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Frequency Weighted Controller Reduction Methods and Loop Transfer Recovery*

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Some apparently quite different procedures for controller reduction turn out to be the same when the controller has been obtained by LQG methods coupled with loop transfer recovery.

Key Words—Model reduction; (controller reduction); (loop transfer recovery); linear optimal regulator.

Abstract—This paper shows that if one designs an LQG controller using the conventional technique of loop transfer recovery (LTR), then two frequency weighted controller reduction methods, the Enns' frequency weighted balanced truncation (Enns, 1984a, Ph.D. Thesis, Stanford University, CA; 1984b, *Proc. 23rd CDC,* Las Vegas, NV, 127–132) and the Bezout identity induced frequency weighted reduction method (Anderson and Liu, 1987, *Proc. Amer. Control Conf.*, MN) will be equivalent under the condition that the plant transfer function is square, nonsingular and minimum phase. We also show that Enns' method is equivalent to the Bezout identity induced frequency weighted reduction method if the controller itself is stable and a particular representation for the controller is assumed.

1. INTRODUCTION

IT IS PRACTICALLY desirable for various reasons to have a simple controller in most real control system designs. Because many (but not all) controller design methods yield a controller with order similar to the plant order, one is faced either with the task of approximating a high order controller by a low order controller, or approximating a high order plant by a low order plant and designing a controller for the approximation. In recent years, a number of different model/controller reduction approaches and techniques have been suggested in the literature (see Anderson and Liu (1987) and the references in it). It should be pointed out that no matter what method and technique are used in the reduction process, one always needs to use the reduced order controller to "control" the real plant, i.e. to form the final closed-loop system by combining the reduced order controller and the high order plant; therefore, it is quite understandable that in controller reduction problems, one should take into account the existence of the plant in the controller reduction procedure itself, or, put another way, the reduction procedure should be based on closed-loop considerations. As strongly argued in Anderson and Liu (1987), one can then sensibly pose the controller reduction problem as a *frequency weighted L^∞ optimal approximation problem* (with the plant reflected in the weight). Unfortunately, the weighted optimal approximation problem cannot in general be easily solved, at least exactly. However, there are many frequency weighted controller reduction methods yielding approximate solutions which are available now (Enns, 1984a,b; Anderson and Liu, 1987; Latham and Anderson, 1985; Anderson, 1986; Al-Saggaf and Franklin, 1986). Among these methods, we mention two of them in particular, viz. the frequency weighted balanced truncation method of Enns and the frequency weighted balanced truncation method based on the Bezout identity. The choices of the frequency weighting for these two methods are both based on consideration of the closed-loop stability margin, although the representations of the controllers (and the plant) used in the methods are quite different. (In the Enns scheme, the controller is represented as a sum of a transfer function matrix with all poles in $\mathrm{Re}[s] < 0$, and one with all poles in $\mathrm{Re}\,[s] \geq 0$. In the method based on the Bezout identity, the controller is represented as a fraction of two specially selected stable proper transfer function matrices.) Furthermore, both methods use the same reduction technique within the procedures, i.e. truncation of a frequency weighted balanced realization of a certain transfer function matrix.

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Also, the computation effort required for both procedures is much less than for some other frequency weighted controller reduction methods.

The prime purpose of this paper is *to show certain connections between these two different reduction methods*. We were led to suspect the existence of connections by certain examples used in Anderson and Liu (1987), which show many similarities for both methods when a certain design parameter (q_2) for the filter becomes very large. (Using large q_2 corresponds to the LTR idea in LQG designs.) A review of both methods is given in the next section. We will show in Section 3 that if one uses the loop transfer recovery (LTR) idea in the linear quadratic Gaussian (LQG) design process to obtain a high order controller for a square plant, *the above mentioned two controller reduction methods will be equivalent under some conditions* (in fact, the usual conditions required for validity of the LTR approach). An example illustrates the conclusion. There is also a related secondary purpose in the paper. We commented above that one of the controller reduction methods proceeds by representing the controller as a fraction of two specially selected stable proper transfer function matrices. Now when the controller is *a priori* stable, there is a natural fractional representation (different from that first referred to): the denominator of the fraction is the identity, and the numerator is the controller transfer function matrix itself. In Section 4, we study the question of whether controller reduction using this fractional representation can be carried out and, if so, what the connection might be with the scheme of Enns (1984a,b). Section 5 contains concluding remarks.

2. FREQUENCY WEIGHTED BALANCED REDUCTION METHODS

In this section, we shall review briefly two frequency weighted balanced controller reduction methods, viz. the frequency weighted balanced truncation method of Enns and the frequency weighted balanced truncation method based on the Bezout identity. Both methods use the same approximation technique, viz. the frequency weighted balanced realization truncation approach as proposed by Enns (1984a,b), but different controller representations.

Consider an asymptotically stable frequency weighting $W_i(s) = E_i + C_i(sI - A_i)^{-1}B_i$ as an input weight to an asymptotically stable system $K(s) = C_k(sI - A_k)^{-1}B_k$ as shown in Fig. 1. Without loss of generality, minimality of (A_i, B_i, C_i, E_i) and (A_k, B_k, C_k) is assumed. An input frequency weighted balanced realization of $K(s)$

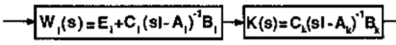

FIG. 1. Introduction of input weighting.

is achieved by changing the co-ordinate basis of (A_k, B_k, C_k) to yield a "frequency weighted" controllability gramian and (unweighted) observability gramian of $K(s)$ which are equal and diagonal. In outline, this is done as follows. We define system matrices associated with $K(s)W_i(s)$ by

$$\mathcal{A} = \begin{bmatrix} A_k & B_kC_i \\ 0 & A_i \end{bmatrix} \quad \text{and} \quad \mathcal{B} = \begin{bmatrix} B_kE_i \\ B_i \end{bmatrix}. \tag{2.1}$$

Suppose

$$\mathcal{U} = \begin{bmatrix} U & U_{12} \\ U_{12}^T & U_{22} \end{bmatrix} \tag{2.2}$$

is the non-negative solution of the following Lyapunov equation

$$\mathcal{A}\mathcal{U} + \mathcal{U}\mathcal{A}^T + \mathcal{B}\mathcal{B}^T = 0. \tag{2.3}$$

Now, regard U in (2.2) as the "frequency weighted" controllability gramian of system $K(s)$ with the input weight $W_i(s)$. Define Y as the positive definite solution of the following Lyapunov equation

$$YA_k + A_k^TY + C_k^TC_k = 0. \tag{2.4}$$

Consider a coordinate basis change to the realization (A_k, B_k, C_k) which makes

$$U_{\text{new}} = Y_{\text{new}} = \Sigma = \text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$$

with $\sigma_i \geq \sigma_{i+1}$, $i = 1, 2, \ldots, n-1$. (The determination of the coordinate transformation matrix is not difficult). There is no change to the realization (A_i, B_i, C_i, E_i). We call this new realization of $K(s)$ an input frequency weighted balanced realizaton.

