poles on the imaginary axis and which are zero at $s = \infty$;

* Received by the editors March 7, 1966, and in revised form November 15, 1966.

† Formerly with Stanford Electronics Laboratories, Stanford University, Stanford, California. Now with the Department of Electrical Engineering, University of Newcastle, New South Wales, Australia. This work was supported by the National Science Foundation under Grant GK 237. The author wishes also to acknowledge the support of the Services Canteens Trust Fund, an agency of the Australian Government, and the United States Education Foundation in Australia for a Fulbright grant.

then matrices with poles only on the imaginary axis; and, finally, general positive real matrices (i.e., those which are not necessarily zero at $s = \infty$, and which have imaginary axis poles permitted). This result appears as Theorem 3, and includes all preceding results as special cases.

Of course Theorem 3 alone could be proved, but the motivation for its proof would then be obscure. As it stands, it is a natural outgrowth of the earlier, more motivated, theorems.

Notation in the paper is straightforward: capital letters will be used for matrices, small letters for vectors. Other symbols will be explained as required.

2. Mathematical and system theoretic preliminaries. In this section we review some of the concepts associated with *linear time-invariant dynamical systems*; these concepts will appear in the discussion of positive real matrices. We shall also state some mathematical results to be used in the sequel.

We assume that $M(s)$ is an $m \times n$ matrix of rational functions, with $M(\infty) = 0$. Then a triple $\{F,, GH\}$ is termed a *realization* of M(see [5], [6]) if

$$M(s) = H'(sI - F)^{-1}G. \tag{1}$$

This is because $M(s)$ is the transfer function matrix relating an input vector u to an output vector y in the following state-space representation of M:

$$\dot{x} = Fx + Gu, \tag{2a}$$

$$y = H'x. \tag{2b}$$

Here x is a p-vector, the state; u is an n-vector, the input; y is an m-vector, the output; F is $p \times p$, G is $p \times n$, and H is $p \times m$.

For a given $M(s)$, there exist infinitely many sets $\{F, G, H\}$ constituting a realization [5]. Clearly there must be a *minimal dimension* which F may have; any realization incorporating F of minimal dimension is termed a *minimal realization*.

LEMMA 1 [7]. *Let $\{F_1, G_1, H_1\}$ and $\{F_2, G_2, H_2\}$ be two minimal realizations of $M(s)$. Then there exists a nonsingular T such that*

$$F_2 = TF_1T^{-1}, \tag{3a}$$

$$G_2 = TG_1, \tag{3b}$$

$$H_2 = (T')^{-1}H_1. \tag{3c}$$

Conversely, if $\{F_1, G_1, H_1\}$ is minimal and T is nonsingular, $\{F_2, G_2, H_2\}$ as given by (3a), (3b) *and* (3c) *is minimal.*

The *dimension of a minimal realization* is the dimension of a minimal F matrix and is termed the *degree* of $M(s)$; it is, naturally, a positive integer number uniquely determined by $M(s)$, written $\delta[M(s)]$ or $\delta[M]$. Actually, numerous definitions of degree have appeared over the last fifteen years [8], [9], [10], but recently these have been reconciled with one another [11].

We shall have occasion to use several properties as set out below.

LEMMA 2 [8], [11]. *If the elements of $M_1(\cdot)$ and $M_2(\cdot)$ have no poles in common,*

$$\delta[M_1 + M_2] = \delta[M_2]. \tag{4}$$

LEMMA 3 [8], [11].

$$\delta[M_1 M_2] \leqq \delta[M_1] + \delta[M_2]. \tag{5}$$

LEMMA 4. *If $M_1(\cdot)$ is $n \times r$, $M_2(\cdot)$ is $r \times n$, $r \leqq n$, the elements of M_1 and M_2 have no common poles,* rank $M_1(s_0) = r$ *at any pole s_0 of an element of M_2, and* rank $M_2(\hat{s}_0) = r$ *at any pole $\hat{s}_0$ of an element of M_1, then*

