

**Sufficient Condition for State-Space Representation of
N-D Discrete-Time Lossless Bounded Real Matrix
and *N*-D Stability of Mansour Matrix**

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Abstract—The definition of an *N*-D discrete time lossless bounded real (DTLBR) matrix and sufficient conditions on a state space representation to correspond to an *N*-D DTLBR matrix are given. Based on these sufficient conditions, it is shown that the transfer function of the *n*-D lattice filter, which is obtained from a stable 1-D lattice filter by replacing each delay element z with z_i ($i = 1, 2, \dots, n$), becomes an *n*-D DTLBR function. From this fact an $n \times n$ Mansour matrix, which is an *A*-matrix of the state space representation of a 1-D lattice filter, is proved to be stable for any dimension up to n , if it is stable in the one-dimensional case.

I. INTRODUCTION

Recently, all-pass functions and matrices have been receiving a great deal of attention in the design of low sensitivity digital and analog filters and filter banks [1]–[6]. This is because

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all-pass functions and matrices are realizable with the particular structure called structurally lossless [6]. In one-dimensional (1-D) digital signal processing, stable all-pass functions and matrices are equivalent to discrete-time lossless bounded real (DTLBR) functions and matrices. Necessary and sufficient conditions for the state-space representation of DTLBR functions and matrices are stated in the DTLBR lemma [7]–[11]. Considering the important properties of the DTLBR functions and matrices in 1-D digital signal processing and recent remarkable progress of multidimensional signal processing [19]–[21], it is natural to ask how 1-D DTLBR functions and matrices can be extended to multidimensional systems.

In this paper, we discuss multidimensional DTLBR matrices from the system theoretical point of view by using generalized Roesser's state space model [18]–[20], [34]. First, the definition of an N -D DTLBR matrix is given, which is a natural extension of the definition of a 1-D DTLBR matrix. Based on this definition, we derive sufficient conditions on an N -D state-space representation for this to correspond to an N -D DTLBR matrix. Next we consider the n -D lattice filter which can be obtained from a 1-D lattice filter by replacing each delay element¹ z with z_i ($i = 1, 2, \dots, n$). We show that the n -D transfer function of this n -D lattice filter is an n -D DTLBR function by using the sufficient condition obtained in this paper. This result means that the well-known Mansour matrix [14], which is the A -matrix of the state equation representation of 1-D lattice filter [15], is always stable in both one dimension and multidimensionally, if the reflection coefficients of the lattice filter are all less than one in magnitude.

Notation: \bar{U}^N denotes the closed unit N -disk:

$$\bar{U}^N = \{(z_1, z_2, \dots, z_N) \mid |z_1| \leq 1, |z_2| \leq 1, \dots, |z_N| \leq 1\}.$$

T^N denotes the distinguished boundary of \bar{U}^N :

$$T^N = \{(z_1, z_2, \dots, z_N) \mid |z_1| = |z_2| = \dots = |z_N| = 1\}.$$

For real square matrix P , $P > 0$ means that P is positive definite symmetric, and for an Hermitian matrix H , $H \geq 0$ means that H is nonnegative definite Hermitian. If z is complex variable z^* denotes the complex conjugate of z . A^T means the transpose of A and \oplus denotes the direct sum of matrices.

II. PRELIMINARIES AND MAIN THEOREM

A 1-D DTLBR matrix is defined as follows [3], [4]:

Definition 1: Let $H(z)$ be a $p \times q$ ($p \geq q$) matrix of real rational functions of the complex variable z . Then $H(z)$ is a 1-D DTLBR matrix if the following conditions hold:

- i) $H(z)$ is analytic in $|z| \leq 1$;
- ii) $I - H(z^*)^T H(z) \geq 0$ in $|z| \leq 1$;
- iii) $I - H(z^{-1})^T H(z) = 0$ for all z .

Then the following lemma holds [7]–[11]:

Lemma 1: Let

$$\begin{aligned} x(i+1) &= Ax(i) + Bu(i) \\ y(i) &= Cx(i) + Du(i) \end{aligned} \quad (1)$$

be a minimal realization of a $p \times q$ transfer function matrix $H(z)$, namely

$$H(z) = D + C[z^{-1}I - A]^{-1}B. \quad (2)$$

¹In order to make the definition of 1-D z -transform conform with the definition of N -D z -transform, in this paper we use complex variable z instead of z^{-1} to express the delay of a 1-D signal.

