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Coprime Factorization Controller Reduction with Bezout Identity Induced Frequency Weighting*

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Low order controllers can be obtained from high order controllers with good approximation of closed-loop behaviour. Controllers are represented using coprime matrix fractions.

Key Words—Approximation theory; controllers; linear optimal regulator; linear systems; model reduction; system order reduction.

Abstract—This paper introduces a frequency-weighted controller reduction scheme which offers good closed-loop stability under use of the reduced order controller. By combining the use of a stable fractional representation of the controller with a frequency weighting in the reduction procedure derived by consideration of the closed-loop stability margin, we formulate a frequency-weighted controller reduction procedure with the weight simplified by the Bezout identity. Some examples are used to illustrate the reduction procedure.

1. INTRODUCTION

THIS PAPER is concerned with the problem of reducing a high order controller to a low order one. For a discussion setting out motivations and alternatives, see, for example, Anderson and Liu (1987).

Controller reduction is distinct from the problem of (open-loop) model reduction, because of the presence of the plant. In the next section, we will show that it is sensible to pose the controller reduction problem as a frequency-weighted L^∞-norm optimization problem, with the plant appearing in the weight. Although the optimization problem cannot easily be solved exactly, several frequency-weighted order reduction methods yielding approximate solutions have been proposed, including frequency-weighted balanced reduction. By combining the frequency-weighted balanced reduction method (Enns, 1984a, b) and the idea of representing a controller as a stable coprime factorization (Liu and Anderson, 1986; Anderson and Liu, 1987), in Section 3 we propose a frequency-weighted controller reduction method which offers many attractive features beyond those of the method reviewed in Section 2. Some examples are used to illustrate the method in Section 4, and Section 5 contains concluding remarks.

2. FREQUENCY-WEIGHTED ORDER REDUCTION TECHNIQUES

The fundamental difference between model reduction and controller reduction is that model reduction is, at least normally, based on *open-loop* considerations; obviously, any controller reduction procedure ought to take into account the existence of the plant, i.e. it should be based on *closed-loop* considerations. Controller reduction should, after all, preserve *closed-loop* stability, and (as far as possible) the *closed-loop* performance and transfer function matrix, with the prime task the maintenance of closed-loop stability.

Making these arguments more precise turns out to produce frequency-weighted approximation problems. The choice of frequency weighting is influenced by the choice of criterion thought most important in the approximation process. In this paper, we focus our attention on the frequency weighting generated by the consideration of closed-loop stability margin [see Anderson and Liu (1987) for other types of frequency weighting].

Let $G(s)$ be the transfer function matrix of a given nth order linear time-invariant plant (with l inputs and m outputs), and let $K(s)$ be a

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stabilizing high order compensator obtained by some standard procedure. Let $K_r(s)$ be a reduced order compensator, which we are seeking. Regard the closed-loop system with $K_r(s)$ replacing $K(s)$ as being equivalent to that of Fig. 1. It can then be concluded (Doyle and Stein, 1981) using this redrawing (and it is now well known) that if

(i) $K(s)$ and $K_r(s)$ have the same number of poles in $\mathrm{Re}\,(s)>0$ and no poles on the imaginary axis (or have identical $\mathrm{j}\omega$-axis poles and residues); and

(ii) either

$$\|[K(s)-K_r(s)]G(s)[I+K(s)G(s)]^{-1}\|_\infty<1 \tag{1}$$

or

$$\|[I+G(s)K(s)]^{-1}G(s)[K(s)-K_r(s)]\|_\infty<1 \tag{2}$$

then $K_r(s)$ is a stabilizing compensator. (The notation $\|A(s)\|_\infty$ means $\sup_\omega \max_i \lambda_i^{\frac{1}{2}}\{A^{\mathrm{T}}(-\mathrm{j}\omega)A(\mathrm{j}\omega)\}$, $\lambda_i\{X\}$ denoting the ith eigenvalue of X.)

This clearly suggests a minimization problem: find a low order $K_r(s)$ satisfying (i) which at the same time minimizes the left-hand side of (1) or (2) for given $G(s)$ and $K(s)$. The matrix $G(I+KG)^{-1}=(I+GK)^{-1}G$ acts as a frequency-weighting matrix in this minimization based on the above closed-loop stability margin argument. The minimization problem just posed flows from a *sufficient* condition for closed-loop stability. There are certainly other sufficient conditions, and later we explore another condition, which requires specification of $G(s)$ as a fraction using stable proper rational matrices. These other conditions lead to a different frequency-weighted optimization problem.

The above frequency-weighted optimization problem cannot, in general, be easily solved. Related problems can, however, be comparatively easily solved. For example, some existing frequency-weighted order reduction methods for the above problem will make the left-hand side of (1) or (2) small, although not necessarily minimal. These include a scheme due to Enns (1984a, b) which generalizes the idea of truncating a balanced realization (Moore, 1982; Pernebo and Silverman, 1982) by incorporating frequency weighting into the balancing procedure.

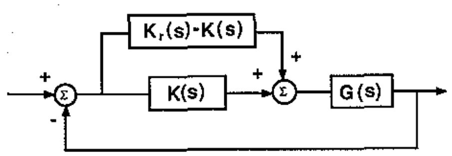

FIG. 1. Rearrangement of the feedback system with a reduced order compensator.

Consider an asymptotically stable frequency-weighting function $W_i(s)=E_i+C_i(sI-A_i)^{-1}B_i$ as an input weighting to the asymptotically stable system $M(s)=\bar{C}_M(sI-\bar{A}_M)^{-1}\bar{B}_M$ as shown in Fig. 2. The basic idea of frequency-weighted balanced realization of $M(s)$ is to find a coordinate basis change which will make a "frequency-weighted" controllability gramian of $M(s)$ equal to the observability gramian of $M(s)$ and both are diagonalized. In outline, this is done as follows. Let system matrices of the cascade system be defined as

$$\tilde{A}=\begin{bmatrix}\bar{A}_M & \bar{B}_M C_i\\ 0 & A_i\end{bmatrix},\qquad \tilde{B}=\begin{bmatrix}\bar{B}_M E_i\\ B_i\end{bmatrix}. \tag{3}$$

Assume that

$$\tilde{U}=\begin{bmatrix}\bar{U} & \bar{U}_{21}^{\mathrm{T}}\\ \bar{U}_{21} & \bar{U}_{22}\end{bmatrix} \tag{4}$$

is the solution of the following Lyapunov equation:

$$\tilde{A}\tilde{U}+\tilde{U}\tilde{A}^{\mathrm{T}}+\tilde{B}\tilde{B}^{\mathrm{T}}=0. \tag{5}$$

Now $\bar{U}$ in (4) can be regarded as the frequency-weighted controllability gramian of $M(s)$. Let $\bar{Y}$ be the observability gramian of $M(s)$, i.e. $\bar{Y}$ satisfies

$$\bar{Y}\bar{A}_M+\bar{A}_M^{\mathrm{T}}\bar{Y}+\bar{C}_M^{\mathrm{T}}\bar{C}_M=0. \tag{6}$$

Consider a coordinate basis change to change the realization $(\bar{A}_M,\ \bar{B}_M,\ \bar{C}_M)$ to a new realization $(A_M,\ B_M,\ C_M)$ with new gramians

$$U=Y=\Sigma=\mathrm{diag}\,\{\sigma_1,\sigma_2,\ldots,\sigma_n\}$$

with $\sigma_i\geq\sigma_{i+1}$, $i=1,2,\ldots,n-1$. There is no change to (A_i, B_i, C_i). We call this new realization a frequency-weighted balanced realization.

The frequency-weighted approximation is achieved by eliminating rows and columns of the new realization (A_M, B_M, C_M) corresponding to smallest $\{\sigma_n,\sigma_{n-1},\ldots,\sigma_{r+1}\}$, so the reduced order system is (A_{11}, B_1, C_1), where A_{11} is the top left $r\times r$ corner of A_M, etc. If $\sigma_r>\sigma_{r+1}$, the approximation is guaranteed stable. No simple formula bounding the approximation error is available.

The dual procedure for output frequency weighting is similar. In controller reduction, we identify the weighting function $W_i(s)$ with

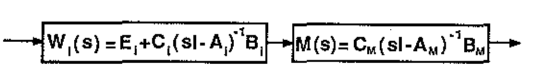

FIG. 2. Introduction of input weighting.

$G(s)[I + K(s)G(s)]^{-1}$ and, if $K(s)$ is stable, we can identify $K(s)$ with $M(s)$. If $K(s)$ is unstable, we decompose it into a stable strictly proper part $K_+(s)$, identified with $M(s)$ and reduced to $K_{+r}(s)$, and an unstable part $K_-(s)$, which is copied into the reduced order approximation $K_r(s)$, which is thus $K_r(s) = K_{+r}(s) + K_-(s)$. From (1) we know that if $\|[K(s) - K_r(s)]W_i(s)\|_\infty < 1$, then $K_r(s)$ is a low order stabilizing controller to the plant $G(s)$.

As it turns out, this controller reduction method works quite well compared with many unweighted controller reduction methods (Liu and Anderson, 1986). However, several qualifications must be made. First, the approximation problems posed in (1) and (2) are not fully appropriate for controllers with unstable or $j\omega$-axis poles. Consider a controller containing a pure integrator. The approximation problem posed demands that any approximation also contain a pure integrator with precisely the same residue of the controller transfer function at the pole $s = 0$. This shows that the approximation scheme is in some way unnecessarily restrictive. Also, the approach suggested above is to copy the unstable part (under additive decomposition) of the controller $K(s)$ into $K_r(s)$ and then just to approximate the stable part of $K(s)$ with the (lower order) stable part of $K_r(s)$. This certainly means we cannot utilize the part of the information contained in the unstable part of controller $K(s)$ in the actual reduction process. Second, if the controller $K(s)$ is stable, of order n, then generically (5) has order $3n$. This means that in solving the above frequency-weighted controller reduction problem, we need to find the solution of the $3n$ order Lyapunov equation (5). Some saving is possible if the controller $K(s)$ is determined by combining an estimator and state feedback law, and is stable, of order n, as then (5) has order $2n$. We have the following lemma (for proof, see Appendix A).

Lemma 1. For a given nth order linear time-invariant system $G(s) = C(sI_n - A)^{-1}B$, suppose L is a state estimator gain and F a state feedback gain, such that $A - BF$ and $A - LC$ are asymptotically stable, and the compensator of $G(s)$ is $K(s) = F(sI_n - A + BF + LC)^{-1}L$, with order n. If $A - BF - LC$ is stable, then the key Lyapunov equation in the frequency-weighted controller approximation method* posed by Enns (1984a, b) i.e. equation (5), has effective order $2n$.

* Recall that if the controller is not open-loop stable, we must reduce the stable part $K_+(s)$ of $K(s)$. The relevant Lyapunov equation involves a realization of $K_+(s)$, not $K(s)$, and will not, in general, have effective order $2n$.

All of these observations motivated us to consider an alternative method to cope with these problems. In the next section, we use the stability margin consideration for frequency weighting in conjunction with the stable fractional representation of the controller to formulate a new controller reduction method which will overcome some of the above problems. In particular, the problem of approximating the sum of a stable and an unstable object is replaced by the problem of approximating two stable objects, and the maximum Lyapunov equation dimension is $n \times n$, not $3n \times 3n$ or $2n \times 2n$.