$$\delta[M_1 M_2] = \delta[M_1] + \delta[M_2]. \tag{6}$$

Proof. We use the Smith-McMillan decomposition [8], [11] for M_1 and M_2. We have $M_1 = A_1 \Gamma_1 B_1$ and $M_2 = A_2 \Gamma_2 B_2$, where A_1, B_2 are $n \times n$ polynomial matrices with constant determinant; B_1, A_2 are $r \times r$ polynomial matrices with constant determinant; Γ_1 has its first r rows given by diag $[\epsilon_1/\psi_1, \cdots, \epsilon_r/\psi_r]$ and its last $n - r$ rows zero; Γ_2 has its first r columns given by diag $[\lambda_1/\mu_1, \cdots, \lambda_r/\mu_r]$ and its last $n - r$ columns zero. The ϵ_i, etc., are polynomials. Denote by $\hat{\Gamma}_1$ the first r rows of Γ_1 and by $\hat{\Gamma}_2$ the first r columns of Γ_2. Then at poles of the elements of $\hat{\Gamma}_1$, $\hat{\Gamma}_2$ is a nonsingular matrix, and vice versa.

This implies (see [8, §§5.26 and 5.4]) that $\delta[\hat{\Gamma}_1 B_1 A_2 \hat{\Gamma}_2] = \delta[\hat{\Gamma}_1] + \delta[\hat{\Gamma}_2]$. The matrix $\Gamma_1 B_1 A_2 \Gamma_2$ is simply $\hat{\Gamma}_1 B_1 A_2 \hat{\Gamma}_2$ with rows and columns of zeros added, and thus [8, §5.45], $\delta[\Gamma_1 B_1 A_1 \Gamma_2] = \delta[\hat{\Gamma}_1 B_1 A_2 \hat{\Gamma}_2]$. Finally, by [8, §5.16], we see that $\delta[M_1 M_2] = \delta[\Gamma_1 B_1 A_2 \Gamma_2]$, and $\delta[M_1] = \delta[\hat{\Gamma}_1]$, $\delta[M_2] = \delta[\hat{\Gamma}_2]$, from which the result immediately follows.

The next lemma, though purely algebraic in character, is of great use in studying the stability of linear systems, being originally due to Lyapunov. For a discussion and proof of the first part of the result, see [12], and for the second part, see [1].

LEMMA 5. *Let F be a $p \times p$ matrix with eigenvalues all possessing negative real parts. Then to each $p \times q$ matrix L (q arbitrary) there corresponds a unique symmetric nonnegative definite solution P to the equation*

$$PF + F'P = -LL'. \tag{7}$$

Moreover if $[F, L']$ is completely observable [5], *or equivalently, if*

$L' \exp(Ft)x = 0$ for all t implies $x = 0$, P is nonsingular, being given by

$$P = \int_0^\infty \exp(F't)LL' \exp(Ft)\, dt.$$

COROLLARY. *The only matrices which commute with*

$$\begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix} \tag{8}$$

are of the form

$$\begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}, \tag{9}$$

where T_1, T_2' commute with F.

Proof of Corollary. Suppose that

$$\begin{bmatrix} T_1 & S \\ R & T_2 \end{bmatrix} \begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix} = \begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix} \begin{bmatrix} T_1 & S \\ R & T_2 \end{bmatrix}.$$

Then

$$FT_1 = T_1F, \tag{10a}$$

$$F'T_2 = T_2F', \tag{10b}$$

$$FS + SF' = 0, \tag{10c}$$

$$RF + F'R = 0. \tag{10d}$$

Clearly $S = 0$ and $R = 0$ satisfy (10c) and (10d). Lemma 5 guarantees these solutions are unique.

To conclude this section we state a lemma on spectral factorization, due to Youla [13]. Let $Z(s)$ be positive real. Let $Y(s) = Z(s) + Z'(-s)$; $Y(s)$ is termed *parahermitian*, i.e., $Y(s) = Y'(-s)$. If $s = j\omega$, $Y(j\omega)$ is nonnegative definite hermitian as a consequence of the positive real character of $Z(s)$. Then [13, Theorem 2] yields a factorization of $Y(s)$ as follows.

LEMMA 6. *Let the $n \times n$ matrix $Z(s)$ be positive real, and suppose that $Z(s) + Z'(-s)$ has rank r almost everywhere. Then there exists an $r \times n$ matrix $W(s)$ such that*

$$Y(s) = Z(s) + Z'(-s) = W'(-s)W(s), \tag{11}$$

and

(i) *W has elements which are analytic for* Re $s > 0$, *and for* Re $s \geqq 0$ *if $Z(s)$ has elements which are analytic for* Re $s \geqq 0$;

(ii) rank $W = r$ *for* Re $s > 0$;

(iii) *W is unique save for multiplication on the left by an arbitrary orthogonal matrix.*

3. Principal results. We shall assume until further notice that $Z(s)$ is positive real, with $Z(\infty) = 0$, and that it possesses no imaginary axis poles, i.e., all poles lie in the half-plane $\operatorname{Re} s < 0$. If $\{F, G, H\}$ is a minimal realization for Z, then F will have eigenvalues with negative real parts.

