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Inner functions and a pseudo-singular-value decomposition in super-optimal H^∞ control

MI-CHING TSAI†, DA-WEI GU†, IAN POSTLETHWAITE† and BRIAN D. O. ANDERSON‡

The inner functions and pseudo-singular-value decomposition (PSVD) which arise in super optimal H^∞ control are studied. New properties of non-square inner matrices are derived and a *structural* inner matrix is defined as an inner matrix with no transmission zeros. The uniqueness properties of the PSVD are derived and structural inner functions are used to characterize particular PSVDs.

1. Introduction

In this note, we study inner functions and a pseudo-singular-value decomposition which arise in H^∞ super-optimal control. Inner functions are relevant to many aspects of control theory and have received careful attention in the literature, e.g. Green and Anderson (1987). Most of the published results, however, are for square matrices and so in this paper we are mainly concerned with non-square inner functions. A structural inner matrix is defined as an inner matrix with no transmission zeros and is useful, for example, in characterizing a particular structure of pseudo-singular-value decomposition (PSVD) which is the other main topic of the paper.

Singular-value decomposition (SVD) is an important tool in numerical linear algebra, and SVDs of frequency-response matrices have an important role to play in the robustness analysis of feedback control systems (Hung and MacFarlane 1982, Klema and Laub 1980). Unfortunately, it is generally not possible formally to decompose a real-rational transfer function matrix into its SVD. Hence design techniques, such as Hung and MacFarlane (1982), which aim to manipulate singular values can do so effectively only at isolated frequencies. In our work and others on super-optimal H^∞ design (Limebeer *et al.* 1988, Postlethwaite *et al.* 1989 a, Tsai *et al.* 1988 b, Yeh and Hwang 1988, Young 1986) we found that a particular real-rational transfer function matrix can always be factorized as the product of two all-pass matrices and a diagonal matrix, similar to the SVD for constant matrices. This factorization, which we call a pseudo-singular-value decomposition, is therefore potentially useful in design. Note, however, that not all real-rational matrices have a PSVD, and the PSVD is not unique. In this note we derive the uniqueness properties of the PSVD.

The paper is organized as follows. In § 2 we study inner functions and define a *structural* inner matrix that is a non-square inner matrix with no transmission zeros. In § 3 we define the PSVD, derive its uniqueness properties and characterize a particular structure of PSVD for super-optimal H^∞ control.

We end this introduction with some mathematical preliminaries and nomencla-

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† Department of Engineering, University of Leicester, Leicester, LE1 7RH, England, U.K.

‡ Department of Systems Engineering, Research School of Physical Science, The Australian National University, GPO Box 4, Canberra, ACT 2601, Australia.

ture. Let $\mathbb{R}\mathbf{L}^{\infty}_{p\times q}$ denote the space of $p \times q$ proper, real-rational matrix functions with no poles on the $j\omega$-axis. $\mathbb{R}\mathbf{H}^{\infty}$ denotes the subspace of $\mathbb{R}\mathbf{L}^{\infty}$ with no poles in the closed right half-plane. G will be called biproper if G is proper and $G(\infty)$ is full rank. The hermitian transpose of a function $G(s) \in \mathbb{R}\mathbf{L}^{\infty}$ is defined by $[G(-s)]^{\mathrm{T}}$ and denoted by $G^{\sim}$. The state-space realization of a real-rational matrix $G(s) = C(sI - A)^{-1}B + D$ is written as

$$G = (A, B, C, D) \quad \text{or} \quad G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

The controllability gramian L_c and observability gramian L_0 of $G \in \mathbb{R}\mathbf{H}^{\infty}$ are defined as the symmetric solutions of the Lyapunov equations

$$AL_c + L_c A^{\mathrm{T}} + BB^{\mathrm{T}} = 0 \tag{1}$$

$$A^{\mathrm{T}}L_0 + L_0 A + C^{\mathrm{T}}C = 0 \tag{2}$$

Let $\lambda_i(\Phi)$ and $s_i(\Phi)$ denote the ith eigenvalue and the ith singular value of the matrix Φ, respectively. The quantities

$$\sigma_i(G) := [\lambda_i(L_c L_0)]^{1/2} \tag{3}$$

are called the Hankel singular values of G. For a given $G(s) \in \mathbb{R}\mathbf{H}^{\infty}_{p\times q}$, the L^{∞} model-matching, or Nehari, problem (Glover 1984) is to find a completely unstable Q such that