Then $H(z)$ is a 1-D DTLBR matrix if and only if there exists a positive definite symmetric matrix P such that

$$\begin{aligned} A^T P A + C^T C &= P \\ B^T P B + D^T D &= I \\ A^T P B + C^T D &= 0. \end{aligned} \quad (3)$$

The proof of the above lemma in [7] and [10] is based on the properties of the autocorrelation function of $H(z)$, the proof in [8] (which imposes the restriction that $p = q$) is based on the existence of a similarity transformation matrix between $H(z^{-1})^T$ and $H(z)^{-1}$, and the proof in [9] is based on an energy balance argument between input and output. The proof of the above lemma in [11] is based on Youla's spectral factorization [12] and skillful matrix manipulation, which is similar to the proof of the continuous time LBR lemma in [13].

Now in order to extend the above results to the N -D case, let us give the definition of an N -D DTLBR matrix as follows:

Definition 2: Let $H(z_1, z_2, \dots, z_N)$ be a $p \times q$ ($p \geq q$) matrix of real rational functions of the complex variables z_1, z_2, \dots, z_N . Then $H(z_1, z_2, \dots, z_N)$ is an N -D DTLBR matrix if the following conditions hold.

- i) $H(z_1, z_2, \dots, z_N)$ is analytic in \bar{U}^N .
- ii) $I - H(z_1^*, z_2^*, \dots, z_N^*)^T H(z_1, z_2, \dots, z_N) \geq 0$ in \bar{U}^N .
- iii) $I - H(z_1^{-1}, z_2^{-1}, \dots, z_N^{-1})^T H(z_1, z_2, \dots, z_N) = 0$ for all z_i .

The condition i) in the above guarantees the n -D stability of $H(z_1, z_2, \dots, z_N)$ [17]. It is clear from the above definition that a scalar N -D DTLBR function is an N -D all-pass function. Now let us assume that the state-space representation of a given transfer matrix $H(z_1, z_2, \dots, z_N)$ is given by the following generalized Roesser's model [18]–[20], [34]:

$$\begin{aligned} &\begin{bmatrix} x_1(i_1+1, i_2, \dots, i_N) \\ x_2(i_1, i_2+1, \dots, i_N) \\ \vdots \\ x_N(i_1, i_2, \dots, i_N+1) \end{bmatrix} \\ &= A \begin{bmatrix} x_1(i_1, i_2, \dots, i_N) \\ x_2(i_1, i_2, \dots, i_N) \\ \vdots \\ x_N(i_1, i_2, \dots, i_N) \end{bmatrix} + BU(i_1, i_2, \dots, i_N) \quad (4a) \\ y(i_1, i_2, \dots, i_N) &= C \begin{bmatrix} x_1(i_1, i_2, \dots, i_N) \\ x_2(i_1, i_2, \dots, i_N) \\ \vdots \\ x_N(i_1, i_2, \dots, i_N) \end{bmatrix} + DU(i_1, i_2, \dots, i_N) \end{aligned} \quad (4b)$$

where $x_i \in R^{m_i}$, $i = 1, 2, \dots, N$ represent the states, $U \in R^q$ is the input, $y \in R^p$ is the output, and $A \in R^{n \times n}$, $B \in R^{n \times q}$, $C \in R^{p \times n}$, $D \in R^{p \times q}$, $n = \sum_{i=1}^N m_i$. The relation between the transfer matrix $H(z_1, z_2, \dots, z_N)$ and the coefficient matrices $\{A, B, C, D\}$ in (4) is given by

$$\begin{aligned} H(z_1, z_2, \dots, z_N) \\ = D + C[\Lambda^{-1} - A]^{-1}B = D + CA[I - A\Lambda]^{-1}B \end{aligned} \quad (5)$$

where

$$\Lambda = [z_1 I_1 \oplus z_2 I_2 \oplus \dots \oplus z_N I_N] \in C^{n \times n} \quad (6)$$

and I_i is a unit matrix of order m_i . Henceforth, we use the

notation $\{A, B, C, D\}$ in order to express the model in (4). Then we can state and prove the following theorem.

Theorem 1: Let $\{A, B, C, D\}$ be a realization of the transfer matrix $H(z_1, z_2, \dots, z_N)$, and suppose that $\{A, B, C, D\}$ satisfies the following conditions:

a)

$$\text{rank} \begin{bmatrix} \dots & C\hat{\Lambda} & \dots \\ \dots & I - A\hat{\Lambda} & \dots \end{bmatrix} = n \text{ (full rank)} \quad (7)$$

for any complex numbers $z_i, i = 1, 2, \dots, N$, where $\hat{\Lambda}$ is a diagonal matrix defined by (6) and n is the size of the matrix A .

b) There exists a positive definite real symmetric block diagonal matrix $P = [P_1 \oplus P_2 \oplus \dots \oplus P_N] > 0$, where $P \in R^{m_i \times m_i}$, such that

$$A^T P A + C^T C = P \quad (8a)$$

$$B^T P B + D^T D = I \quad (8b)$$

$$A^T P B + C^T D = 0. \quad (8c)$$

Then $H(z_1, z_2, \dots, z_N)$ is an N -D DTLBR matrix.

