

On the Markov Stability Criterion for Discrete Systems

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Abstract—A criterion for the stability of discrete systems was derived in [1] that is analogous to the Markov stability criterion for continuous systems [2]. This criterion was simplified in [3]. A further simplification, which considers directly the Markov parameters together with some linear conditions, is presented in this paper.

I. INTRODUCTION

Consider the characteristic polynomial of a discrete system given by

$$f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n. \quad (1)$$

The inverse polynomial is

$$z^n f\left(\frac{1}{z}\right) = f^*(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0. \quad (2)$$

The symmetric and antisymmetric parts of $f(z)$ are given by

$$h(z) = \frac{f(z) + f^*(z)}{2} = \alpha_0 z^n + \alpha_1 z^{n-1} + \cdots + \alpha_1 z + \alpha_0 \quad (3)$$

and

$$g(z) = \frac{f(z) - f^*(z)}{2} = \beta_0 z^n + \beta_1 z^{n-1} + \cdots - \beta_1 z - \beta_0. \quad (4)$$

The stability of (1) is guaranteed if and only if

- i) $h(\cdot)$ and $g(\cdot)$ have simple interlacing roots on the unit circle, and
- ii) $|a_n/a_1| < 1$ [4].

To check the interlacing property on the unit circle of $h(\cdot)$ and $g(\cdot)$, they are projected onto the line $(-1, +1)$ to yield polynomials with real zeros identical to the real parts of the zeros of $h(\cdot)$ and $g(\cdot)$. The projection of $h(z)$ and $g(z)$ gives for $n = 2m$ (even)

$$\begin{aligned} P_1(x) &= \sum_{i=0}^{m-1} \alpha_i T_{m-i} + \frac{\alpha_m}{2} \\ P_2(x) &= \sum_{i=0}^{m-1} \beta_i U_{m-i} \end{aligned} \quad (5)$$

where T_k is the k th Chebyshev polynomial of the first kind and U_k is the k th Chebyshev polynomial of the second kind. T_k and U_k can be obtained by the recursions

$$\begin{aligned} T_{k+1}(x) &= 2xT_k(x) - T_{k-1}(x) \\ U_{k+1}(x) &= 2xU_k(x) - U_{k-1}(x) \end{aligned}$$

where

$$T_0(x) = 1, \quad T_1(x) = x, \quad U_0(x) = 0, \quad U_1(x) = 1.$$

Nour-Eldin has shown in [1] using Cauchy indexes that (1) has

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all its roots inside the unit circle iff

$$\begin{aligned} I_{x=-1}^{x=1} \frac{P_2(x)}{P_1(x)} = m &\Leftrightarrow I_{x=-z}^{x=z} \frac{(x+1)P_2(x)}{P_1(x)} = m \quad \text{and} \\ I_{x=-z}^{x=z} \frac{(x-1)P_2(x)}{P_1(x)} &= -m \\ &\Leftrightarrow S_a > 0 \quad \text{and} \quad S_b > 0 \end{aligned} \quad (6)$$

where S_a and S_b are the Hankel matrices

$$S_a = \begin{pmatrix} s_0 + s_1 & s_1 + s_2 & \cdots & s_{m-1} + s_m \\ s_1 + s_2 & s_2 + s_3 & \cdots & s_m + s_{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{m-1} + s_m & s_m + s_{m+1} & \cdots & s_{2m-2} + s_{2m-1} \end{pmatrix} \quad (7a)$$

$$S_b = \begin{pmatrix} s_0 - s_1 & s_1 - s_2 & \cdots & s_{m-1} - s_m \\ s_1 - s_2 & s_2 - s_3 & \cdots & s_m - s_{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{m-1} - s_m & s_m - s_{m+1} & \cdots & s_{2m-2} - s_{2m-1} \end{pmatrix} \quad (7b)$$

and

$$\frac{P_2(x)}{P_1(x)} = \frac{s_0}{x} + \frac{s_1}{x^2} + \frac{s_2}{x^3} + \cdots \quad (8)$$

It was asserted in [3] that only one of the conditions of Nour-Eldin, i.e., $S_a > 0$ or $S_b > 0$, can be considered together with some linear conditions in order to conclude stability. In this paper, another simplification of the stability criterion is obtained.

II. REDUCED MARKOV STABILITY CRITERION FOR DISCRETE SYSTEMS

From (7), we see that a necessary condition for Schur stability is that $S = S_a + S_b > 0$, i.e.,

$$S = \begin{pmatrix} s_0 & s_1 & \cdots & s_{m-1} \\ s_1 & s_2 & \cdots & s_m \\ \vdots & \vdots & \ddots & \vdots \\ s_{m-1} & s_m & \cdots & s_{2m-2} \end{pmatrix} > 0. \quad (9)$$

We remark that (9) is equivalent to $I_{-\infty}^{\infty}(P_2(x))/(P_1(x)) = m$ [2], which means that all the roots of $P_1(x)$ and $P_2(x)$ are real and interlacing.