Now the frequency weighted approximation can be achieved by eliminating the rows and columns of the new realization $(\hat{A}_k, \hat{B}_k, \hat{C}_k)$ of $K(s)$ corresponding to the smallest singular values, i.e. $\sigma_n, \sigma_{n-1}, \ldots, \sigma_{r+1}$ so that the reduced order system is $(A_{k11}, B_{k1}, C_{k1})$ where A_{k11} is the top left $r \times r$ corner of the new $\hat{A}_k$, etc.

Enns (1984a,b) proposed a frequency weighted controller reduction scheme which is based on the above approximation technique. The choice of the weighting for the scheme is based on stability margin considerations for the closed-loop system. Let $G(s)$ be the transfer function matrix of a given nth order linear time-invariant plant (l inputs and m outputs) and assume $K(s)$ to be a stabilizing compensator (generally of high order) obtained by some standard methods (LQG, H^∞ optimal or other).

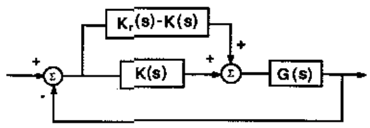

FIG. 2. Rearrangement of feedback system with reduced order compensator.

Let $K_r(s)$ be a reduced order compensator, which we are seeking. Consider the closed-loop system with the reduced order compensator $K_r(s)$, which has been rearranged as shown in Fig. 2. Using this redrawing, we can conclude (Doyle and Stein, 1981)—and it is now well known—that if (i) $K(s)$ and $K_r(s)$ have the same number of poles in Re $(s) > 0$ and no pole on the imaginary axis (or have identical $j\omega$-axis poles and residues); and (ii) either

$$\|[K(s) - K_r(s)]G(s)[I_l + K(s)G(s)]^{-1}\|_\infty < 1, \quad (2.5)$$

or

$$\|[I_m + G(s)K(s)]^{-1}G(s)[K(s) - K_r(s)]\|_\infty < 1, \quad (2.6)$$

then $K_r(s)$ is a stabilizing (low order) compensator. Here the infinity norm $\|A(s)\|_\infty$ means $\sup_\omega \max_i \lambda_i^{1/2}\{A^T(-j\omega)A(j\omega)\}$.

This clearly suggests a minimization problem: find a $K_r(s)$ satisfying (i) which at the same time minimizes the left side of (2.5) or (2.6), and has prescribed degree. The matrix $W_i = G(I_l + KG)^{-1} = (I_m + GK)^{-1}G$ acts as a weighting matrix here. However, the above frequency weighted optimization problem cannot in general be easily solved.

Enns (1984a,b) proposed to use the above mentioned frequency weighted balanced realization truncation technique to solve the minimization problem approximately. As shown by some examples in his work and in Liu and Anderson (1986), Anderson and Liu (1987) and Liu *et al.* (1990), it turns out that this method works well. However, two points should be noted. First, if the controller $K(s)$ itself is unstable, we cannot directly employ the frequency weighted approximation technique. Enns (1984a) suggested decoupling additively the controller $K(s)$ into the stable part $K_+(s)$ and the unstable part $K_-(s)$, then approximating only the stable part $K_+(s)$ by some low order $K_{+r}(s)$ and copying the unstable part $K_-(s)$ into the final approximation $K_r(s)$, i.e. $K_r(s) = K_{+r}(s) + K_-(s)$. Certainly this means that one cannot utilize the part of the information contained in the unstable part $K_-(s)$ in the reduction process. In addition, even if the controller $K(s)$ is stable, of order n, then generically the equation (2.3) has order $3n$. This means that in solving the above frequency weighted controller reduction problem, one needs to solve a $3n$th order Lyapunov equation (2.3). This may not be an easy task, particularly when n is large.

These observations motivated an alternative method for controller reduction proposed in Anderson and Liu (1987). By using the stability margin considerations for frequency weighting in conjunction with special stable fractional representations of the contoller and the plant, Anderson and Liu (1987) formulated a new frequency weighted controller reduction method. In order for the method to be carried through, the full order controller must have been obtained by combination of an estimator and state feedback law, which is of course a very common situation, and certainly includes LQG design.

For a given linear, time-invariant system, $G(s) = C(sI_n - A)^{-1}B$, design the state feedback gain F and the state estimator gain L (by LQG or other non-optimal pole-positioning methods) such that $A - BF$ and $A - LC$ are asymptotically stable. Then the compensator is (see Anderson and Moore, 1971),

$$K(s) = F(sI_n - A + BF + LC)^{-1}L. \quad (2.7)$$

Define

$$\begin{bmatrix} Y(s) & -N(s) \\ X(s) & D(s) \end{bmatrix} = \begin{bmatrix} I_l & 0 \\ 0 & I_m \end{bmatrix} + \begin{bmatrix} -F \\ C \end{bmatrix}(sI_n - A + BF)^{-1}[B\ L] \quad (2.8)$$

and

$$\begin{bmatrix} \tilde{D}(s) & \tilde{N}(s) \\ -\tilde{X}(s) & \tilde{Y}(s) \end{bmatrix} = \begin{bmatrix} I_l & 0 \\ 0 & I_m \end{bmatrix} + \begin{bmatrix} F \\ -C \end{bmatrix}(sI_n - A + LC)^{-1}[B\ L]. \quad (2.9)$$

It has been proved (Nett *et al.*, 1984; Vidyasagar, 1985) that

$$\begin{bmatrix} Y(s) & -N(s) \\ X(s) & D(s) \end{bmatrix}\begin{bmatrix} \tilde{D}(s) & \tilde{N}(s) \\ -\tilde{X}(s) & \tilde{Y}(s) \end{bmatrix} = \begin{bmatrix} \tilde{D}(s) & \tilde{N}(s) \\ -\tilde{X}(s) & \tilde{Y}(s) \end{bmatrix} \times \begin{bmatrix} Y(s) & -N(s) \\ X(s) & D(s) \end{bmatrix} = \begin{bmatrix} I_l & 0 \\ 0 & I_m \end{bmatrix}. \quad (2.10)$$

(This is the so-called Bezout identity.) The transfer function matrices $N(s)$, $D(s)$, $\tilde{N}(s)$, $\tilde{D}(s)$, $X(s)$, $Y(s)$, $\tilde{X}(s)$ and $\tilde{Y}(s)$ are all stable and $D(s)$, $\tilde{D}(s)$, $Y(s)$ and $\tilde{Y}(s)$ are all nonsingular. Hence, the Bezout identity (2.10)

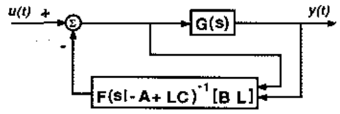

FIG. 3. State feedback law/estimator design of controller.

implies that $N(s)D^{-1}(s)$ $[\tilde{D}^{-1}(s)\tilde{N}(s)]$ is a stable right (left) coprime factorization of the controller $K(s)$, see Vidyasagar (1985). At the same time, $X(s)Y^{-1}(s)[\tilde{Y}^{-1}(s)\tilde{X}(s)]$ is a stable right (left) coprime factorization of the plant $G(s)$.