Remarks: We should note the following: The condition given in the above Theorem 1 is only sufficient and not necessary. The realization $\{A, B, C, D\}$ of $H(z_1, z_2, \dots, z_N)$ is not assumed to be minimal. The procedure to derive minimal realizations of given N -D DTLBR matrices has not been developed yet, except 2-D first-order DTLBR scalar functions [29], [30] and 2-D square DTLBR matrices [35]. Even if a realization $\{A, B, C, D\}$ of $H(z_1, z_2, \dots, z_N)$ does not satisfy the conditions in Theorem 1, there might exist another realization $\{F, G, H, J\}$ satisfying the hypothesis of Theorem 1. The condition a) is an extension of the observability condition for a 1-D system [22] to an N -D system, and can be referred to as a zero coprimeness condition on the pair $(CA, I - A\hat{\Lambda})$ [23]. In [24], Fornasini and Marchesini's second model [25] (it is well known that Fornasini and Marchesini's second model includes Roesser's 2-D state space model [18] as a particular case) is discussed, and a similar condition to the above condition a) with $N = 2$ is proved to be the necessary and sufficient condition for causal reconstructibility and for the existence of a 2-D exact observer.

III. PROOF OF THEOREM 1

We prove Theorem 1 by showing that if the conditions a) and b) in Theorem 1 hold then the conditions i)–iii) in Definition 2 hold.

Proof of a) and b) \Rightarrow i): Suppose that $(\hat{z}_1, \hat{z}_2, \dots, \hat{z}_N)$ makes $I - A\hat{\Lambda}$ singular, where

$$\hat{\Lambda} = [\hat{z}_1 I_1 \oplus \hat{z}_2 I_2 \oplus \dots \oplus \hat{z}_N I_N] \in C^{n \times n} \quad (9)$$

and let

$$\hat{x} \in C^n, \quad \hat{x} \neq 0 \quad (10)$$

be such that

$$[I - A\hat{\Lambda}]\hat{x} = 0. \quad (11)$$

In (11) $\hat{z}_i, i = 1, 2, \dots, N$ are N -D eigenvalues of A and \hat{x} is an N -D eigenvector of A . By applying (11) to (8a) in condition b), we obtain the following:

$$\begin{aligned} \hat{x}^* T \left[(1 - |\hat{z}_1|^2) P_1 \oplus (1 - |\hat{z}_2|^2) P_2 \oplus \dots \oplus (1 - |\hat{z}_N|^2) P_N \right] \hat{x} \\ = -\hat{x}^* T \hat{\Lambda}^* C^T C \hat{\Lambda} \hat{x}. \end{aligned} \quad (12)$$

From the rank condition in condition a) and (11) we should note

that

$$\begin{bmatrix} \dots & C\hat{\Lambda} & \dots \\ \dots & I - A\hat{\Lambda} & \dots \end{bmatrix} \hat{x} = \begin{bmatrix} C\hat{\Lambda} \hat{x} \\ 0 \end{bmatrix} \neq 0. \quad (13)$$

Hence the right side of (12) is negative. Therefore, in the left side of (12), since $P_i, i = 1, 2, \dots, N$ are positive definite, at least one of the eigenvalues $\hat{z}_1, \hat{z}_2, \dots, \hat{z}_N$, say \hat{z}_k , must satisfy the inequality

$$|\hat{z}_k| > 1. \quad (14)$$

It follows from this result and (11) that

$$\det[I - A\hat{\Lambda}] \neq 0 \text{ in } \bar{U}^N. \quad (15)$$

Hence from (5) and (15) we obtain the condition i) in Definition 2.

Proof of a) and b) \Rightarrow ii): From (8) and (5) we have

$$\begin{aligned} I - H(z_1^*, z_2^*, \dots, z_N^*)^T H(z_1, z_2, \dots, z_N) \\ = I - \{D^T + B^T [I - \Lambda^* A^T]^{-1} \Lambda^* C^T\} \{D + CA [I - A\hat{\Lambda}]^{-1} B\} \\ = B^T [I - \Lambda^* A^T]^{-1} \left[(1 - |z_1|^2) P_1 \oplus (1 - |z_2|^2) P_2 \oplus \dots \right. \\ \left. \oplus (1 - |z_N|^2) P_N \right] [I - A\hat{\Lambda}]^{-1} B. \end{aligned} \quad (16)$$

Since

$$\left[(1 - |z_1|^2) P_1 \oplus (1 - |z_2|^2) P_2 \oplus \dots \oplus (1 - |z_N|^2) P_N \right] \geq 0 \text{ in } \bar{U}^N \quad (17)$$

and $[I - A\hat{\Lambda}]$ is nonsingular in \bar{U}^N (see (15)), (16) establishes the condition ii) in Definition 2.