$$\| G - Q \|_{\infty} := \max_{\omega} s_1((G - Q)(j\omega)) \tag{4}$$

is minimized. Then it is well known that the optimal cost (4) is equal to the largest Hankel singular value of G. However, in super-optimal H^{∞} control the following strengthened model-matching problem (SMMP) (Tsai *et al.* 1988 b, Young 1986) is solved by finding a Q such that the sequence

$$s_1^{\infty}(G - Q),\ s_2^{\infty}(G - Q),\ \ldots,\ s_m^{\infty}(G - Q), \quad m = \min(p, q) \tag{5}$$

is minimized with respect to the lexicographic ordering where

$$s_i^{\infty}(G - Q) := \max_{\omega} s_i((G - Q)(j\omega)) \tag{6}$$

The SMMP is of theoretical interest because its solution is unique, but it has some engineering motivation, as argued for example by Foo and Postlethwaite (1985) and Kwakernaak (1986). Further details on engineering motivation are given at the end of § 3. Much effort has been devoted to the development of algorithms for super-optimal control (Gu *et al.* 1989, Kwakernaak 1986, Limebeer *et al.* 1988, Postlethwaite *et al.* 1989 a, Tsai *et al.* 1988 b, Yeh and Hwang 1988, Young 1986).

2. Inner transfer functions

Inner functions are relevant to many aspects of control theory, especially in H^{∞} optimization (Doyle 1984), and have received some careful attention in the literature, e.g. Green and Anderson (1987). Here we explore more properties of inner functions and define a 'structural' inner matrix. The latter is useful in characterizing a particular pseudo-singular-value decomposition, defined in the next section, which arises quite naturally from super-optimal algorithms (Limebeer *et al.* 1988, Tsai *et al.* 1988 b, Young 1986).

The function $G \in \mathbb{R}\mathbf{L}^\infty_{p\times q}$ is called 'all-pass' if $G^\sim G = I$, and is called 'inner' when in addition G is stable. Note that to be an inner matrix, G need not be square but with dimensions $p \times q$ it must have $p \geqslant q$. If G is a non-square inner matrix, then a matrix $G_\perp$ can be found such that $[G \quad G_\perp]$ is square and inner; $G_\perp$ is called a complementary inner factor (CIF) of G. In what follows, we assume that G is a $p \times q$ real-rational function with $p \geqslant q$ and that G is full rank in the field of real-rational functions (i.e. rank $(G) = q$). A realization (A, B, C, D) of G will be assumed to be minimal, if it is not otherwise specified, and the gramians of G are defined as in (1) and (2).

The next lemma is useful in characterizing inner functions in terms of a state-space realization.

Lemma 1 (Glover 1984)

Let $G = (A, B, C, D)$ (not necessarily minimal) be in $\mathbb{R}\mathbf{H}^\infty$. Then G is inner if and only if

$$D^{\mathrm{T}}C + B^{\mathrm{T}}L_0 = 0$$

$$D^{\mathrm{T}}D = I$$

When G is square and inner, the realization (A, B, C, D) is minimal if and only if

$$L_0 L_c = I$$

It is easy to see from this lemma, as in Green and Anderson (1987), that all the non-zero Hankel singular values of a square inner matrix are equal to one. Some properties of square all-pass and square inner functions were discussed by Green and Anderson (1987); here, we are mainly concerned with non-square inner matrices. To define and construct a structural inner function we need to take a closer look at the zeros of non-square inner matrices.

We first define coprime factors and coprime factorizations in $\mathbb{R}\mathbf{H}^\infty$.

Definition 1

Two transfer matrices E and F in $\mathbb{R}\mathbf{H}^\infty$ are *right coprime* over $\mathbb{R}\mathbf{H}^\infty$ if they have an equal number of columns and every greatest *common right* divisor of E and F is invertible in $\mathbb{R}\mathbf{H}^\infty$. Equivalently, E and F are right coprime if they have an equal number of columns and there exist matrices X and Y in $\mathbb{R}\mathbf{H}^\infty$ such that

$$XE + YF = I$$

It is possible to represent any real-rational, proper transfer function in terms of a pair of asymptotically stable, real-rational, proper transfer functions that are right coprime. This is called a right coprime factorization (RCF), and is defined next.