Refer to Fig. 3, and regard the compensator with this estimator and state feedback structure as being defined by a two (vector) input, single (vector) output system with transfer function matrix

$$\tilde{\Gamma}(s) = [\tilde{D}(s) - I_l \quad \tilde{N}(s)] = F(sI_n - A + LC)^{-1}[B \quad L]. \quad (2.11)$$

Regard the "plant" as defined by

$$\tilde{H}(s) = \begin{bmatrix} I_l \\ G(s) \end{bmatrix}, \quad (2.12)$$

so that Fig. 3 is equivalent to Fig. 4. Now, let us seek to approximate the stable $\tilde{\Gamma}(s)$ by a low order stable $\tilde{\Gamma}_r(s)$ as the low order compensator. Taking the closed-loop stability margin point of view, we know that if

$$\tilde{\rho} = \|[\tilde{\Gamma}(s) - \tilde{\Gamma}_r(s)]\tilde{H}(s)[I_l + \tilde{\Gamma}(s)\tilde{H}(s)]^{-1}\|_\infty < 1 \quad (2.13)$$

then the closed-loop system with the low order compensator $\tilde{\Gamma}_r(s)$ is stable [since $\tilde{\Gamma}(s)$ and $\tilde{\Gamma}_r(s)$ are both stable, they clearly have same number of unstable poles, viz. zero]. Defining

$$\tilde{V}(s) = \tilde{H}(s)[I_l + \tilde{\Gamma}(s)\tilde{H}(s)]^{-1},$$

it is easy, using the Bezout identity (2.10), to prove that

$$\tilde{V}(s) = \begin{bmatrix} Y(s) \\ X(s) \end{bmatrix} = \begin{bmatrix} I_l \\ 0 \end{bmatrix} + \begin{bmatrix} -F \\ C \end{bmatrix}(sI_n - A + BF)^{-1}B. \quad (2.14)$$

Hence our goal of reducing the order of the controller becomes one of minimizing or finding a procedure that will approximately minimize

$$\tilde{\rho} = \left\| [\tilde{D}(s) - \tilde{D}_r(s) \quad \tilde{N}(s) - \tilde{N}_r(s)] \begin{bmatrix} Y(s) \\ X(s) \end{bmatrix} \right\|_\infty \quad (2.15)$$

FIG. 4. Redrawing of the scheme of Fig. 3.

over stable $\tilde{\Gamma}_r(s) = [\tilde{D}_r(s) - I_l \quad \tilde{N}_r(s)]$ of prescribed degree. We term this approach the left coprime factorization (weighted) (LCFW) reduction method.

Certainly, one can use frequency weighted balanced realization truncation to approximately achieve the minimization. Note that the transfer function matrices we deal with are all stable [$\tilde{D}(s)$, $\tilde{N}(s)$, $Y(s)$ and $X(s)$], and the frequency weighting is of order n. Hence, in this case, equation (2.3) has order $2n$ generically; actually, simple algebraic transformations allow order reduction to n, see Anderson and Liu (1987) and Liu *et al.* (1990). In fact, the Lyapunov equations needed to be solved for the LCFW method become

$$(A - BF)P + P(A - BF)^T + BB^T = 0 \quad (2.16a)$$

$$(A - LC)^T Q + Q(A - LC) + F^T F = 0. \quad (2.16b)$$

Then a nonsingular coordinate basis transformation T is found from the weighted controllability and observability gramians P and Q which after transformation makes P equal to Q and both diagonal, as say $\Sigma = \text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$. This transformation is then used to change the coordinate basis of the realization of $\tilde{\Gamma}(s)$ into the "frequency-weighted balanced" one. The stable low order $\tilde{\Gamma}_r(s)$ is obtained by directly truncating the subsystem corresponding to the smallest σ_i.

A dual approach is available. We think of the controller as being defined by

$$\Gamma(s) = \begin{bmatrix} D(s) - I_m \\ N(s) \end{bmatrix}.$$

This idea was introduced for controller approximation in Liu and Anderson (1986), where a block diagram interpretation is provided (see also Anderson and Liu, 1987; Liu *et al.* 1990). Carrying through reasoning analogous to the above [and using again the Bezout identity (2.10)] leads to the conclusion that we should use [$\tilde{Y}(s)$, $\tilde{X}(s)$] [defined as in (2.9)] as the output frequency weight for controller reduction, seeking stable $[D_r^T(s), \ N_r^T(s)]^T$ with a prescribed degree, and such that

$$\rho = \left\| [\tilde{Y}(s) \quad \tilde{X}(s)] \begin{bmatrix} D(s) - D_r(s) \\ N(s) - N_r(s) \end{bmatrix} \right\|_\infty \quad (2.17)$$

is minimized. We know that $\rho < 1$ guarantees the closed-loop stability with the reduced order controller $K_r(s) = N_r(s)D_r^{-1}(s)$. This approach is termed the right coprime factorization (weighted) (RCFW) reduction method.

The particular fractional representations assumed above are important because the McMillan degree of the compensator, viz.

$\tilde{D}^{-1}(s)\tilde{N}(s)$ is the same as that of $\tilde{\Gamma}(s) = [\tilde{D}(s) - I_l, \tilde{N}(s)]$. When we reduce the latter to $\tilde{\Gamma}_r(s) = [\tilde{D}_r(s) - I_l\ \tilde{N}_r(s)]$, we then get a reduction in the McMillan degree of the approximating compensator, $\tilde{D}_r^{-1}(s)\tilde{N}_r(s)$. This would not necessarily be the case with an arbitrary fractional representation of the compensator. However, in Section 4, we isolate alternative fractional representations to those used above which can be used for model and controller reduction when either the plant or the controller is open-loop stable.

In the next section, we establish relations between the LCFW and RCFW methods and Enns' method when the LTR idea has been used in the controller design. Section 4 shows the further relationship between the Enns' method and the above LCFW and RCFW methods and different fractional representations are used.