However, we should note that we will be replacing one sufficient condition for stability under controller reduction by a different sufficient condition. As such, there is no *a priori* guarantee that either should be more effective than the other. We are therefore offering an algorithm that is much more in the nature of an alternative than an improvement to an existing algorithm. Also, our algorithm is restricted to the case of high order controllers obtained by combining a state feedback law and estimation.

3. COPRIME FACTORIZATION REDUCTION WITH BEZOUT IDENTITY INDUCED FREQUENCY WEIGHTING

In this section, we will consider controllers formed from a combination of a stabilizing feedback law and an estimator. For a given linear, time-invariant system, $G(s) = C(sI_n - A)^{-1}B$, design the state feedback gain F and the state estimator gain L (by LQG or other non-optimal pole-positioning methods) such that $A - BF$ and $A - LC$ have all eigenvalues in $\mathrm{Re}\,(s) < 0$. Then the compensator is

$$K(s) = F(sI_n - A + BF + LC)^{-1}L. \quad (7)$$

Define

$$\begin{bmatrix} Y(s) & -N(s) \\ X(s) & D(s) \end{bmatrix} \triangleq \begin{bmatrix} I_l & 0 \\ 0 & I_m \end{bmatrix} + \begin{bmatrix} -F \\ C \end{bmatrix}(sI_n - A + BF)^{-1}[B \quad L] \quad (8)$$

and

$$\begin{bmatrix} \tilde{D}(s) & \tilde{N}(s) \\ -\tilde{X}(s) & \tilde{Y}(s) \end{bmatrix} \triangleq \begin{bmatrix} I_l & 0 \\ 0 & I_m \end{bmatrix} + \begin{bmatrix} F \\ -C \end{bmatrix}(sI_n - A + LC)^{-1}[B \quad L]. \quad (9)$$

Then it has been proved by Nett *et al.* (1984) and Vidyasagar (1985) that

$$G(s) = X(s)Y^{-1}(s) = \tilde{Y}^{-1}(s)\tilde{X}(s) \quad (10)$$

$$K(s) = N(s)D^{-1}(s) = \tilde{D}^{-1}(s)\tilde{N}(s) \quad (11)$$

and the Bezout identity holds:

$$\begin{bmatrix} \tilde{D}(s) & \tilde{N}(s) \\ -\tilde{X}(s) & \tilde{Y}(s) \end{bmatrix} \begin{bmatrix} Y(s) & -N(s) \\ X(s) & D(s) \end{bmatrix} = \begin{bmatrix} Y(s) & -N(s) \\ X(s) & D(s) \end{bmatrix} \begin{bmatrix} \tilde{D}(s) & \tilde{N}(s) \\ -\tilde{X}(s) & \tilde{Y}(s) \end{bmatrix} = \begin{bmatrix} I_l & 0 \\ 0 & I_m \end{bmatrix}. \quad (12)$$

The transfer function matrices $N(s)$, $D(s)$, $\tilde{N}(s)$, $\tilde{D}(s)$, $X(s)$, $Y(s)$, $\tilde{X}(s)$ and $\tilde{Y}(s)$ are all stable and $D(s)$, $\tilde{D}(s)$, $Y(s)$ and $\tilde{Y}(s)$ are all nonsingular. Hence, the Bezout identity (12) means $N(s)D^{-1}(s)$ $(\tilde{D}^{-1}(s)\tilde{N}(s))$ is a stable right (left) coprime factorization of the controller $K(s)$. At the same time, $X(s)Y^{-1}(s)$ $(\tilde{Y}^{-1}(s)\tilde{X}(s))$ is a stable right (left) coprime factorization of the plant $G(s)$.

With the controller defined as in (7), we can draw the closed-loop system as shown in Fig. 3.

Now think of the controller as being defined by an interconnection rule together with a stable transfer function matrix (or pair of such matrices), viz.

$$\begin{bmatrix} C \\ F \end{bmatrix} (sI_n - A + BF)^{-1}L = \begin{bmatrix} D(s) - I_m \\ N(s) \end{bmatrix} \quad (13)$$

(see Fig. 4). This suggests that we can approximate

$$\begin{bmatrix} D(s) - I_m \\ N(s) \end{bmatrix} \text{ by some } \begin{bmatrix} D_r(s) - I_m \\ N_r(s) \end{bmatrix}$$

or approximate

$$\begin{bmatrix} D(s) \\ N(s) \end{bmatrix} \text{ by } \begin{bmatrix} D_r(s) \\ N_r(s) \end{bmatrix}$$

with the second matrix having McMillan degree r $(r < n)$, and then recover a reduced order controller by replicating the interconnection rule, i.e. the low order controller $K_r(s) = N_r(s)D_r^{-1}(s)$ (see Liu and Anderson, 1986; Anderson and Liu, 1987). This approach may be termed *the right coprime factorization* (*unweighted*) reduction method.

Notice that we have replace the additive decomposition $K(s) = K_+(s) + K_-(s)$ by a multiplicative decomposition $K(s) = N(s)D^{-1}(s)$ in order to pose a convenient reduction problem.

The heuristic approach to reduction just suggested is further justified in Liu and Anderson, (1986) and Anderson and Liu (1987)

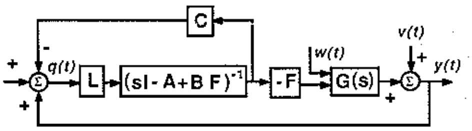

FIG. 3. State feedback/estimation based controller with plant.

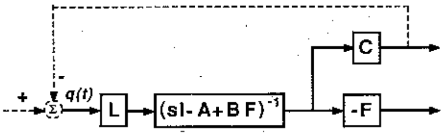

FIG. 4. "Controller" (heavy line) and interconnection rule (dashed line).

by appeal to a different rationale for selecting frequency weighting. The dual method to the above involves starting with the left stable coprime factorization of the controller $K(s) = \tilde{D}^{-1}(s)\tilde{N}(s)$, then approximating $[\tilde{D}(s) - I_l \;\; \tilde{N}(s)] = F(sI_n - A + LC)^{-1}[B \;\; L]$ by some low order $[\tilde{D}_r(s) - I_l \;\; \tilde{N}_r(s)]$ and finally implementing the reduced order controller as $K_r(s) = \tilde{D}_r^{-1}(s)\tilde{N}_r(s)$ (we may term this *the left coprime factorization* (*unweighted*) reduction approach). The significance of the transfer function matrix being approximated is evident from Fig. 5.

As shown by examples in Liu and Anderson (1986), the right coprime factorization (unweighted) controller reduction method can be very efficacious. However, if the controller $K(s)$ is not designed by the LQG method, or if one uses the left coprime factorization approach, then the justification for using a constant weighting in reduction will no longer hold [see Liu and Anderson (1986), Remark (c)], or at least is substantially diminished. Further, examples confirm that the unweighted left coprime factorization approach often results in a low order controller which is not stabilizing. This leads us to consider the introduction of frequency weighting in the above reduction methods, by working with a measure of (robust) stability margin.

Refer again to Fig. 5 and regard the compensator as a two (vector) input, single (vector) output system with transfer function

$$\tilde{\Gamma}(s) = [\tilde{D}(s) - I_l \;\; \tilde{N}(s)] = F(sI_n - A + LC)^{-1}[B \;\; L]. \quad (14)$$

Regard the "plant" model as defined by

$$\tilde{H}(s) = \begin{bmatrix} I_l \\ G(s) \end{bmatrix}$$

so that Fig. 5 is equivalent to Fig. 6. Let us seek to approximate $\tilde{\Gamma}(s)$ by a lower order stable $\tilde{\Gamma}_r(s)$ as in the left coprime factorization approach, but using a closed-loop stability margin point of view for the determination of

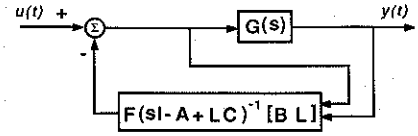

FIG. 5. State feedback law/estimator design of controller.

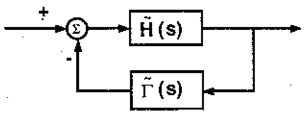

FIG. 6. Redrawing of the scheme of Fig. 5.

the weight. Thus, following the Doyle and Stein (1981) approach which led to (1), we seek to minimize

$$\tilde{\rho} = \|[\tilde{\Gamma}(s) - \tilde{\Gamma}_r(s)]\tilde{H}(s)[I_l + \tilde{\Gamma}(s)\tilde{H}(s)]^{-1}\|_\infty.$$

[If $\tilde{\rho} < 1$, this ensures stability with $\tilde{\Gamma}_r(s)$ replacing $\tilde{\Gamma}(s)$; the smaller $\tilde{\rho}(s)$ is, the better is the quality of the approximation.]

Defining $V(s) = \tilde{H}(s)[I_l + \tilde{\Gamma}(s)\tilde{H}(s)]^{-1}$, and using (8)–(10) and (14), we obtain

$$V(s) = \begin{bmatrix} I_l \\ X(s)Y^{-1}(s) \end{bmatrix} \times \left(I_l + [\tilde{D}(s) - I_l \;\; \tilde{N}(s)]\begin{bmatrix} I_l \\ X(s)Y^{-1}(s) \end{bmatrix}\right)^{-1}$$
$$= \begin{bmatrix} I_l \\ X(s)Y^{-1}(s) \end{bmatrix}[\tilde{D}(s) + \tilde{N}(s)X(s)Y^{-1}(s)]^{-1}$$
$$= \begin{bmatrix} Y(s) \\ X(s) \end{bmatrix}[\tilde{D}(s)Y(s) + \tilde{N}(s)X(s)]^{-1}.$$

Using the Bezout identity (12), we have

$$V(s) = \begin{bmatrix} Y(s) \\ X(s) \end{bmatrix} = \begin{bmatrix} I_l \\ 0 \end{bmatrix} + \begin{bmatrix} -F \\ C \end{bmatrix}(sI_n - A + BF)^{-1}B. \qquad (15)$$

Hence, our goal in controller reduction becomes one of minimizing or finding a procedure that will approximately minimize

$$\tilde{\rho} = \left\|[\tilde{D}(s) - \tilde{D}_r(s) \quad \tilde{N}(s) - \tilde{N}_r(s)]\begin{bmatrix} Y(s) \\ X(s) \end{bmatrix}\right\|_\infty \qquad (16)$$

over stable $\tilde{\Gamma}_r(s) = [\tilde{D}_r(s) \quad \tilde{N}_r(s)]$ of prescribed degree. We implement the lower order controller as $K_r(s) = \tilde{D}_r^{-1}(s)\tilde{N}_r(s)$.

It is very clear from Section 2 that if $\tilde{\rho} < 1$, then the closed-loop system with the reduced order controller based on $\tilde{\Gamma}_r(s)$ is guaranteed stable since $\tilde{\Gamma}(s)$ and $\tilde{\Gamma}_r(s)$ are asymptotically stable (and so have the same number of unstable poles, viz. zero, and no jω-axis poles). Obviously though, the smaller $\tilde{\rho}$ is, the better off we are likely to be.

As already noted, the above L^∞ norm optimization problem generally cannot be solved exactly in a straightforward manner. Hence, we will use the frequency-weighted balanced truncation approach as a tool for approximately solving the optimization problem. To make the connection with Section 2, observe that the weighting matrix $W_i(s) = E_i + C_i(sI - A_i)^{-1}B_i$ and stable system $M(s) = \bar{C}_M(sI - \bar{A}_M)^{-1}\bar{B}_M$ become in this section $V(s)$ of (15) and $\tilde{\Gamma}(s)$ of (14). It follows that the key Lyapunov equation (5) which earlier had $\tilde{A}$, $\tilde{B}$ given by (3) now has

$$\tilde{A} = \begin{bmatrix} A - LC & LC - BF \\ 0 & A - BF \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} B \\ B \end{bmatrix}. \qquad (17)$$

Notice the block triangular structures of $\tilde{A}$.