Definition 2

For a real-rational, proper matrix G, a *right coprime factorization* of G is a factorization $G = NM^{-1}$ where N and M are right coprime matrices in $\mathbb{R}\mathbf{H}^\infty$.

A right coprime factorization is also called of *balanced-degree* if

(*a*) $\deg(N) = \deg(M) = \deg(G)$ (where $\deg(\cdot)$ denotes the McMillan degree) and

(*b*)
$$\begin{bmatrix} N(s) \\ M(s) \end{bmatrix}$$

is full column rank for all s in the complex plane.

There is an arbitrarily large number of coprime factorizations for any one transfer function. However, the coprime factors (N, M) of G are unique to within right multiplication by a unit in $\mathbb{R}\mathbf{H}^\infty$. For any right coprime factor (N, M), we have

$$N^{\sim}N + M^{\sim}M = \Phi^{\sim}\Phi > 0, \quad \forall\, s = j\omega \tag{7}$$

where Φ is a unit in $\mathbb{R}\mathbf{H}^\infty$. When $\Phi = I$, (N, M) is called 'normalized' right coprime. However, for the case $\Phi \neq I$, a normalized coprime factor can easily be constructed as $(N\Phi^{-1}, M\Phi^{-1})$.

Definition 3

Let $G = NM^{-1}$ be a balanced-degree RCF factorization. Then a complex number s_z is said to be a *transmission zero* of G if and only if rank $G(s_z) < q$ or, equivalently, rank $N(s_z) < q$.

Remark 1

The transmission zeros of G are the poles of the left inverse of G with the lowest finite polar degree (Kailath 1982). They can, alternatively, be described as the intersection of the finite pole sets of all left inverses of G (Kailath 1982). Thus G has no transmission zeros if and only if it has two left inverses whose finite pole sets are disjoint. It is also easy to show, from the polynomial Smith–McMillan form (Kailath 1982), that G has no transmission zeros if and only if it has a polynomial (including constant) left inverse.

Remark 2

Given a non-constant G which is biproper and square, it can be shown from the uniqueness of the inverse of a square matrix that G must have one or more transmission zeros.

Our main interest here is in inner matrices, and in particular a 'structural' inner matrix defined as follows.

Definition 4

An inner matrix G having no transmission zeros is called *structural inner*.

The word 'structural' is used here because as we will shortly see some left inverses of this kind of inner function have a special structure. An example of a structural inner function is

$$G(s) = \begin{bmatrix} \dfrac{s}{s+2} & \dfrac{2}{s+2} \end{bmatrix}^{\mathrm{T}}$$

From Remark 2, it is clear that a structural inner matrix cannot be square. Let G be a structural inner matrix. From Remark 1, we know that G has a polynomial left inverse. Therefore, since we are primarily interested in proper real-rational matrices one might ask if G has a constant left inverse. The structural inner example given earlier in this paragraph has a constant left inverse $[1 \quad 1]$. Now since $G^\sim$ is completely unstable and is a left inverse of G, one might also ask if G has a 'stable' left inverse. The following theorem shows the existence of a stable left inverse for any structural inner matrix.

Theorem 1

Let G be an inner matrix. Then G is structural inner if and only if it has a stable left inverse.

Proof

Since the completely unstable $G^\sim$ is a left inverse of G, sufficiency comes directly from Remark 1. Necessity is a little more involved. Since G is inner, it is biproper. We can therefore find a permutation matrix P such that

$$PG = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$$

with G_1 being $q \times q$ and biproper. From the assumption that G has no transmission zeros, G_1 and G_2 must be balanced-degree right coprime. (Otherwise, the right common factor would be biproper and from Remark 2 it would have a transmission zero, which is a contradiction.) Thus there exist stable X and Y such that

$$XG_1 + YG_2 = I$$

Hence, $[X \quad Y]P$ is a stable left inverse of G. □

This characterization of a structural inner matrix is useful when the matrix is taken as an operator and one considers the associated mapping relations. The following corollary is an example. The proof is a direct application of Theorem 1 and thus omitted.

Corollary 1

Let G be a structural inner matrix. Then

(*a*) if $Gx \in \mathbb{R}\mathbf{H}_p^2$ for some $x \in \mathbb{R}\mathbf{L}_q^2$, then $x \in \mathbb{R}\mathbf{H}_q^2$.

(*b*) $G^{\mathrm{T}}\mathbb{R}\mathbf{H}_p^2 = \mathbb{R}\mathbf{H}_q^2$.