To get a set of necessary and sufficient conditions that includes (9), we have to exclude the possibility of real roots of $P_1(x)$ lying on the real axis outside the unit circle. This will then ensure that $I_{-\infty}^{\infty}(P_2(x))/(P_1(x)) = I_{-1}^1(P_2(x))/(P_1(x)) = m$, and stability follows from (6). This exclusion can be studied using the bilinear transformation as follows.

Let $x = s + 1/s - 1$. We have

$$T_k(x) = b_0 x^k + b_1 x^{k-1} + \cdots + b_k = \mathbf{b}_k^T \mathbf{x}_k$$

where

$$\begin{aligned} \mathbf{b}_k^T &= [b_k \cdots b_0] \\ \mathbf{x}_k^T &= [1x \cdots x^k] \end{aligned}$$

e.g., $\mathbf{b}_4^T = [8 \ 0 \ -8 \ 0 \ 1]$.

From (5), simple calculations yield

$$\begin{aligned} \hat{P}_1(s) &\triangleq (s-1)^m P_1\left(\frac{s+1}{s-1}\right) \\ &= \sum_{i=0}^{m-1} \alpha_i \mathbf{b}_{m-i}^T \hat{T}_{m-i} s^{m-i} (s-1)^i + \frac{\alpha_m}{2} (s-1)^m \end{aligned} \quad (10)$$

where $\mathbf{s}_k^T = [1 \ s \ \cdots \ s^k]$ and \hat{T}_k is the bilinear transformation

matrix [5], e.g.,

$$\hat{T}_4 = \begin{pmatrix} 1 & -4 & 6 & -4 & 1 \\ -1 & 2 & 0 & -2 & 1 \\ 1 & 0 & -2 & 0 & 1 \\ -1 & -2 & 0 & 2 & 1 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix}$$

The last row contains the binomial coefficients and the last column has unit elements. Every other element is obtained by subtracting from the element directly below it the sum of the two elements to the right of both elements.

Define the coefficients η_i by

$$\hat{P}_1(s) = \eta_0 s^m + \eta_1 s^{m-1} + \dots + \eta_m \quad (11)$$

Then we have the following main theorem.

Theorem 1: Consider the polynomial $f(z)$ of (1), for which is constructed $g(z), h(z), P_1(x), P_2(x), \hat{P}_1(s)$, and the Markov coefficients s_i in the expansion of $(P_2(x))/(P_1(x))$. Then the conditions

$$\begin{aligned} \text{i) } & S > 0 \\ \text{ii) } & \eta_0, \eta_1, \dots, \eta_m > 0 \end{aligned} \quad (12)$$

are necessary and sufficient for stability of the discrete system.

To see the necessity, observe from (6) that all roots of $P_1(x)$ lie in $(-1, 1)$ and thus by (10) all roots of $\hat{P}_1(s)$ lie in $(-\infty, 0)$. This implies (12). Conversely, if $S > 0$, all roots of $P_1(x)$ are real, and thus all of $\hat{P}_1(s)$ are real. Then (12) ensures that they are necessarily in $(-\infty, 0)$. Thus all roots of $P_1(x)$ lie in $(-1, 1)$.

Remark 1: It can be easily shown that

$$\eta_0 > 0 \equiv \hat{P}_1(0) > 0 \equiv P_1(1) > 1 \equiv f(1) > 0 \quad (13)$$

and

$$\eta_m > 0 \equiv \hat{P}_1(0) > 0 \equiv P_1(-1) > 1 \equiv (-1)^m f(-1) > 0. \quad (14)$$

Both (13) and (14) are well-known necessary conditions for stability.

Remark 2: Suppose that a polynomial is obtained from $f(z)$ by bilinear transformation, $\hat{f}(s) = (s-1)^n f((s+1)/(s-1))$. Then the conditions $\eta_0 > 0, \dots, \eta_m > 0$ are a subset of the (linear) conditions that all the coefficients of $\hat{f}(s)$ are positive. The other subset can be obtained from transforming $P_2(x)$ to form $\hat{P}_2(s)$ and requiring its coefficients are positive, but this is not needed for stability. It is noted that, e.g., for n even the first subset is the coefficients of even powers in $\hat{f}(s)$ and the second one is the coefficients of odd powers of $\hat{f}(s)$.