LEMMA 7. *Let $\{F, G, H\}$ be a minimal realization for $Z(s)$. With Z and W related as in Lemma 6, suppose W has a minimal realization $\{A, K, L\}$. Then the matrices A and F are similar.*

Proof. Because $\{F, G, H\}$ is a realization for $Z(s)$, direct calculation shows that

$$\{F_1, G_1, H_1\} = \left\{\begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}, \begin{bmatrix} G \\ H \end{bmatrix}, \begin{bmatrix} H \\ -G \end{bmatrix}\right\} \tag{12}$$

is a realization for $Z'(-s) + Z(s)$. Because $Z(s)$ and $Z'(-s)$ can have no poles in common (those of $Z(s)$ being in $\operatorname{Re} s < 0$ and those of $Z'(-s)$ in $\operatorname{Re} s > 0$), by Lemma 1,

$$\delta[Z(s) + Z'(-s)] = 2\delta[Z(s)].$$

Hence $\{F_1, G_1, H_1\}$ is minimal.

By direct calculation

$$W'(-s)W(s) = K'(-sI - A')^{-1}LL'(sI - A)^{-1}K \tag{13a}$$

$$= H_2'(sI - F_2)^{-1}G_2, \tag{13b}$$

where

$$\{F_2, G_2, H_2\} = \left\{\begin{bmatrix} A & 0 \\ LL' & -A' \end{bmatrix}, \begin{bmatrix} K \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -K \end{bmatrix}\right\}. \tag{14}$$

By Lemma 4 and (i) and (ii) in Lemma 6, $\delta[W'(-s)W(s)] = 2\delta[W]$, and thus $\{F_2, G_2, H_2\}$ is minimal.

Define P to be the unique positive definite symmetric solution of

$$PA + A'P = -LL'.$$

The existence of P follows by Lemma 5, (i) in Lemma 6 and the minimality of $\{A, K, L\}$. We may now apply Lemma 1, taking

$$T = \begin{bmatrix} I & 0 \\ P & I \end{bmatrix}$$

to obtain the following alternative minimal realization of $W'(-s)W(s)$:

$$\{F_3, G_3, H_3\} = \left\{\begin{bmatrix} A & 0 \\ 0 & -A' \end{bmatrix}, \begin{bmatrix} -K \\ PK \end{bmatrix}, \begin{bmatrix} PK \\ -K \end{bmatrix}\right\}. \tag{15}$$

Since (12) and (15) are minimal realizations of the same matrix, viz., $Y(s)$, and since F has strictly negative eigenvalues, as also has A, from (i) in Lemma 6, the result of the lemma follows by (3a) in Lemma 1.

Corollary A. *Let $Z(s)$ have a minimal realization $\{F, G, H\}$ and let Z and W be related as in Lemma 6. Then there exist matrices K, L such that W has a minimal realization $\{F, K, L\}$.*

Corollary B. *With the notation of Corollary A, two minimal realizations of $Y(s) = Z(s) + Z'(-s) = W'(-s)W(s)$ are given by*

$$\{F_1, G_1, H_1\} = \left\{\begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}, \begin{bmatrix} G \\ H \end{bmatrix}, \begin{bmatrix} H \\ -G \end{bmatrix}\right\} \tag{12}$$

and

$$\{F_3, G_3, H_3\} = \left\{\begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}, \begin{bmatrix} P \\ PK \end{bmatrix}, \begin{bmatrix} PK \\ -K \end{bmatrix}\right\}, \tag{16}$$

where P is uniquely defined (see Lemma 5) *by*

$$PF + F'P = -LL', \tag{17}$$

and K, L are now as defined in Corollary A.

The proofs of these two corollaries follow from Lemma 1.

The next lemma is concerned with making a further identification between minimal realizations of Z and W.