We have

Lemma 2. Consider a controller design for an nth order, linear, time-invariant plant $G(s) = C(sI_n - A)^{-1}B$. Assume that L is the state estimator gain and F is the state feedback gain designed by some standard method such that $A - LC$ and $A - BF$ are both asymptotically stable. Then to solve the controller reduction problem based on minimizing $\tilde{\rho}$ in (16) using frequency-weighted balanced realization truncation, one solves two $n \times n$ Lyapunov equations to find the weighted balanced realization, the same computational effort as for unweighted balancing. These Lyapunov equations are

$$Y(A - LC) + (A - LC)^{\mathrm{T}}Y + F^{\mathrm{T}}F = 0 \quad (18)$$
$$(A - BF)U + U(A - BF)^{\mathrm{T}} + BB^{\mathrm{T}} = 0. \quad (19)$$

The proof of this lemma is in Appendix A. Note that, in the light of (18) and (19), the balancing is not equivalent to the unweighted balancing of any transfer function matrix.

Observe that all frequency weightings are stable, as are the components in the fractional representations of the controller. So, there are no problems associated with open-loop instability (or jω-axis poles) of the controller $K(s)$. Moreover, the methods posed here do not require any underlying LQG assumption for their justification, heuristic or rigorous.

To sum up the similarities and contrasts with the scheme of Enns (1984a, b), we have:

Enns scheme	New scheme
Uses a sufficient condition for stability margin	Same (but a different sufficient condition)
Regards controller as sum of stable and unstable parts	Regards controller as fraction comprised of stable parts
Preserves open-loop controller instabilities exactly	Does not preserve open-loop instabilities exactly
Valid for any controller	Requires controller to be derived from state feedback and estimator
Requires $3n \times 3n$ Lyapunov equation solution (or sometimes $2n \times 2n$), n = dim plant and dim controller, and one $n \times n$ Lyapunov equation solution	Requires two $n \times n$ Lyapunov equation solutions

A dual result is available. We think of the controller is being defined by

$$\Gamma(s) = \begin{bmatrix} D(s) - I_m \\ N(s) \end{bmatrix}$$

(see Fig. 4 again). Carrying through reasoning analogous to the above [and using again the Bezout identity (12)] leads to the conclusion that we should use $[\tilde{Y}(s)\ \ \tilde{X}(s)]$ [defined as in (9)] as the output frequency weighting for controller reduction, seeking stable $D_r(s)$, $N_r(s)$ such that $[D_r^{\mathrm{T}}(s)\ \ N_r^{\mathrm{T}}(s)]^{\mathrm{T}}$ has a prescribed degree, and

$$\hat{\rho} = \left\| [\tilde{Y}(s)\ \ \tilde{X}(s)] \begin{bmatrix} D(s) - D_r(s) \\ N(s) - N_r(s) \end{bmatrix} \right\|_\infty \quad (20)$$

is minimum. We remark that $\hat{\rho} < 1$ [which guarantees the closed-loop stability with the reduced order controller $K_r(s) = N_r(s)D_r^{-1}(s)$] is itself guaranteed if

$$\left\| \begin{matrix} D(s) - D_r(s) \\ N(s) - N_r(s) \end{matrix} \right\|_\infty < \frac{1}{\|[\tilde{Y}(s)\ \ \tilde{X}(s)]\|_\infty}. \quad (21)$$

As a digression, we note that in Liu and Anderson (1986) Lemma 3.2 gave another condition

$$\left\| \begin{matrix} D(s) - D_r(s) \\ N(s) - N_r(s) \end{matrix} \right\|_\infty < \frac{1}{(1 + \|[I_m - D(s)\tilde{Y}(s) - D(s)\tilde{X}(s)]\|_\infty)\, \|D^{-1}(s)\|_\infty}. \quad (22)$$

We claim that (22) is more restrictive than (21). The proof is as follows:

$$\begin{aligned} \|[\tilde{Y}(s)\ \ \tilde{X}(s)]\|_\infty &= \|[D^{-1}(s)\ \ 0] - [D^{-1}(s) - \tilde{Y}(s) - \tilde{X}(s)]\|_\infty \\ &< \|D^{-1}(s)\|_\infty + \|[D^{-1}(s)][I_m - D(s)\tilde{Y}(s) \\ &\quad - D(s)\tilde{X}(s)]\|_\infty < (1 + \|[I_m - D(s)\tilde{Y}(s) \\ &\quad - D(s)\tilde{X}(s)]\|_\infty)\, \|D^{-1}(s)\|_\infty. \end{aligned}$$

Then the conclusion follows.

In many control system design problems, sometimes a single LQG-designed compensator is not enough. For instance, one may need an extra integral control loop to zero the steady-state error, or a shaping filter at input or even output to achieve some performance objective (see Fig. 7 for an illustration that encompasses simultaneously several possibilities). Controller reduction may be required with the constraint that shaping filters or an integral feedback loop are maintained. We now claim this can be easily achieved by using the above idea of combining the fractional representation of the LQG-designed compensator and the closed-loop stability margin consideration for frequency weighting.

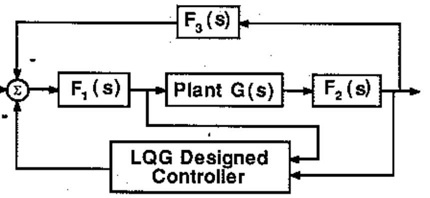

FIG. 7. General set-up allowing integral and/or a shaping filter.

Let us consider a special case to illustrate the basic idea. In this case, we assume that the compensator consists of a shaping filter at the input with an exact state feedback gain and an LQG-designed state estimator (with a state estimate feedback gain), i.e. in Fig. 7, there hold $F_2(s) = I$ and $F_3(s) \equiv 0$. Now assume the plant $G(s) = C(sI_n - A)^{-1}B$ and the input shaping filter (before application of the exact state feedback gain) to be $\hat{C}_f(sI_f - \hat{A}_f)^{-1}\hat{B}_f + \hat{D}_f$. To design an acceptable state feedback gain (at least in the sense of stabilizing the plant), one has to take into account the presence of the shaping filter in the design process. In other words, one should work with the augmented plant (or "frequency shaped plant") when one designs the state feedback gain. Let $\hat{F} \triangleq [\hat{F}_f\ \ F]$ be the state feedback gain for the augmented plant from an LQG design (or a pole positioning design). Here, $\hat{F}_f$ is the part of the feedback gain associated with the states of the shaping filter and F is the part of the feedback gain associated with the states of the original plant. In the case that not all entries of the state vector can be measured directly, or perhaps in the presence of input noise and measurement noise, it becomes necessary to estimate the state vector. In most cases (see Norman *et al.*, 1987), the state vector of the shaping filter is available (if not, we have a situation just like that in the last section but with the order of the system and hence the order of the compensator being $n + f$); hence one only needs to design a state estimator for the original plant. More precisely, if we assume L to be the estimator gain for the original plant, then in Fig. 7, in addition to having $F_2(s) = I$ and $F_3(s) \equiv 0$, we have that the transfer function matrix of the shaping filter with the exact state feedback $\hat{F}_j$ becomes $F_1(s) \triangleq C_f(sI_f - A_f)^{-1}B_f + D_f$ with $C_f = \hat{C}_f - \hat{D}_f\hat{F}_f$, $A_f = \hat{A}_f - \hat{B}_f\hat{F}_f$, $B_f = \hat{B}_f$ and $D_f = \hat{D}_f$, and the transfer function matrix of the state estimator together with the state feedback gain F becomes $\tilde{\Gamma}(s) = F(sI_n - A + LC)^{-1}[B\ \ L]$. The closed-loop system now is shown in Fig. 8.

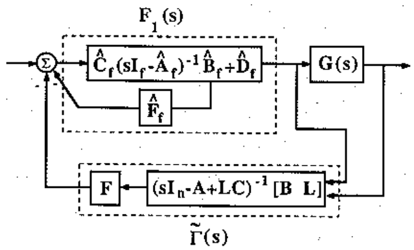

FIG. 8. Closed-loop system with a shaping filter and an LQG compensator.

Hence the closed-loop system of Fig. 7 is equivalent to that of Fig. 6 with now

$$\tilde{H}(s) = \begin{bmatrix} I_l \\ G(s) \end{bmatrix} F_1(s). \quad (23)$$

As before, we seek a stable lower order $\tilde{\Gamma}_r(s)$ which minimizes (or approximately minimizes)

$$\tilde{\rho} = \|[\tilde{\Gamma}(j\omega) - \tilde{\Gamma}_r(j\omega)]V(j\omega)\|_\infty \quad (24)$$

where

$$V(s) = \tilde{H}(s)[I + V(s)\tilde{H}(s)]^{-1} \quad (25)$$

and $\tilde{\Gamma}(s)$ is defined by (14), and $\tilde{H}(s)$ by (23). It is easy to see that

$$\tilde{H}(s) = \begin{bmatrix} I_l \\ G(s) \end{bmatrix} F_1(s)$$

$$= \begin{bmatrix} I_l \\ C(sI_n - A)^{-1}B \end{bmatrix} [C_f(sI_f - A_f)^{-1}B_f + D_f]$$

$$= \begin{bmatrix} D_f \\ 0 \end{bmatrix} + \begin{bmatrix} C_f & 0 \\ 0 & C \end{bmatrix} \left[sI_{n+f} - \begin{bmatrix} A_f & 0 \\ BC_f & A \end{bmatrix} \right]^{-1} \begin{bmatrix} B_f \\ BD_f \end{bmatrix}.$$

Hence, using (14), we have

$$V(s) = \begin{bmatrix} D_f \\ 0 \end{bmatrix} + \begin{bmatrix} C_f & 0 & -D_fF \\ 0 & C & 0 \end{bmatrix} \times \left[sI_{2n+f} - \begin{bmatrix} A_f & 0 & -B_fF \\ BC_f & A & -BD_fF \\ BC_f & LC & A - LC - BD_fF \end{bmatrix} \right]^{-1} \times \begin{bmatrix} B_f \\ BD_f \\ BD_f \end{bmatrix}. \quad (26)$$

This is not a minimal realization. By a simple similarity transformation, say,

$$T = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & I & I \end{bmatrix} \quad \left(T^{-1} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -I & I \end{bmatrix} \right)$$

we can easily find a lower order realization of $V(s)$,

$$V(s) = \begin{bmatrix} D_f \\ 0 \end{bmatrix} + \begin{bmatrix} C_f & -D_fF \\ 0 & C \end{bmatrix} \times \left[sI_{n+f} - \begin{bmatrix} A_f & -B_fF \\ BC_f & A - BD_fF \end{bmatrix} \right]^{-1} \begin{bmatrix} B_f \\ BD_f \end{bmatrix}. \quad (27)$$

Notice that now the frequency weighting $V(s)$ has order $n + f$ and the key Lyapunov equation (5) now has order $2n + f$. Usually, the order of the shaping filter will differ from the order of the LQG-designed compensator, and using the technique in Lemmas 1 and 2, we cannot in general reduce the order of the key Lyapunov equation (5) any further in this case. However, if one were to use the non-factorization procedure of Enns's frequency-weighted reduction method (Enns, 1984a, b), one would face an $(n + f)$th order controller, a $(2n + f)$th order frequency weighting and a $(3n + 2f)$th order key Lyapunov equation. More important, the shaping filter structure is lost after the reductions.

