Robust Strict Positive Realness: Characterization and Construction

BRIAN D. O. ANDERSON, FELLOW, IEEE, SOURA DASGUPTA, MEMBER, IEEE,
PRAMOD KHARGONEKAR, MEMBER, IEEE, FRANTISEK J. KRAUS, MEMBER, IEEE,
AND MOHAMED MANSOUR, FELLOW, IEEE

Abstract —Let $\mathscr{P}$ be a convex set of real polynomials. This paper considers the question of when there exists a real polynomial $b(s)$, or more generally, a real transfer function $b(s)$, such that $p(s)/b(s)$ is strictly positive real for all $p(s) \in \mathscr{P}$. Necessary and sufficient conditions are found for the transfer function $b(s)$ case, and when the degree of the polynomials in $\mathscr{P}$ is restricted, such conditions are also found for the polynomial $b(s)$ case. Closely related results are also obtained for a z-transform version of the problem. The results have application in adaptive systems.

I. INTRODUCTION AND PROBLEM FORMULATION

MOTIVATED by problems of adaptive system theory, and in particular, output error identification and certain adaptive control algorithms [1]–[3], the following problem is addressed in [4]. Consider a set $\mathscr{P}$ of nth degree Hurwitz polynomials. State conditions for the existence of an nth degree Hurwitz polynomial $b(s)$ such that $p(s)/b(s)$ is strictly positive real (Re $p(j\omega)/b(j\omega) > 0$ for all real ω, given the Hurwitz property for $b(s)$) for all $p(s) \in \mathscr{P}$. More generally, one can replace a search for polynomial $b(s)$ by one for rational $b(s)$, with relative degree $-n$. To understand the importance of this problem, consider the adaptive output error identification of a plant whose transfer function has denominator polynomial $p(s)$. Assume degree of $p(s) = n$. Then, to ensure the exponential convergence of the identification algorithm, one must filter certain signals by a filter having transfer function $1/b(s)$, where $1/b(s)$ is rational, has degree $\geqslant n$, has relative degree n, and $b(s)/p(s)$ is strictly positive real (SPR). The degree restrictions apply because of the need to avoid explicitly differentiating certain signals. Notice that the simplest $b(s)$ is a polynomial of degree n. Further, from the definition of SPR

transfer functions, $b(s)/p(s)$ SPR is equivalent to $p(s)/b(s)$ SPR. Notice also that $p(s)$ is unknown. To construct an appropriate $b(s)$, one can make the additional assumption that the coefficients of $p(s)$ lie in some known convex set. The problem then becomes one of finding a single $b(s)$, satisfying the appropriate degree restrictions, such that for all $p(s)$ in this set, $p(s)/b(s)$ is SPR. For a discrete-time plant with stable denominator $p(z^{-1})$, the corresponding design problem is to find $b(z^{-1})$, such that $p(z^{-1})/b(z^{-1})$ is SPR (i.e., it is stable and obeys $\mathrm{Re}[b(e^{-j\omega})/p(e^{-j\omega})] > 0$ for all real ω). Unlike the continuous-time case there are no degree restrictions on $b(z^{-1})$.

Two of the significant contributions of [4] in treating the continuous time problem are the following. First, sets $\mathscr{P}$ are identified with the property that there exists a finite subset $\mathscr{P}^*$ such that p/b is SPR for all $p \in \mathscr{P}^*$ implies p/b is SPR for all $p \in \mathscr{P}$. A most important example of such a set is a "Kharitonov set," so called because of its importance in robust stability [5]. More precisely, with

$$p(s) = s^n + p_1 s^{n-1} + \cdots + p_n, \qquad p_i \in [\alpha_i, \beta_i] \quad (1)$$

defining the set $\mathscr{P}$, the set $\mathscr{P}^*$ is given by the four polynomials

$$p_1(s) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \beta_3 s^{n-3}$$
$$+ \beta_4 s^{n-4} + \alpha_5 s^{n-5} + \cdots \quad (2)$$