III. EXAMPLE

Let

$$f(z) = z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4$$

$$f^*(z) = a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + 1.$$

The symmetric and antisymmetric parts are given by

$$h(z) = \alpha_0 z^4 + \alpha_1 z^3 + \alpha_2 z^2 + \alpha_1 z + \alpha_0$$

$$g(z) = \beta_0 z^4 + \beta_1 z^3 - \beta_1 z - \beta_0$$

where

$$\alpha_0 = \frac{1+a_4}{2}, \quad \alpha_1 = \frac{a_1+a_3}{2}, \quad \alpha_2 = a_2$$

$$\beta_0 = \frac{1-a_4}{2}, \quad \beta_1 = \frac{a_1-a_3}{2}.$$

It follows that

$$\begin{aligned} P_1(x) &= \alpha_0 T_2(x) + \alpha_1 T_1(x) + \frac{\alpha_2}{2} \\ &= \alpha_0(2x^2 - 1) + \alpha_1 x + \frac{\alpha_2}{2} = 2\alpha_0 x^2 + \alpha_1 x + \frac{\alpha_2}{2} - \alpha_0 \end{aligned}$$

and

$$\begin{aligned} P_2(x) &= \beta_0 U_2(x) + \beta_1 U_1(x) \\ &= \beta_0(2x) + \beta_1 = 2\beta_0 x + \beta_1. \end{aligned}$$

Then

$$\begin{aligned} \frac{P_2(x)}{P_1(x)} &= \frac{2\beta_0 x + \beta_1}{2\alpha_0 x^2 + \alpha_1 x + \frac{\alpha_2}{2} - \alpha_0} \\ &= \frac{s_0}{x} + \frac{s_1}{x^2} + \frac{s_2}{x^3} + \dots \end{aligned}$$

Explicitly, we have

$$s_0 = \frac{\beta_0}{\alpha_0} = \frac{1-a_4}{1+a_4}$$

$$s_1 = s_0 \frac{\alpha_1}{2\alpha_0} - \frac{\beta_1}{2\alpha_0} = \frac{a_3 - a_1 a_4}{(1+a_4)^2}$$

$$s_2 = s_1 \frac{\alpha_1}{2\alpha_0} - \frac{s_0}{\alpha_0} \left(\frac{\alpha_2}{4} - \frac{\alpha_0}{2} \right).$$

The transformed polynomial formed from $P_1(x)$ is

$$\hat{P}_1(s) = \alpha_0 b_2^T \hat{T}_2 s_2 + \alpha_1 b_1^T \hat{T}_1 s_1 (s-1) + \frac{\alpha_2}{2} (s-1)^2$$

where

$$b_2^T = [2 \quad 0 \quad -1], \quad \hat{T}_2 = \begin{pmatrix} 1 & -2 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix},$$

$$s_2^T = [1 \quad s \quad s^2]$$

$$b_1^T = [1 \quad 0], \quad \hat{T}_1 = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix},$$

$$s_1^T = [1 \quad s].$$

Thus

$$\begin{aligned} \hat{P}_1(s) &= \frac{1+a_4}{2} (1+6s+s^2) + \frac{a_1+a_3}{2} (s^2-1) \\ &\quad + \frac{\alpha_2}{2} (s^2-2s+1) \\ &= \frac{1+a_1+a_2+a_3+a_4}{2} s^2 + (3-a_2+3a_4)s \\ &\quad + \frac{1-a_1+a_2-a_3+a_4}{2}. \end{aligned}$$

Necessary and sufficient stability conditions are

$$S > 0 \quad \text{and coefficients of } \hat{P}_1(s) > 0$$

i.e.,

$$\begin{aligned} S > 0 &\equiv s_0 > 0, \quad s_0 s_2 - s_1^2 > 0 \\ f(1) > 0, \quad (3-a_2+3a_4) > 0, \quad f(-1) > 0. \end{aligned}$$

IV. CONCLUSION

It has been shown that the Markov stability criterion for discrete systems developed by Nour-Eldin can be simplified to a positivity test of the Hankel matrix S of low dimension $m = n/2$ (for n even) and to a positivity test of certain linear combinations of the coefficients obtained through bilinear transformation. For n odd the polynomial $zf(z)$ is considered instead of $f(z)$.

REFERENCES

- [1] H. Nour-Eldin, "Ein neues Stabilitätskriterium für abgetastete Regelsysteme," *Regelungstechnik und Prozess-Datenverarbeitung*, Heft 7, pp. 301-307, 1971.
- [2] F. R. Gantmacher, *The Theory of Matrices*, vol. 2. New York: Chelsea, 1964.
- [3] B. D. O. Anderson, E. I. Jury, and L. F. Chapparo, "On the root distribution of a real polynomial with respect to the unit circle," *Regelungstechnik*, Heft 3, pp. 101-102, 1976.
- [4] H. W. Schüssler, "A stability theorem for discrete systems," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-24, pp. 87-89, 1976.
- [5] R. Uebenhauen, "Ein Beitrag zur Stabilitätsuntersuchung linearer Abtastsysteme," *Regelungstechnik*, Heft 1, pp. 12-16, 1964.