Proof of a) and b) \Rightarrow iii): Again from (5) and (8) we have

$$\begin{aligned} I - H(z_1^{-1}, z_2^{-1}, \dots, z_N^{-1})^T H(z_1, z_2, \dots, z_N) \\ = B^T [I - \Lambda^{-1} A^T]^{-1} \left[(1 - z_1^{-1} z_1) P_1 \oplus (1 - z_2^{-1} z_2) P_2 \oplus \dots \right. \\ \left. \oplus (1 - z_N^{-1} z_N) P_N \right] [I - A\hat{\Lambda}]^{-1} B \\ = 0. \end{aligned} \quad (18)$$

Therefore, condition iii) holds and the proof of Theorem 1 is complete.

The first part of the above proof is partly similar to the proof of N -D Lyapunov stability in [26]. The remaining part of the above proof may be said to be similar to the sufficiency proof of the 1-D DTLBR lemma in [11] or to the sufficiency proof of the continuous time LBR lemma in [13], in the sense that the matrix manipulations resemble each other. Now we present an illustrative example for Theorem 1.

Example 1: Consider the following 2-input and 2-output 2-D system:

$$\begin{aligned} \begin{bmatrix} x_1(i_1 + 1, i_2) \\ x_2(i_1, i_2 + 1) \end{bmatrix} &= A \begin{bmatrix} x_1(i_1, i_2) \\ x_2(i_1, i_2) \end{bmatrix} + Bu(i_1, i_2) \\ y(i_1, i_2) &= C \begin{bmatrix} x_1(i_1, i_2) \\ x_2(i_1, i_2) \end{bmatrix} + Du(i_1, i_2) \end{aligned}$$

where

$$A = \begin{bmatrix} -\frac{1}{4} & \frac{3}{4} \\ -\frac{3}{10} & -\frac{3}{10} \end{bmatrix}; \quad B = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{16}{15} & \frac{3}{5} \end{bmatrix}$$

$$C = \begin{bmatrix} -\frac{3}{10} & -\frac{3}{10} \\ \frac{3}{4} & 0 \end{bmatrix}; \quad D = \begin{bmatrix} -\frac{3}{5} & \frac{3}{5} \\ 0 & \frac{1}{2} \end{bmatrix}$$

Let

$$P = \begin{bmatrix} 3 & 9 \\ 4 & 16 \end{bmatrix}.$$

It can be easily checked that $\{A, B, C, D\}$ and P in the above satisfy the condition a) and b) in Theorem 1. Hence by Theorem 1 the transfer function matrix of the above system becomes a 2-D DTLBR matrix, which is given by

$$H(z_1, z_2) = \frac{1}{f} \begin{bmatrix} -g & h \\ \tilde{h} & \tilde{g} \end{bmatrix}$$

where

$$f = f(z_1, z_2) = 1 + \frac{1}{4}z_1 + \frac{3}{10}z_2 + \frac{3}{10}z_1z_2$$

$$\tilde{g} = g(z_1, z_2) = \frac{3}{5} + \frac{3}{20}z_1 + \frac{1}{2}z_2 + \frac{1}{2}z_1z_2$$

$$h = h(z_1, z_2) = \frac{3}{5}$$

$$\tilde{f} = z_1z_2f(z_1^{-1}, z_2^{-1}) = z_1z_2 + \frac{1}{4}z_2 + \frac{3}{10}z_1 + \frac{3}{10}$$

$$\tilde{g} = z_1z_2g(z_1^{-1}, z_2^{-1}) = \frac{3}{5}z_1z_2 + \frac{3}{20}z_2 + \frac{1}{2}z_1 + \frac{1}{2}$$

$$\tilde{h} = z_1z_2h(z_1^{-1}, z_2^{-1}) = \frac{3}{5}z_1z_2.$$

Indeed, using Huang's theorem [27] we can prove that the denominator polynomial f of $H(z_1, z_2)$ is devoid of zeros in \bar{U}^2 and direct calculation proves that $H(z_1, z_2)$ has the all-pass property

$$\begin{aligned} H(z_1^{-1}, z_2^{-1})^T H(z_1, z_2) &= \frac{1}{\tilde{f}} \begin{bmatrix} -\tilde{g} & h \\ \tilde{h} & g \end{bmatrix} \frac{1}{f} \begin{bmatrix} -g & h \\ \tilde{h} & \tilde{g} \end{bmatrix} \\ &= \frac{1}{\tilde{f}f} \begin{bmatrix} \tilde{g}g + h\tilde{h} & 0 \\ 0 & \tilde{h}h + g\tilde{g} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Hence $H(z_1, z_2)$ satisfies the condition i) and iii) in Definition 2. It is troublesome to show that $H(z_1, z_2)$ also satisfies the remaining condition ii) in Definition 2. However, a matrix version of the maximum modulus theorem guarantees [16] that the condition ii) in Definition 2 automatically holds if the conditions i) and iii) in Definition 2 hold. Therefore by Definition 2, $H(z_1, z_2)$ is certainly a 2-D DTLBR matrix.