Lemma 8. *Let $Z(s)$ have a minimal realization $\{F, G, H\}$ and let Z and W be related as in Lemma 6. Then there exists a matrix $\hat{L}$ such that $\{F, G, \hat{L}\}$ is a minimal realization for W.*

Proof. By Corollary A of the previous lemma, W has a minimal realization $\{F, K, L\}$ for some K and L. By Corollary B and Lemma 1 there exists a nonsingular matrix T which must commute with

$$\begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}$$

such that

$$T\begin{bmatrix} G \\ H \end{bmatrix} = \begin{bmatrix} K \\ PK \end{bmatrix}$$

By Lemma 5, there exists T_1 commuting with F such that

$$T_1G = K.$$

Moreover, since T is nonsingular, so is T_1. Then by Lemma 1 again, since $\{F, K, L\}$ is a minimal realization for W, so is $\{T_1^{-1}FT_1, T_1^{-1}K, (T_1)'L\} = \{F, G, \hat{L}\}$.

We can now state the following theorem.

Theorem 1. *Let $Z(\cdot)$ be a matrix of rational functions such that $Z(\infty) = 0$ and Z has poles only in* $\operatorname{Re} s < 0$. *Let $\{F, G, H\}$ be a minimal realization of Z. Then $Z(\cdot)$ is positive real if and only if there exist a symmetric positive defi-*

nite matrix P and a matrix L such that

$$PF + F'P = -LL' \tag{17}$$

and

$$PG = H. \tag{18}$$

Proof of sufficiency. Of the three conditions listed in §1 which Z must satisfy, the only one which needs to be verified is the third, viz., $Z'(s^*) + Z(s)$ is nonnegative definite for Re $s > 0$. We have

$$\begin{aligned} Z'(s^*) + Z(s) &= G'(s^*I - F')^{-1}H + H'(sI - F)^{-1}G \\ &= G'\{(s^*I - F')^{-1}P + P(sI - F)^{-1}\}G \\ &= G'(s^*I - F')^{-1}\{P(sI - F) + (s^*I - F')P\}(sI - F)^{-1}G \\ &= G'(s^*I - F')^{-1}\{Ps + s^*P - PF - F'P\}(sI - F)^{-1}G \\ &= G'(s^*I - F')^{-1}P(sI - F)^{-1}G(s + s^*) \\ &\quad + G'(s^*I - F')^{-1}LL'(sI - F)^{-1}G. \end{aligned}$$

Since the right-hand side is of the form $\{2\text{Re } s\}\{A'(s^*)PA(s)\} + B'(s^*)B(s)$, it is clearly nonnegative definite in the right half-plane.

Proof of necessity. Let $W(s)$ be as in Lemma 6, and let $\{F, G, L\}$ be a minimal realization of W, where we are using Lemma 8 and dropping the hat notation. The matrix L is used to define through (17) the matrix P which we know from Lemma 5 to be unique and symmetric positive definite. By Corollary B to Lemma 7,

$$\left\{\begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}, \begin{bmatrix} G \\ H \end{bmatrix}, \begin{bmatrix} H \\ -G \end{bmatrix}\right\} \text{ and } \left\{\begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}, \begin{bmatrix} G \\ PG \end{bmatrix}, \begin{bmatrix} PG \\ -G \end{bmatrix}\right\}$$

are two minimal realizations of $Y(s)$. By Lemma 1, there exists nonsingular T commuting with

$$\begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}$$

such that

$$T\begin{bmatrix} G \\ H \end{bmatrix} = \begin{bmatrix} G \\ PG \end{bmatrix} \quad \text{and} \quad (T^{-1})'\begin{bmatrix} H \\ -G \end{bmatrix} = \begin{bmatrix} PG \\ -G \end{bmatrix}.$$

By Lemma 5, there exists T_1 commuting with F such that $T_1G = G$ and $(T_1^{-1})'H = PG$. Now, since T_1 commutes with F,

$$[G, FG, \cdots] = [T_1G, FT_1G, \cdots] = [T_1G, T_1FG, \cdots] = T_1[G, FG, \cdots].$$

The matrix $[G, FG, \cdots]$ has rank p, where p is the dimension of F, since $\{F, G, H\}$ is a minimal realization of $Z(s)$ and all minimal realizations are completely controllable [5]. Hence $T_1 = I$, and thus $PG = H$.

In preparation for dealing with matrices which may possess imaginary axis poles, we consider now positive real matrices whose *only* poles are on the imaginary axis.