3. REDUCTION METHODS AND LOOP TRANSFER RECOVERY

In this section, we shall assume that the controller $K(s)$ is designed for the system $G(s) = C(sI_n - A)^{-1}B$ by an LQG procedure. More precisely (see Kwakernaak and Sivan, 1972 and Anderson and Moore, 1971), suppose (A, B), (A, C) are completely controllable and observable pairs. Weighting matrices $Q = Q_0 + \hat{q}^2C^TC$ for some $Q_0 = Q_0^T \geq 0$ and $R = R^T > 0$ are chosen to define a quadratic performance index

$$J = \int_0^\infty [u^TRu + x^TQx]\, dt \qquad (3.1)$$

for the plant

$$\dot{x} = Ax + Bu. \qquad (3.2)$$

The minimizing control law is $u = -Fx$ where $F = R^{-1}B^TP_c$ and P_c is the unique positive definite solution of

$$P_cA + A^TP_c - P_cBR^{-1}B^TP_c + Q = 0. \quad (3.3)$$

[Such a solution exists provided $\hat{q} > 0$, or $(A, Q_0^{1/2})$ is observable.] Also, assuming white, zero mean, Gaussian process noise and measurement noise are present (the two noises being independent), there is designed a Kalman filter with the noise intensities $W = W_0 + q^2BB^T$ for some q and $W_0 = W_0^T \geq 0$ and $V = V^T > 0$; the Kalman filter gain is $L = P_fC^TV^{-1}$ where P_f is the unique positive definite solution of

$$AP_f + P_fA^T - P_fC^TV^{-1}CP_f + W = 0. \quad (3.4)$$

[Such a solution exists if $q > 0$ or $(A, W_0^{1/2})$ is controllable.]

The controller implements instead of $u = -Fx$ the law $u = -F\hat{x}$, where $\hat{x}$ is the state estimator output. Combination of the estimator equations with this feedback law leads to a compensator transfer function

$$K(s) = F(sI_n - A + BF + LC)^{-1}L. \quad (3.5)$$

In loop transfer recovery, one regards either Q, R as the primary design parameters, with V, W to be adjusted so that the use of state estimate feedback provides little or no change in robustness from the use of exact state feedback, or one regards V, W as the primary design parameters, with Q, R to be adjusted so that robustness is maintained. More precisely, we have the following well known result (Kwakernaak and Sivan, 1972; Doyle and Stein, 1979; Francis, 1979), which is the basis of loop transfer recovery.

Lemma 1. Assume that

(i) the system $G(s)$ has l inputs and outputs, and $G(s)$ is nonsingular at all except a finite set of s;

(ii) the system $G(s)$ is minimum phase, i.e.

$$\text{rank}\begin{bmatrix} sI_n - A & B \\ C & 0 \end{bmatrix}$$

is constant in $\text{Re}\,(s) \geq 0$.

Then when $q \to \infty$ in the LQG design procedure (3.1) through (3.5),

$$P_f/q^2 \to 0 \quad \text{and} \quad L \to qBUV^{-1/2},$$

where $V^{-1/2}$ is any square root of V^{-1}, i.e. $(V^{-1/2})^T(V^{-1/2}) = V^{-1}$, and U is some $l \times l$ matrix satisfying $UU^T = I_l$.

In order to consider the relation between the controller reduction methods of Enns and that based on coprime factorizations, we first investigate the behaviour of the four factor transfer function matrices defined in (2.8) and (2.9) when the LTR idea has been employed in the LQG design, i.e. when $q \to \infty$. We have

Lemma 2. (Behaviour of left coprime representation of compensator.) When $q \to \infty$ in the LQG design procedure (3.1) through (3.5) and the assumptions of Lemma 1 hold, then pointwise in $s \in \mathbb{C}$ (except at the plant zeros, which lie in $\text{Re}\,(s) < 0$)

$$\tilde{D}(s) = I_l + F(sI_n - A + LC)^{-1}B \to I_l$$

and

$$\tilde{N}(s) = F(sI_n - A + LC)^{-1}B \to F(sI_n - A)^{-1}BG^{-1}(s).$$

Proof. We use the facts from Lemma 1 that $L \to qBUV^{-1/2}$ as $q \to \infty$ and $UV^{-1/2}C(sI_n - A)^{-1}B$ is nonsingular except at the plant zeros.

Evidently,

$$\tilde{D}(s) = I_l + F(sI_n - A + LC)^{-1}B$$
$$\to I_l + F(sI_n - A + qBUV^{-1/2}C)^{-1}B \quad (\text{as } q \to \infty)$$
$$= I_l + F(sI_n - A)^{-1}B \times [I_l + qUV^{-1/2}C(sI_n - A)^{-1}B]^{-1}$$
$\to I_l$ for all s except zeros of the plant, which are finite in number and lie in $\mathrm{Re}\,(s) < 0$, (as $q \to \infty$).

Also

$$\tilde{N}(s) = F(sI_n - A + LC)^{-1}L$$
$$\to F(sI_n - A + qBUV^{-1/2}C)^{-1}qBUV^{-1/2} \quad (\text{as } q \to \infty)$$
$$= F(sI_n - A)^{-1}B[I_l + qUV^{-1/2}C \times (sI_n - A)^{-1}B]^{-1}qUV^{-1/2}$$
$$\to F(sI_n - A)^{-1}B[C(sI_n - A)^{-1}B]^{-1}$$
for all s except zeros of the plant (as $q \to \infty$).

Lemma 3. (Behaviour of left coprime representation of plant.) When $q \to \infty$ in the LQG design procedure (3.1) through (3.5) and the assumptions of Lemma 1 hold, then pointwise in s except at the plant zeros,

$$q\tilde{X}(s) = qC(sI_n - A + LC)^{-1}B \to V^{1/2}U^{-1}$$

and

$$q\tilde{Y}(s) = q[I_l - C(sI_n - A + LC)^{-1}L] \to V^{1/2}U^{-1}G^{-1}(s),$$

where $V^{1/2} = (V^{-1/2})^{-1}$ and U and $V^{-1/2}$ are defined as in Lemma 1.