In the rest of the section we further characterize the structural inner matrix and discuss its relationship with a general inner matrix.

Lemma 2

Let G be inner and define

$$N(s) = \left[\begin{array}{c|c} A^{\mathrm{T}} & -L_c^{-1}B \\ \hline DB^{\mathrm{T}} + CL_c & D \end{array}\right] \tag{8}$$

$$M(s)=\left[\begin{array}{c|c} A^{\mathrm{T}} & -L_c^{-1}B \\ \hline B^{\mathrm{T}} & I \end{array}\right] \tag{9}$$

Then the controllability gramian W_c and observability gramian W_0 of N are

$$W_c = L_c^{-1} \tag{10}$$

$$W_0 = L_c(I - L_0 L_c) \tag{11}$$

and a RCF of $G(-s)$ is given by

$$G(-s) = N(s)M(s)^{-1}$$

where N and M are each inner. Furthermore, the minimal realization of N is a structural inner matrix.

Proof

By direct calculation, we can verify that L_c and L_c^{-1} are the observability gramian and the controllability gramian of M respectively, and $G(-s)M(s) = N(s)$. Then, by Lemma 1, M defined in (9) is inner and minimal and hence N is inner.

Similarly, we can easily verify that the gramians of N are given by

$$W_c = L_c^{-1} \quad \text{and} \quad W_0 = L_c(I - L_0 L_c)$$

Note that N defined in (8) is not minimal if $I - L_0 L_c$ is singular and is equal to D if $L_0 L_c = I$ (since $W_0 = 0$). It is obvious that when G is a square, N is a constant matrix.

Now we prove that N has no transmission zeros. Suppose that N has a transmission zero at s_z. Then, by Remark 1 and the fact that $N^{\sim}$ is a left inverse, we know $\mathrm{Re}\,(s_z) > 0$. Since G is inner, $-s_z$ is not a transmission zero of G. Thus $G(-s_z)$ has full column rank. Hence,

$$G(-s) = N(s)M(s)^{-1}$$

must have a cancellation of $(s - s_z)$ between $N(s)$ and $M(s)^{-1}$. But N is stable, so M^{-1} has poles that coincide with those of $G(-s)$. This means that following the cancellation of $(s - s_z)$ the number of unstable poles of the product NM^{-1} will be less than the number of poles of $G(-s)$. This is a contradiction, since $G(-s)$ is minimal. Hence the above cancellation is impossible. This proves that N has no transmission zeros and hence is structural inner by definition. □

With the above lemma, we can now give another criterion to decide whether an inner matrix is structural inner.

Theorem 2

Let $G(s)$ be an inner matrix. Then G is structural inner if and only if $L_0 L_c$ and $I - L_0 L_c$ are non-singular, where L_0, L_c are gramians of a minimal realization (A, B, C, D) of G.

Proof

Necessity. The invertibility of $L_0 L_c$ follows from the minimality of $G = (A, B, C, D)$. As in Lemma 2,

$$N(s) = G(-s)M(s) \tag{12}$$

where N and M are given by (8) and (9), respectively. By definition, $G(-s)$ has no transmission zeros. Since $G(-s)M(s)$ is stable and M is minimal, the left-hand side of (12) has exactly n poles where n is the dimension of A. Therefore, $N(s)$ is a minimal realization with the gramians of N given by (10) and (11). This proves that $I - L_0 L_c$ is invertible since $L_c(I - L_0 L_c)$ is non-singular.

Sufficiency. Suppose that $L_0 L_c$ and $I - L_0 L_c$ are invertible. Let $G(-s) = N(s)M(s)^{-1}$ as in Lemma 2. Then the structural inner $N(s)$ given by (8) is minimal. Applying Lemma 2 again to $N(-s)$, we have

$$N(-s) = N_N(s)M_N(s)^{-1}$$

where $N_N(s)$ is structural inner. It can easily be verified that

$$N_N(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = G(s)$$

Hence $G(s)$ is structural inner. □

Since $I - L_0 L_c$ is non-singular, we have shown that the maximum Hankel singular value of a structural inner matrix is strictly less than one. In super-optimal H^∞ control, we need to construct a square inner matrix based on a non-square inner matrix. For an inner matrix G there exists an inner matrix $G_\perp$ such that $[G \quad G_\perp]$ is square and inner. A realization of $G_\perp$ is given in the following lemma.