Note here that, generally speaking, if the shaping filter $F_1(s)$ is not constant, we do not readily obtain a dual formulation of the reduction problem (24).

4. EXAMPLES

In this section, we use some examples to illustrate the above reduction methods. Especially because our methods are linked to closed-loop stability considerations, we certainly need to investigate the stability properties of the methods. We compare the effects of different controller reduction procedures in the stability sense.

4.1. *Example* 1

This example is a four-disk drive system. The plant is linear, time-invariant, single input and single output, unstable, non-minimum phase and of eighth order. It was studied by Enns (1984a) and also in Liu and Anderson (1986) and Anderson and Liu (1987). The plant $G(s) = C(sI - A)^{-1}B$ with transfer function described in Liu and Anderson (1986) has a minimal realization (A, B, C), which is defined as in Anderson and Liu (1987). The weightings for an LQG design are given by $Q = q_1 H^T H$, $R = 1$, with $H = [0, 0, 0, 0, 0.55, 11, 1.32, 18]$, $q_1 = 10^{-6}$, for the LQ index and the filter covariance matrices are $W = q_2 BB^T$, $V = 1$, where q_2 is a design parameter.

The following table extends Table 2 of Liu and Anderson (1986) and Table 1 of Anderson and Liu (1987). It depicts the closed-loop stability of the systems with reduced order controllers of different orders obtained by different reduction methods, starting with different LQG designs. These different controller designs are obtained by adjusting the input noise intensity. The left coprime factorization (weighted) method is associated with the measure defined in (16), and the dual method, right coprime factorization (weighted) method is associated with (20).

From Table 1 we can see that in terms of ensuring closed-loop stability with the reduced order controller, the right coprime factorization frequency-weighted reduction method is the best for this example. As it turns out, the RCFW method shows only two failures in stabilizing the system with a low order controller. Enns's method, however, has 13 failures in the table and there are even more failures for other methods in the table. It is also interesting to note that when q_2 is large ($q_2 > 10^4$), the stability properties of Enns's method and the LCFW method are the same for this example. This turns out to be no accident, and is discussed in an unpublished work of the authors that has been submitted for publication. The other methods in the table are drawn from Yousuff and Skelton (1984), Davis and Skelton (1984) and Glover and Limebeer (1983). The first scheme does not take the plant into account, but proposes truncation of a balanced realization of the controller, if the latter is stable. The second involves balancing of the solutions of two Lyapunov equations, similar but not identical to (18) and (19); the argument leading to these equations is not based on stability margin

TABLE 1. CLOSED-LOOP STABILITY OF EXAMPLE 1

Controller order	Controller reduction methods	$q_2=10$	$q_2=100$	$q_2=10^3$	$q_2=2\times10^3$	$q_2=10^4$	$q_2=10^5$	$q_2=10^6$	$q_2=10^7$
7	YS	S	U	U	U	—	—	—	—
	DS	S	S	S	S	—	—	—	—
	GL	S	U	U	S	—	—	—	—
	Enns	S	S	S	S	S	S	S	S
	RCFU	S	S	U	U	U	U	U	U
	LCFW	S	S	S	S	S	S	S	S
	RCFW	U	S	S	S	S	S	S	S
6	YS	S	U	U	U	—	—	—	—
	DS	S	S	S	S	—	—	—	—
	GL	S	U	U	U	—	—	—	—
	Enns	S	S	S	S	S	S	S	U
	RCFU	S	S	S	U	U	U	U	U
	LCFW	S	S	S	S	S	S	S	U
	RCFW	S	S	S	S	S	S	S	S
5	YS	U	U	U	U	—	—	—	—
	DS	S	U	U	U	—	—	—	—
	GL	S	U	U	U	—	—	—	—
	Enns	S	S	S	S	S	S	U	U
	RCFU	S	S	S	S	U	U	U	U
	LCFW	S	S	S	S	S	S	U	U
	RCFW	S	S	S	S	S	S	S	S
4	YS	U	U	U	U	—	—	—	—
	DS	S	S	U	U	—	—	—	—
	GL	S	U	U	U	—	—	—	—
	Enns	S	S	S	U	S	S	U	U
	RCFU	S	S	S	S	U	U	U	U
	LCFW	S	S	U	S	S	S	U	U
	RCFW	S	S	S	S	S	S	S	S
3	YS	U	U	U	U	—	—	—	—
	DS	U	U	U	U	—	—	—	—
	GL	S	U	U	U	—	—	—	—
	Enns	S	S	S	S	U	U	U	S
	RCFU	S	S	U	U	U	U	U	U
	LCFW	S	S	S	U	U	U	U	S
	RCFW	S	S	S	S	U	S	S	S
2	YS	U	U	U	U	—	—	—	—
	DS	U	U	U	U	—	—	—	—
	GL	U	S	U	U	—	—	—	—
	Enns	S	U	U	U	U	S	S	S
	RCFU	S	S	S	S	U	U	U	U
	LCFW	U	U	U	U	U	S	S	S
	RCFW	S	S	S	S	S	S	S	S

YS: Yousuff and Skelton (1984a) method; DS: Davis and Skelton (1984) method; GL: Glover and Limebeer (1983) method; Enns: Enns's method (Enns, 1984a, b); RCFU: right coprime factorization (unweighted) method (Liu and Anderson, 1986); LCFW: left coprime factorization (weighted) method; RCFW: right coprime factorization (weighted) method. S = Stable; U = unstable; — = not available.

considerations, but does take into account the plant. The third involves using Hankel norm reduction to reduce the controller (and does not take into account the plant).

In addition, we should refer to another type of controller reduction due to de Villemagne and Skelton (1988) which performs excellently on this example. Besides controller order, other parameters are adjustable, and so there would be too many table entries to allow good illustration. For q_2 up to 10^6, all designs down to second order were stable. Results for $q_2=10^7$ are not available. The key idea in this method is to eliminate those controller states which are least correlated with the plant states. This is clearly intuitively very appealing; the connection with stability margins and frequency weighting is, however, beyond our capacity to fathom.

Table 2 depicts the closed-loop gain margins and Table 3 the closed-loop phase margins of the system with the reduced order controllers. Again, it is interesting to note that when q_2 is large, the gain margins and phase margins of the closed-loop system with the reduced order controller obtained by Enn's method and by the LCFW method are very similar.

4.2. *Example* 2

This is a flutter control system for a B-767 airplane (Ly and Gangsaas, 1979; Ly *et al.*, 1981). The model is unstable, non-minimum phase, with two inputs and two outputs, and of 55th order. We start with an LQG-designed 55th order controller to compare Enns's reduction method and the LCFW and the RCFW reduction method for the stability properties and certain mean square responses in key variables and certain gain and phase margins. The closed-loop system under consideration when a full order controller is used has the structure shown in Fig. 9. The system data (A, B, C) and the state feedback gain F and Kalman gain L as well as matrices B_N and C_N are shown in the Appendix B. We reduce the 55th order controller down to every order between 1 and 54 using the three reduction methods and compare the closed-loop stability with these controllers. As it turns out, Enns's method yields a stabilizing controller for all orders from 1 to 54. The LCFW method yields a stabilizing controller except for two failures (at order 1 and order 3 cases). The RCFW method has only one failure (at order 1).

TABLE 2. GAIN MARGINS (dB) OF EXAMPLE 1

Controller order	Reduction method*	$q_2=10$	$q_2=100$	$q_2=1000$	$q_2=2000$	$q_2=10^4$	$q_2=10^5$	$q_2=10^6$	$q_2=10^7$
8	Full order	15.94	17.37	19.08	19.44	20.46	21.84	23.37	24.92
7	Enns	15.92	16.54	16.71	16.26	11.36	6.32	7.69	14.83
	RCFU	16.05	16.95	—	—	—	—	—	—
	LCFW	16.00	17.12	16.84	16.60	14.55	6.03	6.68	17.13
	RCFW	—	7.14	6.74	6.36	5.44	5.18	7.99	6.21
6	Enns	15.92	17.22	12.43	11.78	9.13	7.36	8.04	—
	RCFU	16.01	17.28	19.54	—	—	—	—	—
	LCFW	15.97	17.39	9.43	8.98	6.44	5.99	5.42	—
	RCFW	15.53	13.95	6.76	6.80	6.06	10.24	6.59	7.25
5	Enns	15.97	12.04	5.42	5.45	4.57	4.48	—	—
	RCFU	17.13	15.77	5.32	3.23	—	—	—	—
	LCFW	16.63	18.84	5.75	5.86	4.99	4.41	—	—
	RCFW	8.42	4.44	4.16	4.18	4.17	3.89	3.00	2.54
4	Enns	15.31	8.55	1.24	—	5.03	5.02	—	—
	RCFU	16.53	11.90	3.88	1.89	—	—	—	—
	LCFW	14.82	8.85	—	0.78	5.69	5.38	—	—
	RCFW	8.91	5.74	4.77	4.71	5.70	9.98	6.43	3.57
3	Enns	10.93	5.97	5.59	5.25	—	—	—	7.30
	RCFU	10.54	0.50	—	—	—	—	—	—
	LCFW	24.57	7.92	6.89	—	—	—	—	7.41
	RCFW	7.39	5.19	4.03	5.85	—	8.02	7.71	7.47
2	Enns	0.88	—	—	—	—	7.18	7.42	7.23
	RCFU	10.51	4.79	1.39	0.48	—	—	—	—
	LCFW	—	—	—	—	—	7.77	7.83	7.46
	RCFW	24.89	20.84	7.83	5.85	7.26	7.46	6.81	6.73

* For abbreviations see Table 1.
— = Closed-loop system is unstable.

TABLE 3. PHASE MARGINS (DEGREE) OF EXAMPLE 1

Controller order	Reduction method	$q_2=10$	$q_2=100$	$q_2=1000$	$q_2=2000$	$q_2=10^4$	$q_2=10^5$	$q_2=10^6$	$q_2=10^7$
8	Full order	52.32	54.31	55.88	56.32	57.21	58.49	59.62	60.55
7	Enns	52.92	44.99	33.60	31.89	27.45	42.77	51.81	54.02
	RCFU	52.59	52.64	—	—	—	—	—	—
	LCFW	52.87	40.66	32.10	29.09	26.06	41.17	54.38	57.84
	RCFW	—	26.19	27.35	30.44	31.85	30.89	33.00	36.08
6	Enns	52.30	54.03	55.10	55.84	58.66	62.14	63.42	—
	RCFU	52.27	54.33	55.97	—	—	—	—	—
	LCFW	52.31	53.75	54.66	55.30	58.23	58.67	65.99	—
	RCFW	51.84	52.69	36.87	38.98	31.66	34.11	37.48	40.50
5	Enns	48.61	14.50	25.95	26.20	23.95	16.02	—	—
	RCFU	56.52	57.28	57.05	57.06	—	—	—	—
	LCFW	31.33	12.79	29.58	31.05	30.68	22.56	—	—
	RCFW	13.15	12.25	13.31	13.70	14.37	14.90	15.11	16.21
4	Enns	52.04	47.66	60.03	—	15.04	15.61	—	—
	RCFU	52.05	54.02	55.72	56.20	—	—	—	—
	LCFW	51.70	40.20	—	61.37	13.03	14.59	—	—
	RCFW	9.54	14.47	16.62	16.99	18.16	22.70	25.84	29.79
3	Enns	6.39	8.02	8.59	11.54	—	—	—	15.91
	RCFU	52.20	50.81	—	—	—	—	—	—
	LCFW	1.81	10.14	6.51	—	—	—	—	14.34
	RCFW	3.29	4.76	2.91	9.13	—	13.39	14.10	14.71
2	Enns	51.76	—	—	—	—	14.93	14.78	15.28
	RCFU	51.78	54.15	56.06	56.52	—	—	—	—
	LCFW	—	—	—	—	—	13.82	14.06	14.87
	RCFW	14.67	18.58	19.97	18.22	15.79	15.60	15.85	16.20

* For abbreviations see Table 1.
— = Closed-loop system is unstable.