Proof. Again we use the facts that $L \to qBUV^{-1/2}$ as $q \to \infty$ and $UV^{-1/2}C(sI_n - A)^{-1}B$ is nonsingular except at the plant zeros; we have

$$q\tilde{X}(s) = qC(sI_n - A + LC)^{-1}B$$
$$\to qC(sI_n - A + qBUV^{-1/2}C)^{-1}B \quad (\text{as } q \to \infty)$$
$$= qC(sI_n - A)^{-1}B \times [I_l + qUV^{-1/2}C(sI_n - A)^{-1}B]^{-1}$$
$$\to qC(sI_n - A)^{-1}B \times [qUV^{-1/2}C(sI_n - A)^{-1}B]^{-1} \quad (\text{as } q \to \infty)$$
$$= V^{1/2}U^{-1}.$$

Also

$$q\tilde{Y}(s) = q[I_l - C(sI_n - A + LC)^{-1}L]$$
$$\to q[I_l - C(sI_n - A + qBUV^{-1/2}C)^{-1}BUV^{-1/2}q] \quad (\text{as } q \to \infty)$$
$$= q[I_l - qC(sI_n - A)^{-1}BUV^{-1/2} \times [I_l + qC(sI_n - A)^{-1}BUV^{-1/2}]^{-1}]$$
$$= q[I_l + qC(sI_n - A)^{-1}BUV^{-1/2}]^{-1}$$
$$\to V^{1/2}U^{-1}[C(sI_n - A)^{-1}B]^{-1} \quad (\text{as } q \to \infty).$$

The following lemma captures the loop transfer recovery (LTR) property (Doyle and Stein, 1979):

Lemma 4. When $q \to \infty$ in the LQG design procedure (3.1) through (3.5) and the assumptions of Lemma 1 hold, then pointwise in $s \in \mathbb{C}$ (except at a finite number of points in $\mathrm{Re}\,(s) < 0$)

$$K(s) = F(sI_n - A + BF + LC)^{-1}L \to F(sI_n - A)^{-1}BG^{-1}(s)$$

and

$$K(s)G(s) \to F(sI_n - A)^{-1}B.$$

An alternative proof of this lemma to that of Doyle and Stein (1979) follows easily by using the fact that $K(s) = \tilde{D}^{-1}(s)\tilde{N}(s)$, together with the formulas of Lemma 2.

In Lemmas 2 and 3, we have indicated the limiting behaviour of some of the matrices appearing in the Bezout identity, viz. those defining left coprime realizations of the ccontroller and the plant. In fact, we can obtain the behaviour of the remaining matrices, as in the following lemma.

Lemma 5. When $q \to \infty$ in the LQG design procedure (3.1) through (3.5) and the assumptions of Lemma 1 hold, then pointwise in $s \in \mathbb{C}$ except at the plant zeros

$$\begin{bmatrix} Y(s) & -N(s) \\ X(s) & D(s) \end{bmatrix} \to \begin{bmatrix} Y(s) & [I - Y(s)]UV^{-1/2}q \\ X(s) & X(s)UV^{-1/2}q \end{bmatrix}$$

and

$$\begin{bmatrix} \tilde{D}(s) & \tilde{N}(s) \\ -\tilde{X}(s) & \tilde{Y}(s) \end{bmatrix} \to \begin{bmatrix} I_l & F(sI_n - A)^{-1}BG^{-1}(s) \\ -q^{-1}V^{1/2}U^{-1} & q^{-1}V^{1/2}U^{-1}G^{-1}(s) \end{bmatrix}.$$

Furthermore

$$\begin{bmatrix} Y(s) & [I_l - Y(s)]UV^{-1/2}q \\ X(s) & X(s)UV^{-1/2}q \end{bmatrix} \times \begin{bmatrix} I_l & F(sI_n - A)^{-1}BG^{-1}(s) \\ -q^{-1}V^{1/2}U^{-1} & q^{-1}V^{1/2}U^{-1}G^{-1}(s) \end{bmatrix} = \begin{bmatrix} I_l & 0 \\ 0 & I_l \end{bmatrix}$$

where the transfer function matrices $X(s)$, $Y(s)$, $N(s)$, $D(s)$, $\tilde{X}(s)$, $\tilde{Y}(s)$, $\tilde{N}(s)$ and $\tilde{D}(s)$ are defined as in (2.8) and (2.9), and $V^{1/2} = (V^{-1/2})^{-1}$, and $V^{-1/2}$ and U are defined as in Lemma 1.

The proof of this lemma is straightforward.

Now we are in the position to show the relation between Enns' method and the Bezout identity induced frequency weighted reduction methods. We have

Theorem 1. When loop transfer recovery with $q \to \infty$ in (3.1) through (3.5) is used to design a high order controller for the system $G(s)$, and the assumptions of Lemma 1 hold, then the left coprime factorization frequency weighted (LCFW) controller reduction method is equivalent to Enns' frequency weighted balanced truncation controller reduction method (input weighted) in the sense that both methods will yield the same reduced order controller of prescribed dimension.

Proof. For the LCFW method, in order to execute the reduction, the "controller" is defined as $\tilde{\Gamma}(s) = [\tilde{D}(s) - I_l \;\; \tilde{N}(s)]$ and the weighting is $[Y^T(s) \;\; X^T(s)]^T$. By Lemma 2 we know that

$$\tilde{\Gamma}(s) \to [0 \; F(sI_n - A)^{-1}BG^{-1}(s)] \quad \text{as} \quad q \to \infty,$$

and the weighting is not changed. Hence, as $q \to \infty$, the LCFW method aims to reduce the "controller" $[0 \;\; F(sI_n - A)^{-1}BG^{-1}(s)]$ using the frequency weighting $[Y^T(s) \;\; X^T(s)]^T$, or, to reduce $F(sI_n - A)^{-1}BG^{-1}(s)$ using the weighting $X(s)$.

For Enns' method, the controller is $K(s) = F(sI_n - A + BF + LC)^{-1}L$ and the frequency weighting is $W(s) = G(s)[I_l + K(s)G(s)]^{-1} = \tilde{D}^{-1}(s)X(s)$ [by the Bezout identity (2.10)]. Using Lemmas 4 and 2, we have $K(s) \to F(sI_n - A)^{-1}BG^{-1}(s)$ and $W(s) = \tilde{D}^{-1}(s)X(s) \to X(s)$ as $q \to \infty$. Hence, as $q \to \infty$, Enns' method is to reduce $F(sI_n - A)^{-1}BG^{-1}(s)$ using the frequency weighting $X(s)$. Now since both methods yield the same frequency weighted error, they will yield the same reduced order controller, and the conclusion follows.

As the RCFW controller reduction method is a dual of the LCFW method, there is a set of dual results analogous to the above lemmas. We state these dual results without proof.

Lemma 1′. Assume that
- (i) the system $G(s)$ has l inputs and outputs, and $G(s)$ is nonsingular at all except a finite set of s;
- (ii) the system $G(s)$ is minimum phase (as defined in Lemma 1).

Then when $\hat{q} \to \infty$ in the LQG design procedure (3.1) through (3.5),

$$P_c/\hat{q}^2 \to 0 \quad \text{and} \quad F \to \hat{q}R^{-1/2}\hat{U}C,$$

where $R^{-1/2}$ is any square root of R^{-1}, i.e., $(R^{-1/2})(R^{-1/2})^T = R^{-1}$, and $\hat{U}$ is some $l \times l$ matrix satisfying $\hat{U}^T\hat{U} = I_l$.