IV. n -D DTLBR FUNCTION GENERATED BY 1-D LATTICE FILTER

In 1-D digital signal processing, much application of the lattice filters has been done in the areas such as linear prediction, speech synthesis, adaptive signal processing, and so on. When we consider N -D digital signal processing it is natural to ask what the N -D counterpart of a 1-D lattice filter is. In this section we introduce an n -D lattice filter of a particular form, which is generated from a 1-D lattice filter, and by using Theorem 1 we show that the transfer function of this lattice filter becomes an n -D DTLBR function.

It is well known that a 1-D DTLBR function is a stable all-pass function, and it is realizable with the lattice filter shown in Fig. 1 [15], [16], where $\Delta_i, i = 1, 2, \dots, n$ are reflection coefficients and satisfy $|\Delta_i| < 1$. The state-space representation of

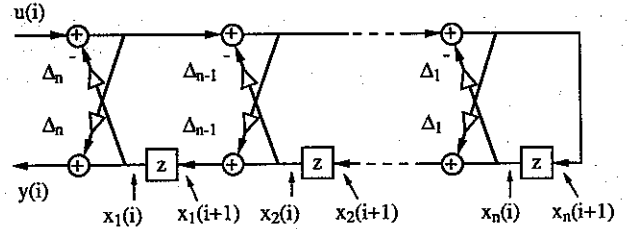


Fig. 1. 1-D n th-order lattice filter.

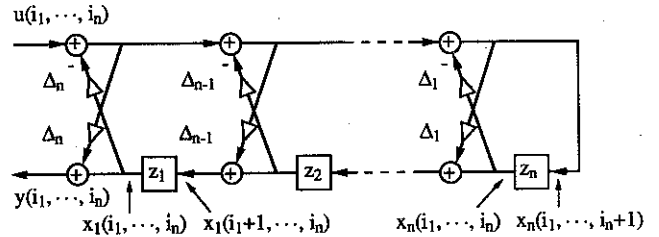


Fig. 2. n -D first-order lattice filter generated from the filter in Fig. 1.

Fig. 1 is given by

$$\begin{aligned} x(i+1) &= Ax(i) + bu(i) \\ y(i) &= cx(i) + du(i) \end{aligned} \quad (19)$$

where

$$x(i) = [x_1(i) \ x_2(i) \ \dots \ x_n(i)]^T \in R^n \quad (20)$$

$$A = \begin{bmatrix} -\Delta_n \Delta_{n-1} & 1 - \Delta_n^2 & 0 & \dots & \dots & 0 \\ -\Delta_n \Delta_{n-2} & -\Delta_{n-1} \Delta_{n-2} & 1 - \Delta_n^2 & 0 & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ -\Delta_n \Delta_1 & -\Delta_{n-1} \Delta_1 & \dots & \dots & -\Delta_2 \Delta_1 & 1 - \Delta_n^2 \\ -\Delta_n & -\Delta_{n-1} & \dots & \dots & -\Delta_2 & -\Delta_1 \end{bmatrix}$$

$$b = [\Delta_{n-1} \ \Delta_{n-2} \ \dots \ \Delta_1 \ 1]^T$$

$$c = [1 - \Delta_n^2 \ 0 \ \dots \ 0 \ 0]$$

$$d = \Delta_n. \quad (21)$$

The matrix A in (21) is called the Mansour matrix and it is known to be stable if and only if $|\Delta_i| < 1, i = 1, 2, \dots, n$ [14]. To $\{A, b, c, d\}$ in (21) with $|\Delta_i| < 1, i = 1, 2, \dots, n$, 1-D DTLBR lemma [9] guarantees that there exists a unique positive definite symmetric matrix P such that

$$A^T P A + c^T c = P \quad (22a)$$

$$b^T P b + d^2 = 1 \quad (22b)$$

$$A^T P b + c^T d = 0. \quad (22c)$$

The explicit solution P to (22) is given by the following diagonal matrix [14].

$$P = \text{diag} \left\{ 1 - \Delta_n^2, (1 - \Delta_n^2)(1 - \Delta_{n-1}^2), \dots, \prod_{i=0}^{n-1} (1 - \Delta_{n-i}^2) \right\}. \quad (23)$$

Clearly P in (23) is positive definite if and only if $|\Delta_i| < 1, i = 1, 2, \dots, n$.