Lemma 3

Let $G(s)$ be an inner matrix with realization (A, B, C, D) (not necessarily minimal). Then a CIF of G is given by

$$G_\perp(s) = \left[\begin{array}{c|c} A & B_\perp \\ \hline C & D_\perp \end{array}\right] \tag{13}$$

where $B_\perp$ and $D_\perp$ satisfy

$$D_\perp D_\perp^{\mathrm{T}} = I - DD^{\mathrm{T}}$$

$$L_0 B_\perp + C^{\mathrm{T}} D_\perp = 0$$

If (C, A) is observable, then the square inner $[G \quad G_\perp]$ given by $(A, [B \quad B_\perp], C, [D \quad D_\perp])$ is minimal and the controllability gramian Y_c of $G_\perp$ is given by

$$Y_c = L_0^{-1} - L_c \tag{14}$$

Proof

The proof is an obvious extension of the corresponding result due to Doyle (1984).

Lemma 3 shows that the realization of $[G \quad G_\perp]$ is always minimal if $G =$

(A, B, C, D) is observable, but that the realization of $G_{\perp}$ given by (13) is not always minimal. In addition, a minimal realization of G does not ensure a minimal realization of $G_{\perp}$. However, when G itself is structural inner, the following corollary shows that $G_{\perp}$ given by (13) is also structural inner and is, of course, minimal. This is computationally useful in super-optimal algorithms (Tsai 1989) where a complementary inner part is required to construct a square inner matrix from the inner part of a maximizing vector. The above tells us that a square matrix of lower degree can be obtained if we begin with the structural inner part of the maximizing vector.

Corollary 2

Let G be structural inner. Then the complementary inner factor $G_{\perp}$ given by Lemma 3 is also structural inner.

Proof

Since G is structural inner, $L_0 L_c$ and $I - L_0 L_c$ are non-singular by Theorem 2. Consequently, $Y_0 Y_c$ and $I - Y_0 Y_c$ are non-singular from Lemma 3, and therefore $G_{\perp}$ is structural inner by Theorem 2. □

To conclude this section, we present a theorem which describes the relationship of structural inner matrices with general inner matrices.

Theorem 3

A $p \times q$ inner matrix $G(s)$ with $p > q$ can always be factorized into

$$G(s) = G_{si}(s) G_i(s)$$

where $G_{si}(s)$ is structural inner and $G_i(s)$ is a square, inner matrix.

Proof

From Lemma 2, we have for a minimal realization of $G(s)$

$$G(s) = N(-s) M_1(-s)^{-1}$$

where $N(s)$ is structural inner and of smaller degree than G, and $M_1(s)$ is (square) inner. Applying Lemma 2 again to a minimal realization of $N(-s)$, we obtain

$$N(-s) = G_{si}(s) M_2(s)^{-1}$$

where $G_{si}(s)$ is structural inner and $M_2(s)$ is (square) inner. Note that if the given $G(s)$ is structural inner, then we can easily prove that $G_{si}(s) = G(s)$.

Let $G_i(s) = M_2^{\mathrm{T}}(-s) M_1^{\mathrm{T}}(s)$. Thus we obtain

$$G(s) = G_{si}(s) G_i(s)$$

Since $G_{si}(s)$ is structural inner, Corollary 1 shows that $G_i(s)$ is stable, and therefore square inner. □

It is clear that if a vector inner function has no common factors in its numerators, then it has no transmission zeros and is structural inner. However, it is not so trivial for the matrix case. Theorem 3 shows that given an inner matrix we can always find its structural inner part. When $G(s) = N(s) M^{-1}(s)$ is a normalized RCF, clearly the stacked matrix $[N(s)^T \quad M(s)^T]^T$ is a structural inner function. Therefore from Theorem 2 we obtain that all Hankel singular values of $[N^T \quad M^T]^T$ are strictly less than one. This property was also proved by McFarlane (1988) (using a different approach) to characterize an optimal stability margin.

In the next section we examine the pseudo-singular-value decomposition that arises in super-optimal algorithms, and show how structural inner functions can be used to characterize the uniqueness properties of the decomposition.