Lemma 2′. (Behaviour of right coprime realization of controller.) When $\hat{q} \to \infty$ in the LQG design procedure (3.1) through (3.5) and the assumptions of Lemma 1′ hold, then pointwise in $s \in \mathbb{C}$ [except at the plant zeros, which lie in $\mathrm{Re}(s) < 0$]

$$D(s) = I_l + C(sI_n - A + BF)^{-1}L \to I_l$$

and

$$N(s) = F(sI_n - A + BF)^{-1}L \to G^{-1}(s)C(sI_n - A)^{-1}L.$$

Lemma 3′. (Behaviour of right coprime realization of plant.) When $\hat{q} \to \infty$ in the LQG design procedure (3.1) and the assumptions of Lemma 1′ hold, then pointwise in $s \in \mathbb{C}$ except at the plant zeros

$$\hat{q}X(s) = \hat{q}C(sI_n - A + BF)^{-1}B \to \hat{U}^{-1}R^{1/2}$$

and

$$\hat{q}Y(s) = \hat{q}[I_l - F(sI_n - A + BF)^{-1}B] \to G^{-1}(s)\hat{U}^{-1}R^{1/2},$$

where $R^{1/2} = (R^{-1/2})^{-1}$ and $\hat{U}$ and $R^{-1/2}$ are defined as in Lemma 1′.

Lemma 4′. When $\hat{q} \to \infty$ in the LQG design procedure (3.1) through (3.5) and the assumptions of Lemma 1′ hold, then pointwise in $s \in \mathbb{C}$ except at the plant zeros

$$K(s) = F(sI_n - A + BF + LC)^{-1}L \to G^{-1}(s)C(sI_n - A)^{-1}L$$

and

$$G(s)K(s) \to C(sI_n - A)^{-1}L.$$

Lemma 5′. When $\hat{q} \to \infty$ in the LQG design procedure (3.1) through (3.5) and the assumptions of Lemma 1′ hold, then pointwise in $s \in \mathbb{C}$ except at the plant zeros

$$\begin{bmatrix} Y(s) & -N(s) \\ X(s) & D(s) \end{bmatrix} \to \begin{bmatrix} \hat{q}^{-1}G^{-1}(s)\hat{U}^{-1}R^{1/2} & -G^{-1}(s)C(sI_n-A)^{-1}L \\ \hat{q}^{-1}\hat{U}^{-1}R^{1/2} & I_l \end{bmatrix}$$

and

$$\begin{bmatrix} \tilde{D}(s) & \tilde{N}(s) \\ -\tilde{X}(s) & \tilde{Y}(s) \end{bmatrix} \to \begin{bmatrix} \hat{q}R^{-1/2}\hat{U}\tilde{X}(s) & \hat{q}R^{-1/2}\hat{U}[I-\tilde{Y}(s)] \\ -\tilde{X}(s) & \tilde{Y}(s) \end{bmatrix}.$$

Furthermore

$$\begin{bmatrix} \hat{q}^{-1}G^{-1}(s)\hat{U}^{-1}R^{1/2} & -G^{-1}(s)C(sI_n-A)^{-1}L \\ \hat{q}^{-1}\hat{U}^{-1}R^{1/2} & I_l \end{bmatrix} \times \begin{bmatrix} \hat{q}R^{-1/2}\hat{U}\tilde{X}(s) & \hat{q}R^{-1/2}\hat{U}[I_l-\tilde{Y}(s)] \\ -\tilde{X}(s) & \tilde{Y}(s) \end{bmatrix} = \begin{bmatrix} I_l & 0 \\ 0 & I_l \end{bmatrix}$$

where the transfer function matrices $X(s)$, $Y(s)$, $N(s)$, $D(s)$, $\tilde{X}(s)$, $\tilde{Y}(s)$, $\tilde{N}(s)$ and $\tilde{D}(s)$ are defined as in (2.8) and (2.9), and $R^{1/2} = (R^{-1/2})^{-1}$, and $R^{-1/2}$ and $\hat{U}$ are defined as in Lemma 1′.

Theorem 1′. When loop transfer recovery with $\hat{q} \to \infty$ in (3.1) through (3.5) is used to design a high order controller for the system $G(s)$, and the assumptions of Lemma 1′ hold, then the right coprime factorization frequency weighted (RCFW) controller reduction method is equivalent to Enns' frequency weighted balanced truncation controller reduction method (output weighted) in the sense that both methods will yield the same reduced order controller of prescribed dimension.

Now we make an important side remark concerning the LCFW frequency-weighted controller reduction method. Yousuff and Skelton (1984) proposed an unweighted Balanced Controller Reduction Algorithm (BCRA) for the controller reduction problem, which is merely a direct truncation of a balanced realization of the controller $K(s)$ [defined as in (3.5)]. The method requires the controller to be open loop stable; one finds a coordinate basis change to $K(s) = F(sI_n - A + BF + LC)^{-1}L$ which makes

$$A_cP + PA_c^T + LL^T = 0 \qquad (3.6a)$$

$$A_c^TQ + QA_c + F^TF = 0 \qquad (3.6b)$$

where $A_c = A - BF - LC$ and $P = Q = \Sigma = \mathrm{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ with $\sigma_i \geq \sigma_{i+1} \geq 0$. Then one deletes the subsystem of $K(s)$ corresponding to the smallest σ_i to obtain the low order controller. However, since the controller $K(s)$ itself may be open-loop unstable, a modification to the above BCRA is also suggested in Yousuff and Skelton (1984), this being known as BCRAM. In this modified algorithm, one balances the following two Lyapunov equations instead of (3.6a) and (3.6b):

$$(A - BF)P + P(A - BF)^T + LL^T = 0 \quad (3.7a)$$

$$(A - LC)^TQ + Q(A - LC) + F^TF = 0 \quad (3.7b)$$

where $P = Q = \Sigma = \mathrm{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ with $\sigma_i \geq \sigma_{i+1} \geq 0$. Now compare (3.7) with (2.16); it is clear that (3.7a) differs from (2.16a) while (3.7b) and (2.16b) are the same. If we assume the controller $K(s)$ is designed through the LTR procedure (when $q \to \infty$), then by Lemma 1 we have (3.7) is equivalent to (2.16) [to within a scalar multiplier and assuming $V = I$]. That is to say, the frequency weighted controller reduction method LCFW is equivalent to the unweighted controller reduction method BCRAM when the LTR is used in the controller design; of course the BCRAM scheme depends on a totally different reasoning in its derivation.