THEOREM 2. *Let a positive real $Z(s)$ have all pure imaginary poles, with $Z(\infty) = 0$, and let $\{F, G, H\}$ be a minimal realization for Z. Then there exists a symmetric positive definite P such that*

$$PF + F'P = 0, \tag{19}$$

$$PG = H. \tag{20}$$

Proof. First note that if P satisfies the above equations, then $P^* = (T')^{-1}PT^{-1}$ satisfies the corresponding equations for the minimal realization $\{TFT^{-1}, TG, (T^{-1})'H\}$. *Consequently if we exhibit a symmetric positive definite P for any one minimal realization of Z, it follows that a symmetric positive definite P exists for all minimal realizations.* Our procedure will in fact be to choose a minimal realization $\{F, G, H\}$ for which P has a particularly obvious form.

The form of $Z(s)$ has been established (see, for example, [2]) as

$$Z(s) = \sum_i \frac{A_i s + B_i}{s^2 + \omega_i^2}, \tag{21}$$

where the ω_i are all different and the matrices A_i and B_i satisfy certain requirements. By realizing separately each term $(A_i s + B_i)(s^2 + \omega_i^2)^{-1}$ with minimal $\{F_i, G_i, H_i\}$ and selecting a P_i such that (19) and (20) are satisfied, one can obtain a minimal $\{F, G, H\}$ and a P satisfying (19) and (20) with $F = \dot{+}_i F_i$ (where $\dot{+}$ denotes direct sum), $G' = [G_1', G_2', \cdots]$, $H' = [H_1', H_2', \cdots]$ and $P = \dot{+}_i P_i$. Consequently we shall consider the realization of the simpler

$$Z(s) = \frac{As + B}{s^2 + \omega_0^2}. \tag{22}$$

In [2, Chap. 6] it is pointed out that if $2k$ is the degree of Z in (22), there exist k complex vectors x_i such that

$$x_i^{*\prime} x_i = 1, \quad x_i' x_i = \mu_i, \quad 0 < \mu_i \leqq 1, \quad \mu_i \text{ real}, \tag{23}$$

and

$$Z(s) = \sum_{i=1}^{k} \left[\frac{x_i x_i^{*\prime}}{s - j\omega_0} + \frac{x_i^* x_i'}{s + j\omega_0} \right]. \tag{24}$$

Direct sum techniques then allow us to restrict consideration to obtaining a minimal realization for the degree 2

$$Z(s) = \frac{xx^{*\prime}}{s - j\omega_0} + \frac{x^* x'}{s + j\omega_0}. \tag{25}$$

It is easy now to verify that if

$$y_1 = \frac{x + x^*}{\sqrt{2}}, \qquad y_2 = j\frac{x - x^*}{\sqrt{2}}, \tag{26}$$

we have

$$Z(s) = [y_1, y_2] \frac{1}{s^2 + \omega_0^2} \begin{bmatrix} s & \omega_0 \\ -\omega_0 & s \end{bmatrix} \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}, \tag{27}$$

and then

$$\{F, G, H, P\} = \left\{ \begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix}, \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}, \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \tag{28}$$

defines a minimal realization for (25), with (19) and (20) being satisfied.

All preliminaries are now in hand to give the final theorem, which applies to general positive real matrices. Since any positive real matrix can be written as the sum of Ds and $Z(s)$, where D is nonnegative definite and $Z(s)$ is also positive real, but with $Z(\infty)$ finite, we shall lose no generality in restricting the theorem statement to such $Z(s)$ (see [2, Chap. 5]).

By way of notation, we shall say that $\{F, G, H, Z(\infty)\}$ is a minimal realization of $Z(s)$ if $\{F, G, H\}$ is a minimal realization of $Z(s) - Z(\infty)$.

THEOREM 3. *Let $Z(\cdot)$ be a matrix of rational transfer functions such that $Z(\infty)$ is finite and Z has poles which lie in $\operatorname{Re} s < 0$ or are simple on $\operatorname{Re} s = 0$. Let $\{F, G, H, Z(\infty)\}$ be a minimal realization of Z. Then $Z(\cdot)$ is positive real if and only if there exist a symmetric positive definite P and matrices W_0 and L such that*

$$PF + F'P = -LL', \tag{29}$$

$$PG = H - LW_0, \tag{30}$$

$$W_0'W_0 = Z(\infty) + Z'(\infty). \tag{31}$$

Proof of sufficiency. It only remains to verify the positive real behavior of $Z'(s^*) + Z(s)$ in the right half-plane. We have