Now let us replace the i th delay element z from the left of the lattice filter in Fig. 1 with z_i for each $i (i = 1, 2, \dots, n)$. Then we obtain the n -D lattice filter in Fig. 2. The state-space representation of Fig. 2 is given by

$$\begin{aligned} x'(i_1, i_2, \dots, i_n) &= Ax(i_1, i_2, \dots, i_n) + bu(i_1, i_2, \dots, i_n) \\ y(i_1, i_2, \dots, i_n) &= cx(i_1, i_2, \dots, i_n) + du(i_1, i_2, \dots, i_n) \end{aligned} \quad (24)$$

where A , b , c , and d in (24) is given by (21) and

$$x'(i_1, i_2, \dots, i_n) = \begin{bmatrix} x_1(i_1+1, i_2, \dots, i_n) \\ x_2(i_1, i_2+1, \dots, i_n) \\ \vdots \\ x_n(i_1, i_2, \dots, i_n+1) \end{bmatrix} \in R^n \quad (25)$$

$$x(i_1, i_2, \dots, i_n) = \begin{bmatrix} x_1(i_1, i_2, \dots, i_n) \\ x_2(i_1, i_2, \dots, i_n) \\ \vdots \\ x_n(i_1, i_2, \dots, i_n) \end{bmatrix} \in R^n. \quad (26)$$

The associated n -D transfer function of (24), which is the transfer function of the n -D lattice filter in Fig. 2, becomes

$$H(z_1, z_2, \dots, z_n) = d + c[\Lambda^{-1} - A]^{-1}b = d + c\Lambda[I - A\Lambda]^{-1}b \quad (27)$$

where

$$\Lambda = \text{diag}\{z_1, z_2, \dots, z_n\} \in C^{n \times n}. \quad (28)$$

Then we can state and prove the following theorem.

Theorem 2: The n -D transfer function $H(z_1, z_2, \dots, z_n)$ of the n -D lattice filter in Fig. 2, which is given by (27), becomes an n -D DTLBR function if and only if $|\Delta_i| < 1$, $i = 1, 2, \dots, n$.

Theorem 2 can be proved easily by using Theorem 1 and the state space representation of Fig. 2, namely (24), where $\{A, b, c, d\}$ in (24) is given by (21).

Proof Sufficiency: It will suffice to show that $\{A, b, c, d\}$ in (21) satisfies the conditions (a) and (b) in Theorem 1 if $|\Delta_i| < 1$, $i = 1, 2, \dots, n$. Note that the condition (b) is already satisfied by (22) and (23). Hence it remains to show that the condition (a) holds.

Suppose that $|\Delta_i| < 1$ for $i = 1, 2, \dots, n$ and for some z_i , $i = 1, 2, \dots, n$, there exists a nonzero column vector such that

$$x = [x_1, x_2, \dots, x_n] (\neq 0) \in C^n \quad (29a)$$

$$\begin{bmatrix} c\Lambda \\ \dots \\ I - A\Lambda \end{bmatrix} x = \begin{bmatrix} (1 - \Delta_n^2)z_1 & 0 & \dots & \dots & 0 \\ 1 + \Delta_n\Delta_{n-1}z_1 & (-1 + \Delta_{n-1}^2)z_2 & 0 & \dots & 0 \\ \vdots & 1 + \Delta_{n-1}\Delta_{n-2}z_2 & (-1 + \Delta_{n-2}^2)z_3 & \dots & \vdots \\ \vdots & \vdots & \vdots & \dots & 0 \\ \Delta_n\Delta_1z_1 & \Delta_{n-1}\Delta_1z_2 & \vdots & \dots & (-1 + \Delta_1^2)z_n \\ \Delta_nz_1 & \Delta_{n-1}z_2 & \vdots & \dots & 1 + \Delta_1z_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = 0 \quad (29b)$$

where Λ is given by (28).