Now, we will use an example to illustrate the result of Theorem 1. It should be pointed out here that in Theorem 1, one of the conditions to guarantee the equivalence of the two frequency weighted controller reduction methods is that the plant is minimum phase. However, in the following example, with the plant being non-minimum phase, we are still able to show the equivalence of the two controller reduction methods if the LTR technique is used to design the full order controller. The reason for this is that the frequency corresponding to the unstable zeros of the plant is located far outside the band width of the closed-loop system. Hence, when the LTR technique is used to design the full order controller, the effect of these unstable zeros of the plant on the closed-loop system remains minor; thus we can expect that this non-minimum phase plant will behave like a minimum phase plant in the controller reduction process.

Example. We consider the example used in Anderson and Liu (1987). The plant $G(s) = C(sI_n - A)^{-1}B$ with transfer function described in Liu and Anderson (1986) and minimal realization described in Anderson and Liu (1987) is a four disk system originally studied by Enns (1984a). The plant is linear, time-invariant, single input and output, unstable, non-minimum phase and of eighth order. The zeros of the plant are

$$\{2.26+5.1783j, 2.26-5.1783j, -4.84, -0.02+0.9998j, -0.02-0.9998j\}.$$

Now, for two different LQG designed (with and without employing LTR technique) full order controllers, we will compare the Enns' frequency weighted and the left coprime frequency weighted (LCFW) controller reduction methods to show the result of Theorem 1. The weightings for the LQR designs are given by $Q = H^TH$, $R = 1$ with $H = [0, 0, 0, 0, 0.00055, 0.011, 0.00132, 0.018]$ in (3.1) and the weightings for filter design are given by $W = W_0 + q^2BB^T$, $V = 1$ with $W_0 = 0$ where q is a design parameter. In the first case, we take $q = 1$. It corresponds to a normal LQG design without the LTR for the full order controller (8th order). It should be noted for this design that the cross-over frequency of the closed-loop system ($[I + G(s)K(s)]^{-1}G(s)$) is about 1.4 rad s^{-1}, some four times smaller than the frequency corresponding to the unstable zeros of the plant. The frequency corresponding to the unity loop gain $G(s)K(s)$ is about 0.1 rad s^{-1}. We now use the Enns' and LCFW controller reduction methods to obtain a second order and a third order controller. Figure 5 depicts the comparison of the Nyquist plots of the loop gain $G(s)\hat{K}(s)$ with $\hat{K}(s)$ the full order controller, the second order controller by Enns' method and the second order controller by the LCFW method. (We elect to show Nyquist plots of loop gain rather than simply the Nyquist plots of the controllers themselves; the loop gain Nyquist plots are in effect weighted versions of the controller plots. Because loop transfer recovery is intrinsically concerned with loop gain, it seems appropriate to include this weighting). Figure 6 shows the similar comparison but with the reduced order controllers having order 3. It is easy to see from these two figures that these two controller reduction methods yield different low order controllers in this case. In the second case, we take the design parameter $q^2 = 10^7$. This corresponds to an LQG design with LTR. Once again, we reduce the full order controller to a second and a third order controller by the Enns' method and the LCFW method and show the comparison of the reductions in Fig. 7 and Fig. 8. In this case now, it is very clear that the reduced order controllers obtained by Enns' method and the LCFW method are very close to each other and look almost the same for the second order reduction. Note that every plot shows the part of the Nyquist plot corresponding to a limited frequency range ($0.04 \le \omega \le 1.5$) for a clear comparison.

In the next section, we shall show a further connection between Enns' method and the LCFW or RCFW method.

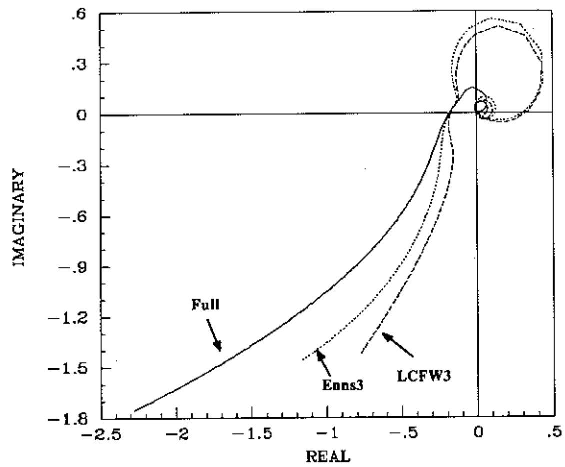

FIG. 6. Comparison of Nyquist plots (3rd order, with out LTR).

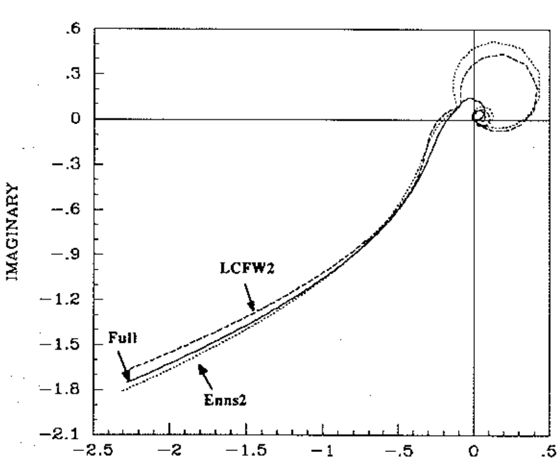

FIG. 5. Comparison of Nyquist plots (2nd order, without LTR).

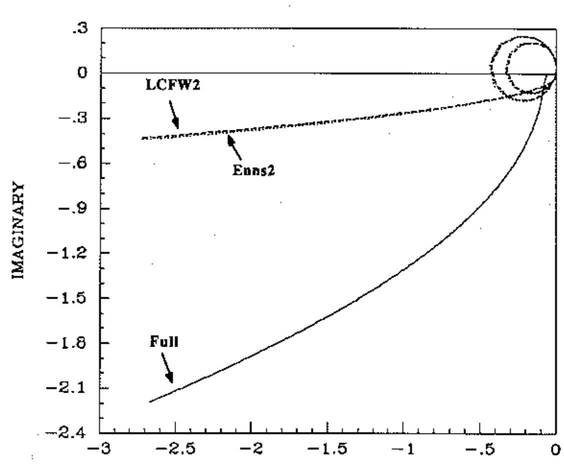

FIG. 7. Comparison of Nyquist plots (2nd order, with LTR).