$$\begin{aligned} Z'(s^*) + Z(s) &= Z'(\infty) + Z(\infty) + G'(s^*I - F')^{-1}H + H'(sI - F)^{-1}G \\ &= W_0'W_0 + G'[(s^*I - F')^{-1}P + P(sI - F)^{-1}]G \\ &\qquad + G'(s^*I - F')^{-1}LW_0 + W_0'L'(sI - F)^{-1}G \end{aligned}$$

$$
\begin{aligned}
&= W_0'W_0 + G'(s^*I - F')^{-1}[P(s + s^*) - PF - F'P](sI - F)^{-1}G \\
(32) \qquad &\qquad + G'(s^*I - F')^{-1}LW_0 + W_0'L'(sI - F)^{-1}G \\
&= W_0'W_0 + G'(s^*I - F')^{-1}LW_0 + W_0'L'(sI - F)^{-1}G \\
&\qquad + G'(s^*I - F')^{-1}LL'(sI - F)^{-1}G \\
&\qquad + G'(s^*I - F')^{-1}P(sI - F)^{-1}G(s + s^*) \\
&= [W_0' + G'(s^*I - F')^{-1}L][W_0 + L'(sI - F)^{-1}G] \\
&\qquad + G'(s^*I - F')^{-1}P(sI - F)^{-1}G(s + s^*),
\end{aligned}
$$

which is plainly nonnegative definite for Re $s > 0$.

This completes the proof of sufficiency.

Proof of necessity. Initially, suppose $Z(s)$ has strict left half-plane poles. We shall consider the general case later, with the aid of Theorem 2. Let $W(s)$ be the matrix defined in Lemma 6. Then the arguments used to establish Lemmas 7 and 8 carry through in essentially the same fashion to establish that there exist matrices L and $W_0 = W(\infty)$ such that W has a minimal realization $\{F, G, L, W_0\}$, with two minimal realizations for $Z(s) + Z'(-s) = W'(-s)W(s)$ being given by

$$
(33) \quad \{F_1, G_1, H_1, W_0'W_0\} = \left\{\begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}, \begin{bmatrix} G \\ H \end{bmatrix}, \begin{bmatrix} H \\ -G \end{bmatrix}, W_0'W_0\right\}
$$

and

$$
\begin{aligned}
&\{F_3, G_3, H_3, W_0'W_0\} \\
(34) \qquad &= \left\{\begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}, \begin{bmatrix} G \\ PG + LW_0 \end{bmatrix}, \begin{bmatrix} PG + LW_0 \\ -G \end{bmatrix}, W_0'W_0\right\}.
\end{aligned}
$$

Here P is the unique symmetric positive definite solution of (29).

The arguments of Theorem 1 can now be followed directly to conclude that (30) holds. Equation (31) follows by setting $s = \infty$ in $Z(s) + Z'(-s) = W'(-s)W(s)$.

Now let us relax the restriction on the poles of $Z(s)$. Then $Z(s)$ may be decomposed as

$$
(35) \qquad Z(s) = Z_1(s) + Z_2(s),
$$

where Z_1 has purely imaginary axis poles, Z_2 has poles only in Re $s < 0$, and Z_1 and Z_2 are both positive real [2, §5.2].

Now select for Z_1 a minimal $\{F_1, G_1, H_1\}$ and P_1, using Theorem 2, such that

$$
(19') \qquad P_1F_1 + F_1'P_1 = 0,
$$

$$P_1 G_1 = H_1\,; \tag{20$'$}$$

and for Z_2 select $\{F_2, G_2, H_2\}$ and P_2, using the material just proved, such that

$$P_2 F_2 + F_2' P_2 = -L_2 L_2', \tag{29$'$}$$

$$P_2 G_2 = H_2 - L_2 W_0, \tag{30$'$}$$

$$W_0' W_0 = Z_2(\infty) + Z_2'(\infty). \tag{31$'$}$$

Then it is easily verified that (29), (30) and (31) are satisfied by taking

$$\begin{aligned} P &= P_1 \dotplus P_2, \\ F &= F_1 \dotplus F_2, \\ G' &= [G_1', G_2'], \\ H' &= [H_1', H_2'], \\ L' &= [0, L_2']. \end{aligned} \tag{36}$$