Now since x in (29) is a nonzero vector there exists at least one nonzero element among x_1, x_2, \dots, x_n . Let us check which element is nonzero from x_1 in the order of x_1, x_2, \dots, x_n and suppose that there exist k nonzero elements, say, $x_{i_1}, x_{i_2}, \dots, x_{i_k}$, where

$$1 \leq i_1 < i_2 < \dots < i_k \leq n. \quad (30)$$

Then since the matrix in (29b) is in a lower triangular form, it follows from the i_1 th row in (29b) that

$$(-1 + \Delta_{n+1-i_1}^2)z_{i_1}x_{i_1} = 0 \quad \text{and} \quad x_{i_1} \neq 0. \quad (31)$$

Since $|\Delta_{n+1-i_1}| < 1$ it follows from (31) that

$$z_{i_1} = 0. \quad (32)$$

Next, substitute (32) into (29b). Then it follows from the i_2 th

row in (29b) that

$$(-1 + \Delta_{n+1-i_2}^2)z_{i_2}x_{i_2} = 0 \quad \text{and} \quad x_{i_2} \neq 0. \quad (33)$$

Since $|\Delta_{n+1-i_2}| < 1$ we obtain

$$z_{i_2} = 0. \quad (34)$$

Repeating the above discussion to each row in (29b), finally we have

$$z_{i_1} = z_{i_2} = \dots = z_{i_k} = 0. \quad (35)$$

On the other hand, it follows from (22a) and the nonzero column vector x in (29) that

$$x^{*T}\Lambda^*(A^T P A - P)\Lambda x = -x^{*T}\Lambda^*c^T c \Lambda x. \quad (36)$$

Substitution of the relations $x = A\Lambda x$ and $c\Lambda x = 0$ in (29b) and the positive definite diagonal matrix P in (23) into (36) yields

$$\sum_{i=1}^n (1 - |z_i|^2) p_i |x_i|^2 = 0 \quad (37)$$

where p_i denotes the i th diagonal element of the positive definite diagonal matrix P in (23), namely,

$$p_i = \prod_{j=0}^{i-1} (1 - \Delta_{n-j}^2). \quad (38)$$

Substituting (35) into (37) and noting that $x_i = 0$ except $x_{i_1}, x_{i_2}, \dots, x_{i_k}$, (37) becomes

$$\sum_{s=1}^k p_{i_s} |x_{i_s}|^2 = 0. \quad (39)$$

Since p_{i_s} is positive we have from (39) that $x_{i_1} = x_{i_2} = \dots = x_{i_k} = 0$, which contradicts the previous supposition that $x_{i_1}, x_{i_2}, \dots, x_{i_k}$ are all nonzero. This implies that no nonzero column vector x satisfying (29b) can exist if $|\Delta_i| < 1$ for $1 \leq i \leq n$.

Therefore the condition (b) also holds if $|\Delta_i| < 1$, $i = 1, 2, \dots, n$.

Necessity: If $H(z_1, z_2, \dots, z_n)$ in (27) is an n -D DTLBR function, it is easy to see that

$$H(z, z, \dots, z) = d + c[z^{-1}I - A]^{-1}b \quad (40)$$

becomes a 1-D n th-order DTLBR function, where I is an $n \times n$ identity matrix. Hence the matrix A in (21) is stable. Then it follows from the property of the Mansour matrix that $|\Delta_i| < 1$, $i = 1, 2, \dots, n$ and the proof is complete.

From the above proof of Theorem 2 we can easily obtain the following Theorem 3.

Theorem 3: The $n \times n$ Mansour matrix A in (21) defines a stable system of any dimension up to n if and only if it defines a stable 1-D system, or equivalently $|\Delta_i| < 1$, $i = 1, 2, \dots, n$.

Proof: If $|\Delta_i| < 1$, $i=1,2,\dots,n$, as we have shown in the proof of Theorem 2, the pair (c, A) in (21) satisfies the rank condition a) in Theorem 1. Then by similar discussion to that used in the proof of Theorem 1, we obtain

$$\det[I - A\Lambda] \neq 0 \quad \text{in } \bar{U}^n \quad (41)$$

where A is the $n \times n$ Mansour matrix in (21), Λ is a diagonal matrix in (28), and in this case $N=n$. It follows from (41) that the $n \times n$ Mansour matrix defines a stable system with any dimension up to n .

Remark: From the combination of Theorems 2 and 3 we can state the following: Replacing each delay element of the 1-D stable lattice filter in Fig. 1 with z_1 or z_2 arbitrarily, we can obtain a high order 2-D lattice filter. Similarly if we replace the delay elements z in Fig. 1 with k -types ($k \geq 2$) of delay elements z_1, z_2, \dots, z_k , we can obtain various k -D high order lattice filters. These k -D lattice filters are all stable, structurally lossless, and the transfer functions become k -D DTLBR functions. In addition, they could be expected to inherit good coefficient sensitivity from the original 1-D lattice filter. However, in view of the filter structure, the structure of the k -D lattice filter obtained by the above method is too simple to realize an arbitrary k -D DTLBR function. The well-posed structure of the n -D lattice filter which covers the whole n -D DTLBR functions should be more complicated one. Regarding the meaningful and natural extension of 1-D lattice filters and the Mansour matrix to multidimensional systems, we have few results [26], [37], and a deeper discussion is needed.