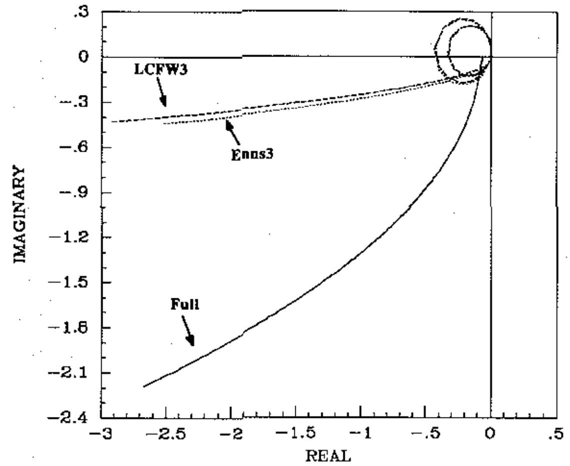

FIG. 8. Comparison of Nyquist plots (3rd order, with LTR).

4. FURTHER RELATIONS FOR OPEN-LOOP STABLE CONTROLLERS

Let $G(s)$ be the transfer function matrix of a linear, time-invariant system with l inputs and m outputs. Assume $K(s)$ is a stabilizing controller for $G(s)$ designed by some standard procedure. In addition, we suppose in this section that the controller $K(s)$ is itself (open-loop) stable. The standard Enns' method is to reduce the controller $K(s)$ to a low order one $K_r(s)$ with weighting $[I_m + G(s)K(s)]^{-1}G(s)$. Now consider what happens with the LCFW (RCFW) method, but where we use a different fractional representation from that of the last section.

In view of the stability of the controller, one notes a pair of possible fractional coprime representations that is provided by $\tilde{N}(s) = K(s)$, $\tilde{D}(s) = I_l$, $N(s) = K(s)$ and $D(s) = I_m$ (if coprimeness is not immediately obvious, it will be established via displaying below a Bezout identity). Now in order to use the LCFW (RCFW) controller reduction method, we need to find a right (left) comprime factorization of the plant $G(s) = X(s)Y^{-1}(s)$ $\{G(s) = \tilde{Y}^{-1}(s)\tilde{X}(s)\}$ which will satisfy the Bezout identity (2.10). It is easy to show that if we define $X(s) = G(s)[I_l + K(s)G(s)]^{-1}$ $\{\tilde{X}(s) = [I_m + G(s)K(s)]^{-1}G(s)\}$ and define $Y(s) = [I_l + K(s)G(s)]^{-1}$ $\{\tilde{Y}(s) = [I_m + G(s)K(s)]^{-1}\}$, then $G(s) = X(s)Y^{-1}(s)$ $\{G(s) = \tilde{Y}^{-1}(s)\tilde{X}(s)\}$ and the Bezout identity (2.10) holds. Furthermore, since the controller $K(s)$ is stabilizing for $G(s)$, the above defined $X(s)$ and $Y(s)$ $\{\tilde{X}(s)$ and $\tilde{Y}(s)\}$ are stable and the nonsingularity of $Y(s)\{\tilde{Y}(s)\}$ is obvious. [Hence, $X(s)$ and $Y(s)$ $\{\tilde{X}(s)$ and $\tilde{Y}(s)\}$ is a right (left) coprime factorization of the plant $G(s)$.] Now, the LCFW (RCFW) controller reduction task is to approximate $[\tilde{D}(s) - I_l \;\; \tilde{N}(s)] = [0\; K(s)]$ $\{[D^T(s) - I_m \;\; N^T(s)]^T = [0\; K^T(s)]^T\}$ with the weighting $[Y^T(s) \;\; X^T(s)]^T = [[(I_l + K(s)G(s))^{-1}]^T \;\; [G(s)(I_l + K(s)G(s))^{-1}]^T]^T$ $\{[\tilde{Y}(s) \;\; \tilde{X}(s)] = [[I_m + G(s)K(s)]^{-1} \;\; [I_m + G(s) \times K(s)]^{-1}G(s)]\}$. Obviously, approximating $[0\, K(s)]$ with weighting $[Y^T(s) \;\; X^T(s)]^T$ is equivalent to approximating $K(s)$ with weighting $X(s) = G(s) \times [I_l + K(s)G(s)]^{-1}$ and approximating $[0\, K^T(s)]^T$ with (left) weighting $[\tilde{Y}(s)\; \tilde{X}(s)]$ is equivalent to approximating $K(s)$ with left weighting $\tilde{X}(s) = [I_m + G(s)K(s)]^{-1}G(s)$. These are precisely the approximation tasks tackled using Enns' controller reduction method. Hence we have established:

Theorem 2. Let $G(s)$ be the $m \times l$ transfer function matrix of a system for which $K(s)$ is the $l \times m$ transfer function matrix of a stabilizing controller. Suppose that the controller is itself stable, i.e. all poles of elements of $K(s)$ lie in $\mathrm{Re}\,(s) < 0$. Then the input (output) weighted Enns' controller reduction method is a special case of the LCFW (RCFW) controller reduction method when one uses the following left (right) coprime factorization of the controller $K(s) = \tilde{D}^{-1}(s)\tilde{N}(s)$ $\{K(s) = N(s)D^{-1}(s)\}$ where

$$\tilde{D}(s) = I_l\{D(s) = I_m\} \quad \text{and}$$
$$\tilde{N}(s) = K(s)\{N(s) = K(s)\},$$

and the following right (left) coprime factorization of the plant $G(s) = X(s)Y^{-1}(s)\{G(s) = \tilde{Y}^{-1}(s)\tilde{X}(s)\}$ where

$$X(s) = G(s)[I_l + K(s)G(s)]^{-1}$$
$$\{\tilde{X}(s) = [I_m + G(s)K(s)]^{-1}G(s)\}$$

and

$$Y(s) = [I_l + K(s)G(s)]^{-1}$$
$$\{\tilde{Y}(s) = [I_m + G(s)K(s)]^{-1}\}.$$

Analogously to the final remark in the end of the Section 2, one can obtain a result similar to that of the above theorem for the frequency weighted *plant model* reduction problem. We remark that Enns' frequency weighted model reduction method (Enns, 1984a,b) is a special case of the LCFW (or RCFW) model reduction method when one considers a stable plant model $G(s)$ and uses certain fractional representations of the plant and the controller.

5. CONCLUSIONS

This paper has shown some relations between two frequency weighted controller reduction methods, i.e. the Enns' frequency weighted balanced truncation approach and the Bezout identity induced frequency weighting approach (LCFW or RCFW method). When a system model $G(s)$ is minimum phase and nonsingular, we have shown that Enns' method is equivalent to the LCFW method if the loop transfer recovery (LTR) idea has been used in determining the state estimator gain in the LQG controller design procedure, and is equivalent to the RCFW method if LTR has been used to design the stable feedback gain in the LQG controller design procedure. We have also shown that when the full order controller is open-loop stable, Enns' method is equivalent to the LCFW or RCFW method when one chooses a particular set of the coprime factorizations of the plant and the controller in the latter methods. Certain examples in Anderson and Liu (1987) and Liu and Anderson (1988) confirm the principal conclusions reached in this paper.

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