Moreover, with $\{F_1, G_1, H_1\}$ and $\{F_2, G_2, H_2\}$ minimal realizations for $Z_1(s)$ and $Z_2(s) - Z_2(\infty)$, $\{F, G, H\}$ is a minimal realization for $Z(s) - Z(\infty)$. This is because, by Lemma 2, the degree of Z is the sum of the degrees of Z_1 and Z_2, the latter having no common poles, while the dimension of F is the sum of the dimensions of F_1 and F_2. One should at this stage verify that (29), (30) and (31) are valid under a state space coordinate transformation, as they have merely been established for a particular class of F (i.e., those of the form $F_1 \dotplus F_2$). This is easy to do, however, along the lines given in Theorem 2 for a more particular case.

4. Conclusions. The significance of the theorems in their own right is self-evident. They provide a conceptual link between basic concepts of control theory and network theory. Their proofs have a number of interesting features, such as the necessity to use the particular $W(s)$ in the factorization of $Z(s) + Z'(-s)$, the heavy reliance on the concept of degree in the network [8] and control theory [11] senses, and the canonical $\{F, G, H\}$ representation (believed new) of a "lossless" Z. Hopefully the results themselves as well as their proofs will help forge another link in the growing chain [3], [14] between control and network theory.

There are several immediate applications of the theory. The stability of control systems containing multiple nonlinearities is discussed in [15], a new passive network synthesis of positive real functions and matrices in [16], and the properties of multivariable control systems with linear feedback laws in [17].

REFERENCES

[1] R. E. Kalman, *Lyapunov functions for the problem of Lur'e in automatic control*, Proc. Nat. Acad. Sci. U. S. A., 49 (1963), pp. 201–205.

[2] R. W. Newcomb, *Linear Multiport Synthesis*, McGraw-Hill, New York, 1966.

[3] R. E. Kalman, *On a new characterization of linear passive systems*, Proc. First Allerton Conference on Circuit and System Theory, University of Illinois, Urbana, 1963, pp. 456–470.

[4] V. M. Popov, *Hyperstability and optimality of automatic systems with several control functions*, Rev. Roumaine Sci. Tech. Electrotechn et Energ., 9 (1964), pp. 629–690.

[5] R. E. Kalman, *Mathematical description of linear dynamical systems*, this Journal, 1 (1963), pp. 152–192.

[6] B. D. O. Anderson and R. W. Newcomb, *A canonical simulation of a transfer function matrix*, IEEE Trans. Automatic Control, AC-11 (1966), pp. 319–320.

[7] B. L. Ho and R. E. Kalman, *Effective construction of linear state-variable models from input/output data*, Proc. Third Allerton Conference on Circuit and System Theory, University of Illinois, Urbana, 1965.

[8] B. McMillan, *Introduction to formal realizability theory*, Bell System Tech. J., 31 (1952), pp. 217–279; 31 (1952), pp. 541–600.

[9] R. J. Duffin and D. Hazony, *The degree of a rational matrix function*, J. Soc. Indust. Appl. Math., 11 (1963), pp. 645–658.

[10] D. Hazony, *Elements of Network Synthesis*, Reinhold, New York, 1963.

[11] R. E. Kalman, *Irreducible realizations and the degree of a matrix of rational functions*, J. Soc. Indust. Appl. Math., 13 (1965), pp. 520–544.

[12] J. LaSalle and S. Lefschetz, *Stability by Liapunov's Direct Method with Applications*, Academic Press, New York, 1961.

[13] D. C. Youla, *On the factorization of rational matrices*, IRE Trans. Information Theory, IT-7 (1961), pp. 172–189.

[14] D. C. Youla and P. Tissi, *N-port synthesis via reactance extraction, Part I*, Electrophysics Memo PIBMRI-1309-66, Polytechnic Institute of Brooklyn, Brooklyn, New York, 1966.

[15] B. D. O. Anderson, *Stability of control systems with multiple nonlinearities*, J. Franklin Inst., 281 (1966), pp. 155–160.

[16] B. D. O. Anderson and R. W. Newcomb, *Impedance synthesis via state space techniques*, Rept. SEL-66-024 (TR 6558-5), Stanford Electronics Laboratories, Stanford, California, 1966.

[17] B. D. O. Anderson, *The inverse problem of optimal control*, Rept. SEL-66-038 (TR 6560-3), Stanford Electronics Laboratories, Stanford, California, 1966.