3. Pseudo-singular-value decompositions

The solution of SMMP shown by Limebeer *et al.* (1988), Tsai *et al.* (1988 b), Yeh and Hwang (1988) and Young (1986) relies upon diagonalizing the cost function $(G-Q)$ using all-pass matrices. We find from the algorithm in Tsai *et al.* (1988 b) that there exist all-pass matrices U_w and U_v, such that

$$U_w^{\sim}(G-Q_{\text{sup}})U_v = \Lambda \tag{15}$$

where

$$\Lambda = [\text{diag}\,(\bar{s}_1^\infty g_1, \ldots, \bar{s}_q^\infty g_p) \quad 0], \quad \text{for } p<q$$

or

$$\Lambda = \begin{bmatrix} \text{diag}\,(\bar{s}_1^\infty g_1, \ldots, \bar{s}_q^\infty g_q) \\ 0 \end{bmatrix}, \quad \text{for } p \geqslant q$$

the g_i are all-pass, and

$$\bar{s}_i^\infty = s_i^\infty[G-Q_{\text{sup}}], \quad i=1,2,\ldots,\min\,(p,q)$$

and Q_{sup} is the super-optimal solution. The decomposition

$$G-Q_{\text{sup}} = U_w \Lambda U_v^{\sim} \tag{16}$$

is called pseudo-singular-value decomposition (PSVD). We define a PSVD as follows.

Definition 5

A pseudo-singular-value decomposition of $G \in \mathbb{R}\mathbf{L}^\infty_{p\times q} (p \geqslant q)$ is a factorization

$$G = W\begin{bmatrix} \Lambda \\ 0 \end{bmatrix} V^{\sim} \tag{17}$$

where W, V, Λ are in $\mathbb{R}\mathbf{L}^\infty$ with dimensions $p\times q$, $q \times q$ and $q\times q$, respectively, and W, V are all-pass; $\Lambda = \text{diag}\,[\lambda_1, \lambda_2, \ldots, \lambda_q]$, $\Lambda^{\sim}\Lambda$ is a constant matrix, and

$$|\lambda_1(j\omega)| \geqslant |\lambda_2(j\omega)| \geqslant \ldots \geqslant |\lambda_q(j\omega)|, \quad \forall\, \omega \in \mathbb{R}_e$$

Remark 3

We are interested in real-rational decompositions. The above decomposition differs from the traditional singular value decomposition (SVD) by not requiring $\Lambda(j\omega)$ to have non-negative real values for each ω.

Remark 4

For any ω, it is trivial to construct a conventional SVD of $G(j\omega)$. However, given a collection of SVDs for $G(j\omega)$ as ω varies from $-\infty$ to ∞, the construction of (17) cannot easily be achieved, and may not be achievable; an arbitrary $G \in \mathbb{R}\mathbf{L}^{\infty}$ may not have a PSVD.

Remark 5

The rank of G is constant for all s in the $j\omega$-axis, i.e. the number of identically zero elements on the diagonal of Λ is fixed, and λ_i is either non-zero for all ω or identically zero.

The following theorem describes the uniqueness properties of the PSVD.

Theorem 4 (Tsai 1989)

Let $G \in \mathbb{R}\mathbf{L}^{\infty}_{p\times q}$ with rank $(G) = q$. If G has two PSVDs, namely

$$G = W_1 \begin{bmatrix} \Lambda_1 \\ 0 \end{bmatrix} V_1^{\sim} \tag{18}$$

$$= W_2 \begin{bmatrix} \Lambda_2 \\ 0 \end{bmatrix} V_2^{\sim} \tag{19}$$

where

$$\Lambda_1 \Lambda_1^{\sim} = \Lambda_1^{\sim} \Lambda_1 = \begin{bmatrix} s_1^2 & & \\ & \ddots & \\ & & s_q^2 \end{bmatrix}, \quad s_i > 0, \quad i = 1, \ldots, q$$

(*a*) $s_i \neq s_j$, for all $i \neq j$: then there exist diagonal all-pass D_i, $i = 1, 2$ and an all-pass U that is either a diagonal D_3 for $p = q$ or in the form

$$\begin{bmatrix} D_3 & 0 \\ 0 & U_{22} \end{bmatrix}$$

for $p > q$ with D_3 diagonal, such that

(i) $\Lambda_2 = \Lambda_1 D_1$

(ii) $V_2 = V_1 D_2$

(iii) $W_2 = W_1 U$

(*b*) $s_i = s_j$, for some $i \neq j$: then the matrices D_2 and D_3 are block diagonal where the size of the ith block corresponds to the multiplicity of s_i.