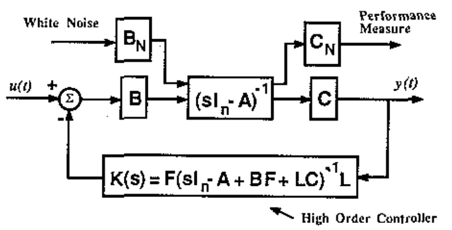

FIG. 9. Closed-loop system structure of Example 2.

The most interesting issue in this example is the numerical problem. Since the order of the original controller is very large now, i.e. $n = 55$, if we use Enns's method directly without using the result of Lemma 1, we are forced to solve a 165 order Lyapunov equation. For many Lyapunov equation solvers, the dimension is too large, or a numerically accurate result is not obtainable. Even when we employ the result of Lemma 1, we are still faced with solving a 110 order Lyapunov equation, which is often not an easy task. In contrast, the key Lyapunov equation in the LCFW or the RCFW method has only 55th order. From the closed-loop stability point of view, both methods are not much worse off than Enns's method in this example.

In this example, we implemented Enns's method by solving a 110 order Lyapunov equation, and almost surely the procedure introduces certain numerical errors. For the LCFW method, solution of the 55th order Lyapunov equations will also have introduced numerical errors. Whether these errors are affecting the results on closed-loop stability is not yet clear.

We have also applied certain scalings to both the input and the output of the controller (and also the plant when formulating the frequency weighting). More precisely, we use the scaled B_s, C_s, F_s and L_s in the balancing process instead of the unscaled B, C, F and L defined in Appendix B. Here $B_s = B(R^{1/2})^{-1}$, $C_s = (V^{1/2})^{-1}C$, $F_s = R^{1/2}F$ and $L_s = LV^{1/2}$, where

$$R = \begin{bmatrix} 364.8 & 0 \\ 0 & 14.59 \end{bmatrix}$$

$$\text{and} \quad V = \begin{bmatrix} 6.85 \times 10^{-6} & 0 \\ 0 & 373 \end{bmatrix}.$$

Note that R is the control weighting matrix in the LQ index and V is the measurement noise intensity matrix in the full order LQG design.

Table 4 depicts the simulation results of the closed-loop systems with different reduced order controllers. In the table, "Boeing 10" means the controller is of tenth order and obtained by a Boeing engineer using a modal reduction method (Ly and Gangsaas, 1979) which matches

TABLE 4. SIMULATION RESULTS ON CLOSED-LOOP SYSTEMS OF EXAMPLE 2

Reduction method	Damping	BMOMI	TORI	GM 1 (dB)	PM 1 (degree)	GM 2 (dB)	PM 2 (degree)
Full order	0.0743	234800	44369	15.86	180	13.99	58.6
Boeing 10	0.0699	234900	45969	36.40	180	25.12	−138.9
Enns 4	0.0290	252190	46267	20.60	−60.43	7.51	66.66
LCFW 4	0.0169	257230	57225	19.47	180	12.83	72.82
LCFU 4	0.0158	245320	65695	11.39	180	16.58	85.06
RCFW 4	0.0070	268160	96521	25.95	180	−7.68	38.39
Enns 5	0.0329	239200	45282	9.83	87.48	8.66	64.07
LCFW 5	0.0501	262380	49689	18.17	−102	14.40	82.30
LCFU 5	—	—	—	—	—	—	—
RCFW 5	0.0070	267740	96155	25.94	180	−7.7	38.55
Enns 6	0.0334	238490	45147	9.99	180	9.78	64.05
LCFW 6	0.0354	253440	63977	0.34	−6.16	−0.26	4.94
LCFU 6	0.0448	238010	47335	14.96	180	12.19	54.16
RCFW 6	0.0108	251460	76700	13.47	108.7	14.88	56.13
Enns 9	0.0668	240370	45600	8.66	180	10.85	61.85
LCFW 9	0.0380	255140	46960	8.37	180	6.58	62.91
LCFU 9	0.0430	235230	47104	15.03	180	13.45	−38.88
RCFW 9	0.0122	236040	68364	13.49	180	10.47	59.24
Enns 10	0.0570	242410	46403	8.41	180	11.78	60.94
LCFW 10	0.0443	241980	46229	9.085	69.53	10.36	62.97
LCFU 10	0.0423	236570	44599	5.25	110.4	11.39	74.86
RCFW 10	0.0115	235890	66836	7.48	−49.54	9.17	49.54

Frequency range considered is: $0.01 \le \omega \le 100$ rad/s.

Damping: closed-loop damping of the flutter mode; BMOMI: bending moment at the wing root station (vertical turbulence is 10 ft/s); TORI: torsion moment at the wing root station (vertical turbulence is 10 ft/s); GM 1 and PM 1: gain and phase margins of the elevator control loop (classical single loop sense); GM 2 and PM 2: gain and phase margins of the aileron control loop (classical single loop sense).

the DC gains of the full order controller and this tenth order controller as well. "Enns 4" means a fourth order controller obtained by Enns's method, and so on. In Table 4, the first column displays the closed-loop damping of the flutter mode, which is identified by its frequency at around 20 rad/s and its light damping. (A minimum closed-loop damping of 0.015 is required for an acceptable design.) The second and third columns show the bending and the torsion moments, BMOMI and TORI, corresponding to the first and second diagonal elements of the 2×2 output intensity matrix (standard deviation) through C_N respectively at the wing root station. (The values achieved by the full order LQG design would serve as design goals.) The intensity of exciting white noise (corresponding to the vertical turbulence) injected into the closed-loop system from B_N is 10 (ft/s). The last four columns of the table show the gain and phase margins evaluated one-loop-at-a-time using classical single-loop analysis. The minimum gain margin of ± 6 dB and the minimum phase margin of $\pm 45°$ are required for an acceptable design. As we mentioned before, since all these reduction methods require solution of a high order Lyapunov equation to form a certain transformation for balancing, the introduction of numerical errors is unavoidable. Hence the simulation results are based on "not so accurate" lower order controllers.

5. CONCLUSIONS

Throughout this paper we have argued that the controller reduction problem is appropriately viewed as a frequency-weighted order reduction problem. By a combination of the closed-loop stability margin consideration and stable fractional representations of the controller, we proposed two frequency-weighted controller reduction schemes, namely, the LCFW and the RCFW approaches. As shown in Lemma 2, these two methods require the same computational effort as the unweighted internally balanced realization truncation method. Compared with Enns's method, this offers a numerical advantage with little sacrifice of the stability properties, particularly when the original controller order is large (as in Example 2). In fact, in Example 1, the RCFW method has the best stability properties. It is also interesting to note that for some examples the LCFW method works better and for some the RCFW methods work better. However, we consider that the LCFW method is the more natural method. This is because the left coprime factorization representation of the controller is directly related to the structure of the estimator and state feedback form compensator, and it can be easily extended to the more general cases (controller with integral feedback loop and shaping filter or, introducing an extra output into the balancing to improve the closed-loop performance, as shown in Anderson and Liu (1987)). This is not the case for the right coprime factorization representation of the controller.

Certainly it would be desirable for any controller reduction method to have an easily calculable frequency error bound. For the frequency-weighted balanced realization truncation reduction method, up to now, no satisfactory frequency error bound is available. This is also the case for the LCFW and the RCFW method.

Last, we mention that when the controller order n is very large, almost all controller reduction procedures will introduce numerical errors. It is not clear how sensitive the reduction results will be with respect to these numerical errors.

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APPENDIX A: PROOFS OF LEMMA 1 AND 2

Proof of Lemma 1

For any given controller $K(s) = C_K(sI - A_K)^{-1}B_K$ of order n, the frequency weighting is given by

$$W_i(s) = G(s)[I + K(s)G(s)]^{-1} = [C\ \ 0]\left\{sI - \begin{bmatrix} A & -BC_K \\ B_KC & A_K \end{bmatrix}\right\}^{-1}\begin{bmatrix} B \\ 0 \end{bmatrix}.$$

When $K(s)$ is stable, $M(s) = K_+(s) = K(s)$, and in equation (5),

$$\tilde{A} = \begin{bmatrix} A_K & B_KC & 0 \\ 0 & A & -BC_K \\ 0 & B_K & A_K \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} 0 \\ B \\ 0 \end{bmatrix}$$

i.e. the Lyapunov equation (5) generically has order $3n$.

Now, by assumption that the controller has the structure of a state estimator and the state feedback law, the controller $K(s) = F(sI_n - A + BF + LC)^{-1}L$, where L is the estimator gain and F the feedback gain. Define $A_L \triangleq A - LC$, $A_F \triangleq A - BF$, and $A_C \triangleq A - BF - LC$. Note that both A_L and A_F are stable in any acceptable compensator design, while the lemma hypothesis requires A_C to be stable.

It follows that

$$\tilde{A} = \begin{bmatrix} A_C & LC & 0 \\ 0 & A & -BF \\ 0 & LC & A_C \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} 0 \\ B \\ 0 \end{bmatrix}.$$

Let

$$T = \begin{bmatrix} I_n & I_n & I_n \\ 0 & I_n & 0 \\ 0 & I_n & I_n \end{bmatrix}, \quad \text{then} \quad T^{-1} = \begin{bmatrix} I_n & 0 & -I_n \\ 0 & I_n & 0 \\ 0 & -I_n & I_n \end{bmatrix}.$$

Define $\hat{A} = T^{-1}\tilde{A}T$, $\hat{B} = T^{-1}\tilde{B}$ and $\hat{U} = T^{-1}\tilde{U}(T^{\mathrm{T}})^{-1}$. Then it is easy to see that equation (5) becomes

$$\hat{A}\hat{U} + \hat{U}\hat{A}^{\mathrm{T}} + \hat{B}\hat{B}^{\mathrm{T}} = 0 \tag{A1}$$

and in fact

$$\hat{A} = \begin{bmatrix} A_C & 0 & 0 \\ 0 & A_F & -BF \\ 0 & 0 & A_L \end{bmatrix}, \qquad \hat{B} = \begin{bmatrix} 0 \\ B \\ -B \end{bmatrix}.$$

Let

$$\hat{U} = \begin{bmatrix} U & U_{12} & U_{13} \\ U_{12}^{\mathrm{T}} & U_{22} & U_{23} \\ U_{13}^{\mathrm{T}} & U_{23}^{\mathrm{T}} & U_{33} \end{bmatrix}.$$

Then the first row of (A1) yields:

$$A_CU + UA_C^{\mathrm{T}} = 0$$

$$A_CU_{12} + U_{12}A_F^{\mathrm{T}} - U_{13}F^{\mathrm{T}}B^{\mathrm{T}} = 0$$

$$A_CU_{13} + U_{13}A_L^{\mathrm{T}} = 0.$$

Since A_C, A_F and A_L are all stable, we have $U \equiv 0$, $U_{13} \equiv 0$ and $U_{12} \equiv 0$. Hence, to solve equation (5), we only need to solve the following $2n$ order Lyapunov equation:

$$\begin{bmatrix} A_F & -BF \\ 0 & A_L \end{bmatrix}\begin{bmatrix} U_{22} & U_{23} \\ U_{23}^{\mathrm{T}} & U_{33} \end{bmatrix} + \begin{bmatrix} U_{22} & U_{23} \\ U_{23}^{\mathrm{T}} & U_{33} \end{bmatrix}\begin{bmatrix} A_F^{\mathrm{T}} & 0 \\ -F^{\mathrm{T}}B^{\mathrm{T}} & A_L^{\mathrm{T}} \end{bmatrix} + \begin{bmatrix} B \\ -B \end{bmatrix}[B^{\mathrm{T}} - B^{\mathrm{T}}] = 0. \tag{A2}$$

We obtain the solution of (5) by setting

$$\tilde{U} = T\begin{bmatrix} 0 & 0 & 0 \\ 0 & U_{22} & U_{23} \\ 0 & U_{23}^{\mathrm{T}} & U_{33} \end{bmatrix} T^{\mathrm{T}}$$

and so obtain the frequency-weighted controllability gramian of $K(s)$ [defined as in (4)]

$$\bar{U} = U_{22} + U_{23} + U_{23}^{\mathrm{T}} + U_{33}$$

for balancing purpose.