Example 2: Assume that $n=2$ in Fig. 2. Then the state-space representation of the corresponding lattice filter becomes

$$\begin{aligned} & \begin{bmatrix} x_1(i_1+1, i_2) \\ x_2(i_1, i_2+1) \end{bmatrix} \\ &= \begin{bmatrix} -\Delta_2\Delta_1 & 1-\Delta_1^2 \\ -\Delta_2 & -\Delta_1 \end{bmatrix} \begin{bmatrix} x_1(i_1, i_2) \\ x_2(i_1, i_2) \end{bmatrix} + \begin{bmatrix} \Delta_1 \\ 1 \end{bmatrix} u(i_1, i_2) \\ y(i_1, i_2) &= [1-\Delta_2^2 \quad 0] \begin{bmatrix} x_1(i_1, i_2) \\ x_2(i_1, i_2) \end{bmatrix} + \Delta_2 u(i_1, i_2) \end{aligned}$$

where $|\Delta_1| < 1$ and $|\Delta_2| < 1$. Then by Theorem 2 the transfer function of this lattice filter, which is given by

$$H(z_1, z_2) = \frac{\Delta_2 + \Delta_1 z_1 + \Delta_1 \Delta_2 z_2 + z_1 z_2}{1 + \Delta_1 \Delta_2 z_1 + \Delta_1 z_2 + \Delta_2 z_1 z_2}$$

is a 2-D DTLBR function. Clearly $H(z_1, z_2)$ has the all-pass property. Using Huang's theorem [27] it can be easily proved that the denominator polynomial of $H(z_1, z_2)$ is devoid of zeros in \bar{U}^2 . Furthermore, $H(z_1, z_2)$ satisfies

$$\begin{aligned} & 1 - H(z_1^*, z_2^*) H(z_1, z_2) \\ &= \frac{(1 - \Delta_2^2) \{ |1 + \Delta_1 z_2|^2 - |z_1|^2 |\Delta_1 + z_2|^2 \}}{|f|^2} \\ &\geq \frac{(1 - \Delta_2^2) \{ |1 + \Delta_1 z_2|^2 - |\Delta_1 + z_2|^2 \}}{|f|^2} \\ &= \frac{(1 - \Delta_2^2)(1 - \Delta_1^2)(1 - |z_2|^2)}{|f|^2} \geq 0 \quad \text{in } \bar{U}^2 \end{aligned}$$

where f denotes the denominator polynomial of $H(z_1, z_2)$. Therefore the conditions i)–iii) in Definition 2 are satisfied, which proves that certainly $H(z_1, z_2)$ is a 2-D DTLBR function.

V. CONCLUSION AND DISCUSSION

We have defined an N -D DTLBR matrix as the extension of a 1-D DTLBR matrix and given a sufficient condition on a state space representation of the matrix for the N -D DTLBR property. Using this sufficient condition, we have proved that the transfer function of the n -D lattice filter which is generated from a 1-D lattice filter becomes an n -D DTLBR function. We have also shown that the Mansour matrix, which appears as the A -matrix of the state space representation of a 1-D lattice filter, is always stable in the sense of any dimension if it is stable in the sense of 1-D.

The important problem which remains untreated in this paper is how to derive necessary conditions (or better, necessary and sufficient conditions) on the state space representation of an N -D DTLBR matrix. Considering the case of the 1-D DTLBR matrix [7]–[11], it is easy to imagine that this problem is closely related to the minimal delay realization problem for an N -D DTLBR matrix and the problem of the existence of a similarity transformation matrix between any two minimal delay realization of an N -D DTLBR matrix. General algorithms for the minimal delay realization of a multidimensional system have not been found yet, and indeed, do not exist in general [28]. But since an N -D DTLBR matrix is a particular matrix, there is a possibility of finding such a minimal delay realization algorithm. For 2-D first-order DTLBR functions and 2-D square DTLBR matrices, such a minimal delay realization algorithm has already been obtained [29], [30], [35]. Alternatively, it may be that special nonminimal realizations could be used for which the derived results could be established.

Finally we would like to refer to the continuous time version of the results of this paper. Basically it is possible to translate the whole results of this paper into continuous time systems. The definition of N -D continuous time lossless bounded real matrix can be easily obtained from Definition 2 with some minor changes. It is not difficult to derive a theorem which corresponds to Theorem 1. Its proof will be similar to that of Theorem 1. Regarding Theorems 2 and 3, if we consider the tridiagonal matrix called the Schwarz form [31], it is possible to derive the continuous time version of Theorems 2 and 3. This is because, under a certain condition, just as in the case of Mansour matrix (see (22a) and (23)), the solution to the Lyapunov matrix equation for a stable Schwarz matrix becomes a positive diagonal matrix [32], [33].

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