Remark 6

The above definition and the related properties for the PSVD can be generalized so that the diagonal matrix $\Lambda^{\sim}\Lambda$ need not be constant but with the constraint that, for all i, j pairs,

$$|\lambda_i(j\omega)| \neq |\lambda_j(j\omega)| \quad \text{or} \quad |\lambda_i(j\omega)|, \quad \forall\, \omega \in \mathbb{R}_e$$

and

$$|\lambda_i(j\omega)| \neq 0 \quad \text{or} \quad |\lambda_i(j\omega)| = 0, \quad \forall\, \omega \in \mathbb{R}_e$$

We adopted Definition 5 because the PSVDs which arise in super-optimal control have the property that $\Lambda^{\sim}\Lambda$ is constant.

In super-optimal H^∞ control, the algorithms due to Limebeer *et al.* (1988) and Tsai *et al.* (1988 b) rely on diagonalizing the cost function by norm-preserving transformations. These transformations are not the same. However, for the 1-block SMMP there does exist a unique Q_{sup} as shown by Young (1986) and it is such that $G - Q_{\text{sup}}$ has a PSVD. The PSVD is not unique, but by using transformations whose columns (or rows transposed) are each structural inner, a particular structure of PSVD is obtained.

Definition 6

An inner matrix G is called *column structural inner* if each column of G is structural inner, and is called *row structural inner* if the transpose of each row of G is structural inner.

With this definition, we can obtain the following uniqueness result on the super-optimal PSVD.

Theorem 5

Let $G \in \mathbb{R}\mathbf{H}^\infty_{p\times q}(p \geqslant q)$ and $\text{rank}(G) = q$. Then there exists a super-optimal solution Q_{sup} that is unique, and a column structural inner matrix U_w and a row structural inner matrix $U_v^{\sim}$ such that

$$F - Q_{\text{sup}} = U_w \Lambda U_v^{\sim}$$

where

$$\Lambda = \begin{bmatrix} \text{diag}\,(\bar{s}_1^\infty g_1, \ldots, \bar{s}_q^\infty g_q) \\ 0 \end{bmatrix}$$

if $p = q$ the PSVD of $G - Q_{\text{sup}}$ is unique up to diagonal unitary constant matrices for $\bar{s}_i^\infty \neq \bar{s}_j^\infty$, for all $i \neq j$, and to block diagonal all-pass matrices for $\bar{s}_i^\infty = \bar{s}_j^\infty$ for some $i \neq j$.

The above result is based on the work of Tsai *et al.* (1988 b) and Young (1986) and the treatment of inner functions given in the previous section of this paper. Essentially any transmission zeros in the columns of U_w and any transmission zeros in the transposed rows of U_v are taken out via inner functions and placed in the appropriate diagonal elements of Λ. The corresponding result for $p < q$ is obvious. Note that for scalar problems, the column and row structural inner functions are simply $U_w = \pm 1$ and $U_v = \pm 1$. However, for the matrix case, U_w and U_v will not in general be constant matrices and will include structural information at each frequency on the $j\omega$-axis.

Remark 7

The diagonalization properties of Theorem 5 could be used in *multivariable* control system design, e.g. by Tsai *et al.* (1988 a) where the design objectives are specified in terms of the *shape* (magnitude) of a desired loop transfer function and then a controller in a two-feedback-loop scheme is found using H^∞ theory to meet the desired *loop shape*.

We call the optimal sequence $(\bar{s}_1^\infty, \bar{s}_1^\infty, \ldots, \bar{s}_q^\infty)$ the *s*-number of F and next give an interpretation of the *s*-number in terms of energy gains (Postlethwaite *et al.* 1989 b). Let the cost function corresponding to $F-Q$ have an input $d(t)$ and output $y(t)$, and let $d(t)$ be bounded in energy. Then

$$\sup_{d(t)\neq 0,\, d(t)\in\mathscr{B}} \| y(t) \|_2 = s_1^\infty(F - Q_{\text{sup}}) = \bar{s}_1^\infty \tag{20}$$

where

$$\mathscr{B} \triangleq \{d(t): \| d(t) \|_2 \leqslant 1\}$$

Now let $\hat{d}_1(s)$ be the Laplace transform of an input $d_1(t)$ which produces maximum energy in the output; then the direction (structural inner part) of $\hat{d}_1(s)$ is the direction (structural inner part) of a maximizing vector of the Hankel operator generated by F (Tsai 1989), and we have

$$\sup_{d(t)\neq 0,\, d(t)\in\mathscr{B}_1} \| y(t) \|_2 = s_2^\infty(F - Q_{\text{sup}}) = \bar{s}_2^\infty \tag{21}$$

where $\mathscr{B}_1$ is the subset of $\mathscr{B}$ whose elements have Laplace transforms that are pointwise orthogonal to $\hat{d}_1(s)$ for each s on the $j\omega$-axis.