Proof of Lemma 2

In this case, we must prove that with identification of (17) the key Lyapunov equation (5) is equivalent to one of order n.

Assume $\tilde{U} = \begin{bmatrix} \bar{U} & \bar{U}_{12} \\ \bar{U}_{12}^{\mathrm{T}} & \bar{U}_{22} \end{bmatrix}$ [the same as in (4)] and define $T \triangleq \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}$, and

$$\hat{A} \triangleq T^{-1}\tilde{A}T = \begin{bmatrix} A - BF & LC - BF \\ 0 & A - LC \end{bmatrix}$$

$$\hat{B} \triangleq T^{-1}\tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

$$\hat{U} \triangleq T^{-1}\tilde{U}(T^{\mathrm{T}})^{-1} \triangleq \begin{bmatrix} U & U_{12} \\ U_{12}^{\mathrm{T}} & U_{22} \end{bmatrix}.$$

Then equation (5) is equivalent to

$$\hat{A}\hat{U} + \hat{U}\hat{A}^{\mathrm{T}} + \hat{B}\hat{B}^{\mathrm{T}} = 0 \qquad \text{(A3)}$$

or

$$\begin{aligned} (A - BF)U + (LC - BF)U_{12}^{\mathrm{T}} \\ + U(A - BF)^{\mathrm{T}} + U_{12}(LC - BF)^{\mathrm{T}} + BB^{\mathrm{T}} = 0 \\ (A - BF)U_{12} + (LC - BF)U_{22} + U_{12}(A - LC)^{\mathrm{T}} = 0 \\ (A - LC)U_{22} + U_{22}(A - LC)^{\mathrm{T}} = 0. \end{aligned}$$

Since $A - BF$ and $A - LC$ are both asymptotically stable, we have $U_{22} \equiv 0$, $U_{12} \equiv 0$ and

$$(A - BF)U + U(A - BF)^{\mathrm{T}} + BB^{\mathrm{T}} = 0. \qquad \text{(A4)}$$

Hence, to solve the Lyapunov equation (5), we only need to solve (A4), and construct the solution of (5) by

$$\tilde{U} = T\begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix}T^{\mathrm{T}} = \begin{bmatrix} U & U \\ U & U \end{bmatrix}$$

or simply us $\bar{U} = U$ to construct the weighted balanced realization in the reduction process.

APPENDIX B: DATA OF EXAMPLE 2

The system matrices of Example 2 are (A, B, C), with F the state feedback gain matrix and L the Kalman gain matrix for a certain LQG design; the matrices B_N and C_N arise in connection with the closed-loop performance measures (defined as in Fig. 9). For any matrix A, $A(i, j)$ stands for the ith row and the jth column element of A. $A(:, j)$ is the jth column vector of A. There holds:

$$A \triangleq \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}$$

with the dimension of A_1, A_2 and A_3 being 45×45, 45×10, 10×10,

$$A_1 \triangleq \operatorname{diag}\{\Lambda_i\}, \quad i = 1, 2, \ldots, 23.$$

with Λ_i all 2×2 except Λ_{15}, which is 1×1.

$$A_2 \triangleq [A_{12} \quad A_{13}]$$

with A_{12} and A_{13} of dimension 45×6 and 45×4.

$$A_3 \triangleq \begin{bmatrix} \Lambda_{24} & 0 & A_{14} \\ 0 & \Lambda_{24} & A_{15} \\ 0 & 0 & A_{25} \end{bmatrix}$$

with A_{14} and A_{15} both of dimensions 3×4. Also

$$A_{14}(3, 3) = A_{15}(3, 4) = -1.6 \times 10^7$$

and

$$A_{14}(i, j) = 0 \quad \text{and} \quad A_{15}(i, j) = 0 \quad \text{for other } i, j.$$

Further,

$$\Lambda_1 = \begin{bmatrix} 1.0155 \times 10^{-1} & -1.9771 \times 10 \\ 1.9771 \times 10 & 1.0155 \times 10^{-1} \end{bmatrix}$$

$$\Lambda_2 = \begin{bmatrix} -2.3202 \times 10^{-2} & -9.2543 \times 10^{-2} \\ 19.2543 \times 10^{-2} & -2.3202 \times 10^{-2} \end{bmatrix}$$

$$\Lambda_3 = \begin{bmatrix} -3.1651 \times 10^{-1} & -1.4325 \times 10 \\ 1.4325 \times 10 & -3.1651 \times 10^{-1} \end{bmatrix}$$

$$\Lambda_4 = \begin{bmatrix} -3.8919 \times 10^{-1} & -2.2292 \times 10 \\ 2.2292 \times 10 & -3.8919 \times 10^{-1} \end{bmatrix}$$

$$\Lambda_5 = \begin{bmatrix} -9.8829 \times 10^{-1} & -3.6158 \times 10 \\ 3.6158 \times 10 & -9.829 \times 10^{-1} \end{bmatrix}$$

$$\Lambda_6 = \begin{bmatrix} -1.3422 & -2.5473 \\ 2.5473 & -1.3422 \end{bmatrix}$$

$$\Lambda_7 = \begin{bmatrix} -2.3120 & -2.1514 \times 10 \\ 2.1514 \times 10 & -2.3120 \end{bmatrix}$$

$$\Lambda_8 = \begin{bmatrix} -2.7924 & -2.6708 \times 10 \\ 2.6708 \times 10 & -2.7924 \end{bmatrix}$$

$$\Lambda_9 = \begin{bmatrix} -2.7957 & -6.3792 \times 10 \\ 6.3792 \times 10 & -2.7957 \end{bmatrix}$$

$$\Lambda_{10} = \begin{bmatrix} -3.3321 & -5.6368 \times 10 \\ 5.6368 \times 10 & -3.3321 \end{bmatrix}$$

$$\Lambda_{11} = \begin{bmatrix} -3.4179 & -8.8668 \times 10 \\ 8.8668 \times 10 & -3.4179 \end{bmatrix}$$

$$\Lambda_{12} = \begin{bmatrix} -4.4407 & -6.9454 \times 10 \\ 6.9454 \times 10 & -4.4407 \end{bmatrix}$$

$$\Lambda_{13} = \begin{bmatrix} -5.1083 & -5.3390 \times 10 \\ 5.3390 \times 10 & -5.1083 \end{bmatrix}$$

$$\Lambda_{14} = \begin{bmatrix} -5.1986 & -5.0642 \times 10 \\ 5.0642 \times 10 & -5.1986 \end{bmatrix}$$

$$\Lambda_{15} = [-5.3010],$$

$$\Lambda_{16} = \begin{bmatrix} -5.6705 & -9.2964 \times 10 \\ 9.2964 \times 10 & -5.6705 \end{bmatrix}$$

$$\Lambda_{17} = \begin{bmatrix} -6.1975 & -4.000 \times 10 \\ 4.000 \times 10 & -6.1975 \end{bmatrix}$$

$$\Lambda_{18} = \begin{bmatrix} -8.1771 & -1.6614 \times 10 \\ 1.6614 \times 10 & -8.1771 \end{bmatrix}$$

$$\Lambda_{19} = \begin{bmatrix} -8.2205 & -1.3906 \times 10^2 \\ 1.3906 \times 10^2 & -8.2205 \end{bmatrix}$$

$$\Lambda_{20} = \begin{bmatrix} -1.0346 \times 10 & -1.6305 \times 10^2 \\ 1.6305 \times 10^2 & -1.0346 \times 10 \end{bmatrix}$$

$$\Lambda_{21} = \begin{bmatrix} -1.720 \times 10 & -1.0934 \times 10^2 \\ 1.0934 \times 10^2 & -1.1720 \times 10 \end{bmatrix}$$

$$\Lambda_{22} = \begin{bmatrix} -1.1794 \times 10 & -3.0459 \times 10^2 \\ 3.0459 \times 10^2 & -1.1794 \times 10 \end{bmatrix}$$

$$\Lambda_{23} = \begin{bmatrix} -3.3270\times 10 & 0.0000 \\ 0.0000 & -2.2120\times 10^2 \end{bmatrix}$$

$$\Lambda_{24} = \begin{bmatrix} 0.0000 & 1.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 \\ -8.00\times 10^5 & -6.08\times 10^4 & -1.06\times 10^3 \end{bmatrix}$$

$$\Lambda_{25} = \begin{bmatrix} 0.0000 & -2.6685\times 10^{-1} & 0.0000 & 0.0000 \\ 1.0000 & -1.0331 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & -2.000\times 10 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & -2.0000\times 10 \end{bmatrix}.$$

And furthermore, we have

$$A_{12} = \begin{bmatrix}
-1.9703\times 10^3 & 6.7959 & -1.3345\times 10^{-1} & -1.4288\times 10^3 & 1.8092\times 10 & -1.5168\times 10^{-1} \\
-2.6654\times 10^3 & 7.1957 & -1.5685\times 10^{-1} & -2.2217\times 10^3 & -2.6844 & -5.0393\times 10^{-2} \\
5.8687\times 10^2 & 1.7334\times 10^{-3} & -4.3119\times 10^{-5} & 1.1897\times 10^2 & 4.8611\times 10^{-4} & -7.4279\times 10^{-6} \\
-5.0115\times 10^2 & 4.1488\times 10^{-3} & -8.0848\times 10^{-5} & -1.0587\times 10^2 & 1.0274\times 10^{-3} & -1.1816\times 10^{-5} \\
1.3447\times 10^2 & -2.6899\times 10^{-1} & 4.9694\times 10^{-3} & -1.4684\times 10^2 & -7.0967\times 10^{-1} & 1.2317\times 10^{-3} \\
-1.6447\times 10^2 & 4.6275\times 10^{-1} & -8.4320\times 10^{-3} & 1.9597\times 10^2 & 1.3449 & -3.9645\times 10^{-3} \\
6.0846\times 10^2 & -1.8346 & 3.6978\times 10^{-2} & 1.7513\times 10^2 & -1.0825 & 1.1391\times 10^{-2} \\
-7.7545\times 10 & 1.5948\times 10^{-1} & -2.0525\times 10^{-3} & 9.6157 & 6.3698\times 10^{-1} & -2.8823\times 10^{-3} \\
-1.1167\times 10^3 & 4.6147 & -8.4286\times 10^{-2} & -3.4141\times 10 & 6.428 & -4.9071\times 10^{-2} \\
1.7902\times 10^3 & -3.7516 & 7.1871\times 10^{-2} & -2.2269\times 10^2 & 1.8987 & -1.5192\times 10^{-2} \\
-1.3858\times 10^4 & 5.5496\times 10^{-1} & -1.2727\times 10^{-2} & -2.0140\times 10^3 & -5.0439\times 10^{-2} & 1.0263\times 10^{-3} \\
1.5010\times 10^4 & -3.1376 & 6.5016\times 10^{-2} & 2.6684\times 10^3 & -1.0603 & 1.1647\times 10^{-2} \\
-2.1697\times 10^4 & 6.1457\times 10 & -1.2860 & -3.6519\times 10^3 & 5.0738 & -1.2766\times 10^{-1} \\
1.9966\times 10^4 & -5.3558\times 10 & 1.1209 & 2.8813\times 10^3 & -2.6771\times 10 & 2.5400\times 10^{-1} \\
-2.8609\times 10^3 & -1.3140\times 10 & -2.4260\times 10^{-1} & -1.0840\times 10^3 & 4.8489\times 10 & -3.7124\times 10^{-1} \\
6.5827\times 10^3 & -1.9317\times 10 & 3.8409\times 10^{-1} & -3.8832\times 10^3 & -1.2933\times 10 & 1.0825\times 10^{-2} \\
-2.2116\times 10^3 & -1.2284\times 10 & 1.2287\times 10^{-1} & -1.6467\times 10^3 & -5.1298 & 2.0187\times 10^{-2} \\
6.3210\times 10^3 & -2.7168\times 10 & 2.0485\times 10^{-1} & -1.0449\times 10^3 & -1.5589\times 10 & 6.4588\times 10^{-2} \\
1.3964\times 10^4 & -1.7242\times 10 & 3.5851\times 10^{-2} & 5.6204\times 10^2 & -2.5482\times 10 & 1.0715\times 10^{-1} \\
2.3816\times 10^3 & -2.0076\times 10 & 3.6049\times 10^{-1} & 9.0259\times 10^2 & -1.16230 & 7.7575\times 10^{-2} \\
-3.0369\times 10^2 & 2.6237 & -1.12076\times 10^{-1} & 1.1292\times 10^3 & 4.3990 & -2.7240\times 10^{-2} \\
5.4284\times 10^2 & 5.7885\times 10^{-1} & 1.1941\times 10^{-1} & -1.2585\times 10^3 & -6.9929 & 2.9297\times 10^{-2} \\
-5.2710\times 10^2 & -3.6012\times 10 & 3.0717\times 10^{-1} & -1.2673\times 10^3 & 7.5832 & -6.5626\times 10^{-3} \\
2.6578\times 10^3 & 3.8124 & -3.7101\times 10^{-2} & -4.9109\times 10^3 & -4.3474 & -8.4969\times 10^{-3} \\
2.5341\times 10^4 & -3.6447\times 10 & 4.5139\times 10^{-1} & 4.3599\times 10^3 & -1.3129\times 10^2 & 8.3944\times 10^{-1} \\
6.5383\times 10^3 & -2.8557\times 10 & 3.2445\times 10^{-1} & 3.7905\times 10^3 & 9.0713\times 10 & -3.4954\times 10^{-1} \\
4.2048\times 10^4 & -5.9432\times 10 & 7.5575\times 10^{-1} & 2.5014\times 10^3 & -1.7393\times 10 & 1.1132\times 10^{-1} \\
2.1059\times 10^2 & -3.9508\times 10 & 5.0765\times 10^{-1} & -4.7035\times 10^2 & -1.8754\times 10 & 1.3955\times 10^{-1} \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
9.9417\times 10^2 & 1.3574\times 10 & -2.1971\times 10^{-2} & 8.2837\times 10^2 & 1.2898\times 10 & -3.6967\times 10^{-2} \\
7.6923\times 10^2 & 1.0602\times 10 & -9.2043\times 10^{-2} & 8.1002\times 10^2 & 1.9073\times 10 & -8.1012\times 10^{-2} \\
4.2702\times 10^4 & -1.2068\times 10^2 & 1.9403 & -3.0773\times 10^3 & 3.5895\times 10 & -2.4668\times 10^{-1} \\
-7.6982\times 10^4 & 1.4062\times 10^2 & -2.7694 & -3.4827\times 10^2 & -3.5862\times 10 & 1.9813\times 10^{-1} \\
-2.8880\times 10^2 & -9.4283\times 10^{-1} & 2.3831\times 10^{-2} & -1.4991\times 10^3 & 9.7579\times 10^{-1} & -4.1943\times 10^{-2} \\
-2.2374\times 10^3 & 8.4543 & -1.5646\times 10^{-1} & -6.5778\times 10^3 & 4.9499\times 10 & -4.9803\times 10^{-1} \\
-1.6942\times 10^4 & 9.4433\times 10 & -1.1443 & -6.0893\times 10 & 2.2208\times 10^{-1} & -1.1955\times 10^{-2} \\
-1.2513\times 10^4 & 6.8017\times 10 & -6.8207\times 10^{-1} & -2.9369\times 10 & 4.6390\times 10^{-1} & -9.9611\times 10^{-3} \\
5.7478\times 10^3 & -3.5777\times 10 & 2.0673\times 10^{-1} & -3.9341\times 10 & -6.8298\times 10^{-1} & 7.8189\times 10^{-3} \\
-1.4657\times 10^4 & 2.1677\times 10^2 & -6.6149\times 10^{-1} & 1.2338\times 10^2 & 9.0426 & -8.7434\times 10^{-2} \\
-4.0892\times 10^3 & -2.5806\times 10^2 & -1.0595 & 3.3034\times 10 & -2.0396 & 2.3536\times 10^{-2} \\
7.9403\times 10^3 & 3.0902\times 10^2 & 1.1753 & 6.1044\times 10 & 2.3109 & -2.8729\times 10^{-2} \\
9.5816\times 10^3 & 3.7301\times 10 & -3.1126\times 10^{-2} & 1.4094\times 10^2 & 2.6275 & -1.8542\times 10^{-2} \\
9.4736\times 10^3 & 4.1398\times 10 & -5.0827\times 10^{-2} & 1.5630\times 10^2 & 3.0982 & -2.2172\times 10^{-2} \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000
\end{bmatrix}$$

$$
A_{13}(:,2)=\begin{bmatrix}
-3.1583\\
1.7788\\
5.3132\times10^{-3}\\
8.0111\times10^{-3}\\
1.0664\times10^{-1}\\
-4.0783\times10^{-1}\\
1.009\times10^{-1}\\
-1.5708\times10^{-1}\\
-1.0604\\
4.8545\times10^{-1}\\
-2.7331\\
2.0674\\
3.0012\\
3.5442\\
-7.1989\\
7.2067\\
6.1859\times10^{-1}\\
1.0761\\
2.8214\\
-7.8657\times10^{-1}\\
-1.1110\times10^{-1}\\
1.5094\\
-3.6592\times10^{-1}\\
8.6350\times10^{-1}\\
7.4328\\
-1.5356\times10\\
4.5705\\
-2.7714\times10^{-1}\\
-1.2510\\
-5.0311\times10^{-1}\\
-4.5236\times10^{-1}\\
-1.5918\\
1.9560\\
-5.9813\\
-1.0883\times10\\
-1.1212\\
1.5316\times10^{-1}\\
-4.2540\times10^{-1}\\
-7.8857\times10^{-1}\\
-1.0526\times10^{-1}\\
-2.0074\\
1.8865\times10^{-1}\\
9.2049\times10^{-3}\\
-1.7068\times10\\
-3.7825\times10
\end{bmatrix}\quad\text{and } A_{13}(i,j)=0 \text{ for other } i,j.
$$

B is a 55×2 matrix defined as follows:

$$B(48,1)=B(52,2)=800\,000$$

$$\text{and}\quad B(i,j)=0\quad\text{for other } i,j.$$

B_N is a 55×1 matrix defined as follows:

$$B_N(52,1)=0.37127,\quad B_N(53,1)=1.2449$$

$$\text{and}\quad B_N(i,j)=0\quad\text{for other } i,j.$$

The matrix $-C$ is defined as follows:

$$
-C^{\mathrm{T}}=\begin{bmatrix}
-4.4247\times10^{-5} & -3.5919\times10\\
-4.3403\times10^{-5} & 1.2459\times10\\
-4.9173\times10^{-5} & 5.3317\times10^{-1}\\
-4.5556\times10^{-5} & 5.6586\times10^{-1}\\
4.1311\times10^{-6} & 9.6643\times10^{-1}\\
-7.9201\times10^{-6} & -1.2746\times10\\
-7.2496\times10^{-6} & 2.1454\times10\\
6.7385\times10^{-5} & -6.8223\\
1.8236\times10^{-4} & -3.3882\times10\\
-2.0657\times10^{-5} & -1.6997\times10\\
-1.3194\times10^{-4} & 1.3802\times10^{-1}\\
1.5849\times10^{-4} & 8.8031\times10^{-1}\\
6.7655\times10^{-5} & -5.4653\\
8.7378\times10^{-6} & 1.6138\\
9.8119\times10^{-5} & -5.5182\times10\\
-3.1119\times10^{-5} & -3.0101\times10\\
5.7479\times10^{-5} & 4.8783\\
-3.1600\times10^{-4} & -1.4357\times10\\
5.7750\times10^{-4} & -6.2542\times10\\
7.1000\times10^{-4} & -5.5179\times10\\
9.6843\times10^{-5} & 1.3879\times10^{2}\\
-2.9934\times10^{-5} & 1.0435\times10^{2}\\
-5.4014\times10^{-4} & -6.8306\\
4.7620\times10^{-4} & 1.1775\times10\\
4.1630\times10^{-6} & 5.5036\times10\\
2.1609\times10^{-4} & 3.0663\times10\\
1.0541\times10^{-4} & -1.4018\\
3.1883\times10^{-5} & 3.0335\times10\\
1.5062\times10^{-4} & -4.8688\\
2.7714\times10^{-4} & 7.4275\times10\\
2.4361\times10^{-4} & -7.4144\times10\\
-2.7990\times10^{-5} & -1.6300\times10\\
1.6592\times10^{-4} & -3.1184\\
-1.2748\times10^{-5} & -8.7279\\
2.0762\times10^{-5} & 7.4525\times10\\
1.2546\times10^{-4} & -1.4605\\
4.4360\times10^{-5} & 3.3338\\
1.0147\times10^{-5} & 7.3667\\
-1.5343\times10^{-4} & 1.0325\\
-2.1956\times10^{-5} & -2.8667\\
2.9386\times10^{-5} & -6.5826\\
-6.5513\times10^{-5} & 1.3609\\
-6.1813\times10^{-5} & -1.3498\\
-1.9995\times10^{-5} & -1.0707\times10\\
-8.9674\times10^{-6} & 2.5912\times10\\
0.0000 & -4.8520\times10\\
0.0000 & 7.9324\\
0.0000 & -7.9915\times10^{-2}\\
0.0000 & 2.5618\times10^{4}\\
0.0000 & 1.6739\times10^{2}\\
0.0000 & -2.2129\times10^{-1}\\
0.0000 & 0.0000\\
0.0000 & -1.8803\\
0.0000 & 0.0000\\
0.0000 & 0.0000
\end{bmatrix}
$$