Similarly, let $\hat{d}_2(s)$ be the Laplace transform of an input $d_2(t)$ that produces the energy gain $s_2^\infty(F - Q_{\text{sup}})$. The inner part of $\hat{d}_2(s)$ is characterized from the inner part of a maximizing vector of a Hankel operator generated by a matrix which now is a function of F but with dimensions each reduced by one; details can be found in the paper by Tsai (1989). We then have

$$\sup_{d(t)\neq 0,\, d(t)\in\mathscr{B}_2} \| y(t) \|_2 = s_3^\infty(F - Q_{\text{sup}}) = \bar{s}_3^\infty \tag{22}$$

where $\mathscr{B}_2$ is the subset of $\mathscr{B}_1$ whose elements have Laplace transforms that are pointwise orthogonal to $\hat{d}_2(s)$ for each s on the $j\omega$-axis; etc.

The *s*-numbers can therefore be interpreted as the largest energy gains from appropriately defined input spaces to the output and the corresponding directions constitute a column structure inner matrix (i.e., U_v). Intuitively then, if disturbance rejection is an objective reflected in the cost function, it is better (more robust) to minimize all the singular values (not just the maximum), and to be able to characterize (via structural inner functions) the directions of worst energy gain.

4. Conclusions

In this paper we have studied inner matrices and presented some new results for non-square inner functions. A structural inner matrix was defined as an inner matrix with no transmission zeros. This was seen to be useful in deriving the uniqueness properties of the PSVD which arises in super-optimal H^∞ control.

References

Doyle, J. C., 1984, Lecture notes. ONR/Honeywell Workshop on Advances in Multivariable Control, Minneapolis.

Foo, Y. K., and Postlethwaite, I., 1985, *Systems and Control Letters*, **7**, 261.

Glover, K., 1984, *International Journal of Control*, **39**, 115.

Green, M., and Anderson, B. D. O., 1987, *International Journal of Control*, **45**, 215.

Gu, D.-W., Tsai, M. C., and Postlethwaite, I., 1989, *Automatica*, **25**, 215.

Hung, Y. S., and MacFarlane, A. G. J., 1982, *Multivariable Feedback: a Quasi-classical Approach* (Berlin: Springer-Verlag).

Kailath, T., 1982, *Linear Systems* (Englewood Cliffs, NJ: Prentice Hall).

Klema, V. C., and Laub, A. J., 1980, *I.E.E.E. Transactions on Automatic Control*, **25**, 164.

Kwakernaak, H., 1986, *International Journal of Control*, **44**, 117.

Limebeer, D. J. N., Halikias, G. D., and Glover, G., 1988, A state-space algorithm for the computation of superoptimal matrix interpolating functions. *Proceedings of the Mathematical Theory of Networks and Systems Conference on Linear Circuits, Systems and Signal Processing: Theory and Applications*, Phoenix, Arizona (Amsterdam: North-Holland).

McFarlane, D., 1988, Robust controller design using normalized coprime factor plant descriptions. Ph.D. thesis, University of Cambridge.

Postlethwaite, I., Tsai, M. C., and Gu, D.-W., 1989 a, *International Journal of Control*, **49**, 247; 1989 b, Super-optimal H^∞ design. *International Symposium on the Mathematical Theory of Networks and Systems*, Amsterdam (Amsterdam: North-Holland).

Tsai, M. C., 1989, Super-optimal control system design for multivariable plants. D.Phil. thesis, University of Oxford.

Tsai, M. C., Gu, D.-W., O'Young, S. D., and Postlethwaite, I., 1988 a, An application of H^∞ super optimization in multivariable control system design. *Proceedings of the American Control Conference*, Atlanta, Georgia.

Tsai, M. C., Gu, D.-W., and Postlethwaite I., 1988 b, *I.E.E.E. Transactions on Automatic Control*, **33**, 833.

Yeh, F. B., and Hwang, T. S., 1988, *Systems and Control Letters*, **11**, 203.

Young, N. J., 1986, *Journal of Operator Theory*, **15**, 239.