C_N is a 2×55 matrix defined as follows:

$$
C_N^{\mathrm{T}} = \begin{bmatrix}
-1.016 \times 10^4 & -3.5918 \times 10^4 \\
-9.2139 \times 10^3 & -1.8298 \times 10^4 \\
1.9756 \times 10^4 & 1.4438 \times 10^3 \\
2.1545 \times 10^4 & 2.6748 \times 10^3 \\
1.3993 \times 10^4 & 1.6733 \times 10^4 \\
1.7694 \times 10^4 & 7.9888 \times 10^3 \\
-2.9838 \times 10^3 & 3.4725 \times 10^4 \\
-3.8424 \times 10^3 & 1.1489 \times 10^4 \\
-1.7279 \times 10^4 & 3.5282 \times 10^3 \\
1.3145 \times 10^3 & 9.1701 \times 10^3 \\
3.2923 \times 10^4 & 3.8491 \times 10^3 \\
3.3817 \times 10^4 & 3.8252 \times 10^3 \\
-1.0817 \times 10^3 & -3.3694 \times 10^3 \\
3.0762 \times 10^3 & -1.0423 \times 10^4 \\
-3.8871 \times 10^4 & -4.7616 \times 10^3 \\
-3.6494 \times 10^4 & -1.0333 \times 10^3 \\
-7.0376 \times 10^3 & -8.8330 \times 10^{-1} \\
1.1182 \times 10^4 & 4.6190 \times 10^3 \\
8.0022 \times 10^3 & 3.4482 \times 10^3 \\
1.0350 \times 10^4 & 1.3034 \times 10^3 \\
-5.7674 \times 10^4 & -1.6288 \times 10^4 \\
-5.8520 \times 10^4 & 9.1742 \times 10^2 \\
-9.6781 \times 10^3 & -6.5198 \times 10^3 \\
-1.7572 \times 10^4 & -4.4202 \times 10^3 \\
-4.1362 \times 10^2 & 1.3953 \times 10^4 \\
-4.9296 \times 10^2 & 1.0935 \times 10^4 \\
1.5269 \times 10^2 & -1.2971 \times 10^4 \\
-2.3913 \times 10^3 & 2.2653 \times 10^3 \\
-5.4007 \times 10^4 & -6.6966 \times 10^3 \\
1.4065 \times 10^4 & -6.6463 \times 10^4 \\
-1.8204 \times 10^4 & 5.9803 \times 10^4 \\
-4.2189 \times 10^3 & -4.7485 \times 10^3 \\
-1.6171 \times 10^3 & -3.5247 \times 10^3 \\
4.2055 \times 10^4 & -6.1477 \times 10^3 \\
3.3610 \times 10^4 & 2.4890 \times 10^4 \\
9.5825 \times 10 & 1.3765 \times 10^2 \\
-5.3033 \times 10^2 & -1.5953 \times 10^3 \\
-4.2395 \times 10^2 & -2.9115 \times 10^3 \\
-8.0471 \times 10 & -2.0361 \times 10^2 \\
5.8053 \times 10^2 & 2.4331 \times 10^3 \\
6.6739 \times 10^2 & 3.1340 \times 10^3 \\
-2.0593 \times 10 & -1.7063 \times 10^2 \\
2.8404 \times 10 & 2.0237 \times 10^2 \\
-3.2517 \times 10^4 & -1.4519 \times 10^4 \\
-1.3449 \times 10^3 & -6.9120 \times 10^2 \\
0.0000 & 0.0000 \\
0.0000 & 0.0000 \\
0.0000 & 0.0000 \\
0.0000 & 0.0000 \\
0.0000 & 0.0000 \\
0.0000 & 0.0000 \\
0.0000 & 0.0000 \\
0.0000 & 0.0000 \\
0.0000 & 0.0000 \\
0.0000 & 0.0000
\end{bmatrix}
$$

The state feedback gain and the Kalman gain (with a negative sign) are respectively

$$F^{\mathrm{T}} = \begin{bmatrix} -1.5832\times10^{-4} & -3.0315\times10^{-3} \\ 2.3582\times10^{-4} & -2.6654\times10^{-3} \\ -1.1387\times10^{-4} & -4.1081\times10^{-4} \\ -1.8312\times10^{-4} & -7.8271\times10^{-4} \\ 4.1426\times10^{-5} & -1.4589\times10^{-4} \\ -4.8678\times10^{-5} & 8.5377\times10^{-4} \\ -2.26735\times10^{-4} & 1.2540\times10^{-3} \\ -1.7564\times10^{-4} & -4.6786\times10^{-4} \\ 9.1525\times10^{-6} & -1.3814\times10^{-6} \\ 1.9235\times10^{-5} & 6.7161\times10^{-5} \\ -2.1995\times10^{-4} & -9.5182\times10^{-4} \\ 4.5889\times10^{-5} & -3.5894\times10^{-4} \\ 1.7184\times10^{-6} & -6.2767\times10^{-5} \\ 4.3851\times10^{-5} & -3.7742\times10^{-4} \\ 8.0336\times10^{-5} & 9.2428\times10^{-5} \\ 2.5128\times10^{-5} & -6.5354\times10^{-4} \\ 6.4003\times10^{-6} & -4.6382\times10^{-5} \\ -9.7488\times10^{-6} & 2.8777\times10^{-4} \\ 2.4859\times10^{-6} & 1.6007\times10^{-4} \\ -1.1835\times10^{-6} & 1.7710\times10^{-4} \\ 3.6349\times10^{-6} & -2.4929\times10^{-4} \\ -7.3601\times10^{-6} & -8.8603\times10^{-5} \\ 1.4385\times10^{-5} & -3.1912\times10^{-4} \\ 8.0966\times10^{-6} & -2.5139\times10^{-4} \\ 2.9199\times10^{-5} & 1.1683\times10^{-4} \\ -7.3229\times10^{-6} & 9.9825\times10^{-4} \\ -2.0059\times10^{-5} & -2.7410\times10^{-4} \\ -1.9455\times10^{-6} & -7.8487\times10^{-4} \\ 5.0617\times10^{-5} & 9.1330\times10^{-4} \\ 1.4392\times10^{-5} & 6.2633\times10^{-4} \\ -2.9402\times10^{-5} & -5.0017\times10^{-4} \\ -6.3263\times10^{-6} & 5.6772\times10^{-5} \\ -2.3606\times10^{-6} & -1.5107\times10^{-5} \\ 7.1657\times10^{-5} & -1.5403\times10^{-3} \\ -1.0550\times10^{-5} & -1.0277\times10^{-3} \\ 2.3694\times10^{-8} & 9.3928\times10^{-8} \\ -6.8456\times10^{-8} & 5.9855\times10^{-7} \\ -1.2641\times10^{-8} & 3.5760\times10^{-7} \\ -3.0596\times10^{-10} & -3.5076\times10^{-7} \\ -4.0774\times10^{-7} & -5.9781\times10^{-6} \\ 2.9662\times10^{-7} & -6.4990\times10^{-6} \\ -9.0481\times10^{-10} & 2.7729\times10^{-9} \\ 5.4855\times10^{-10} & -5.3934\times10^{-9} \\ -3.9594\times10^{-6} & 3.8792\times10^{-4} \\ -1.6945\times10^{-9} & 3.7903\times10^{-8} \\ 4.1324\times10^{-1} & 6.3293\times10^{-1} \\ 6.8441\times10^{-3} & 7.2630\times10^{-3} \\ 6.4513\times10^{-6} & 7.0158\times10^{-6} \\ 2.1111\times10^{-2} & 1.3601 \\ 2.7671\times10^{-4} & 2.1233\times10^{-2} \\ 2.8063\times10^{-7} & 2.0544\times10^{-4} \\ -8.0010\times10^{-5} & -2.2294\times10^{-4} \\ -1.0842\times10^{-4} & -1.4034\times10^{-3} \\ -3.5408 & -2.6766 \\ -1.9284\times10^{-1} & -8.3273 \end{bmatrix}$$

and

$$-L = \begin{bmatrix} -1.7850\times10^{3} & -2.2716 \\ -3.3192\times10^{3} & -2.6879\times10^{-2} \\ -7.0878\times10 & -2.3474\times10^{-2} \\ -5.5433\times10^{2} & -2.5270\times10^{-2} \\ 6.8964\times10^{2} & 2.2660\times10^{-1} \\ 4.3744\times10 & -1.8839\times10^{-1} \\ 3.9378\times10 & 9.2326\times10^{-2} \\ 1.0827\times10^{2} & -4.5916\times10^{-2} \\ 1.0555\times10^{2} & -2.7936\times10^{-1} \\ -7.7792\times10^{2} & -2.0914\times10^{-1} \\ -1.8692\times10^{4} & -4.8365\times10^{-2} \\ 9.1119\times10^{3} & 1.1950 \\ -2.4830\times10^{3} & 3.0818\times10^{-1} \\ -8.4329 & 2.6924 \\ -3.3319\times10^{3} & -4.7184 \\ -6.8920\times10^{3} & -1.2168\times10^{-1} \\ -1.8901\times10^{2} & -1.7660\times10^{-1} \\ 1.4074\times10 & 3.1361\times10^{-1} \\ 6.9803\times10^{2} & 6.6731\times10^{-1} \\ -9.6051\times10^{2} & 7.3644\times10^{-1} \\ -9.0556\times10 & -2.2644\times10^{-1} \\ 6.1330 & -9.7789\times10^{-2} \\ -3.0625\times10^{2} & -1.5529\times10^{-1} \\ -4.4035\times10 & -5.8742\times10^{-2} \\ 2.7989\times10^{3} & 5.2259 \\ -6.3422\times10^{2} & -6.3972\times10^{-1} \\ -1.7563\times10^{3} & 1.1495 \\ -3.5783\times10^{3} & 9.0801\times10^{-1} \\ -3.1221\times10^{3} & -3.9626\times10^{-1} \\ -1.3313\times10 & 2.7231\times10^{-2} \\ 3.0984\times10 & -1.2380\times10^{-1} \\ -6.6832\times10^{3} & -1.2586 \\ 8.0685\times10^{3} & 6.2340\times10^{-1} \\ 7.8283\times10^{3} & -6.8463\times10^{-1} \\ -8.3137\times10^{3} & -4.3423 \\ -8.6051\times10 & -1.0998\times10^{-2} \\ 2.0950\times10 & -9.2414\times10^{-2} \\ -4.3464\times10 & 9.7768\times10^{-2} \\ -8.2233\times10 & -5.1786\times10^{-2} \\ 2.9630\times10^{2} & 2.1551\times10^{-1} \\ -6.0504\times10 & -1.2169\times10^{-2} \\ 2.0619\times10 & 9.3656\times10^{-3} \\ -1.8696\times10 & -4.0892\times10^{-4} \\ -7.8301\times10^{3} & -3.8145 \\ -2.3541\times10^{3} & -1.5864 \\ -8.8779\times10^{-1} & 4.1170\times10^{-6} \\ -1.1429\times10 & 1.2973\times10^{-3} \\ -3.8086\times10^{2} & -1.2976 \\ 3.5575 & -1.2770\times10^{-3} \\ 9.8838\times10 & 5.5995\times10^{-1} \\ -1.7817\times10^{4} & -4.0628\times10 \\ 4.6004\times10^{3} & 2.5972 \\ 1.3620\times10^{4} & 8.3563 \\ 3.9461\times10^{-2} & -3.3999\times10^{-8} \\ 3.8782\times10^{-1} & 3.7939\times10^{-4} \end{bmatrix}$$