$$p_2(s) = s^n + \alpha_1 s^{n-1} + \beta_2 s^{n-2} + \beta_3 s^{n-3}$$
$$+ \alpha_4 s^{n-4} + \alpha_5 s^{n-5} + \cdots \quad (3)$$

$$p_3(s) = s^n + \beta_1 s^{n-1} + \alpha_2 s^{n-2} + \alpha_3 s^{n-3}$$
$$+ \beta_4 s^{n-4} + \beta_5 s^{n-5} + \cdots \quad (4)$$

$$p_4(s) = s^n + \beta_1 s^{n-1} + \beta_2 s^{n-2} + \alpha_3 s^{n-3}$$
$$+ \alpha_4 s^{n-4} + \beta_5 s^{n-5} + \cdots. \quad (5)$$

(The coefficient pattern involves alternation of two minimum values and two maximum values.)

The fact that the infinite set $\mathscr{P}$ can be replaced by the finite subset $\mathscr{P}^*$ of course makes the search for $b(s)$ a much easier task.

The second contribution of [4] to be noted here is that a sufficient condition is derived for the existence of a Hurwitz nth degree $b(s)$ such that $p_i(s)/b(s)$, $i = 1, \cdots, 4$, is SPR, the $p_i(s)$ being the four corner polynomials of a Kharitonov set, i.e., they are given by (2)–(5). Such a $b(s)$ yields $p(s)/b(s)$ SPR for all $p(s)$ defined by (1).

Our main contribution in this paper is to find conditions for the existence of $b(s)$ that are necessary and sufficient, and to present a constructive procedure. We present results for the case when $b^{-1}(s)$ is a relative degree n transfer function, but $b(s)$ is not polynomial, and results applying with polynomial $b(s)$, when $n \leqslant 4$.

The first problem considered in this paper is posed in discrete time.

We work with the nth degree polynomial $p(z^{-1})$ lying in a known convex polytope $\mathscr{P}$. It has been shown in [6] that with $\mathscr{P}^*$ denoting the set of corners of $\mathscr{P}$, $p(z^{-1})/b(z^{-1})$ is SPR for all $p(z^{-1}) \in \mathscr{P}$ and some fixed $b(z^{-1})$ if and only if $p(z^{-1})/b(z^{-1})$ is SPR for all $p(z^{-1}) \in \mathscr{P}^*$. Given a finite set $\mathscr{P}^*$ of polynomials $p_i(z^{-1})$ in z^{-1}, we seek a polynomial $b(z^{-1})$ in z^{-1} such that p_i/b is SPR for all i. It is shown that such a $b(z^{-1})$ exists iff for all p in $\mathscr{P}$, $p(z_0^{-1}) = 0$ implies $|z_0| < 1$. We also give a constructive procedure to find such ab if one exists; this constructive procedure uses the polynomials in $\mathscr{P}^*$.

Two equivalent conditions are established: the first involves the phases of $p_i(e^{j\omega})$ for different i, and the second is that for all $p(z^{-1}) \in \mathscr{P}$, $p(z_0^{-1}) = 0$ implies $|z_0| < 1$.

What makes the discrete-time problem somewhat easier than the continuous-time problem is the fact that the degree of b (as a polynomial in z^{-1}) is not constrained by the degrees of the p_i, so that there are in fact arbitrarily many coefficients that can be adjusted in b to secure the SPR property. In contrast, if the continuous-time transfer function $p(s)/b(s)$ is SPR, the degrees of b and p necessarily differ by at most 1, and thus the freedom in choosing b is more limited.

The second issue tackled in the paper is the continuous-time problem. We now work with a convex polytope $\mathscr{P}$ of nth degree monic polynomials $p(s)$ and seek an operator $b(s)$, satisfying the continuous-time SPR requirement that the relative degree of an SPR function is necessarily ± 1, or 0. We shall work with functions of relative degree 0. Again [6], with $\mathscr{P}^*$ the set of corners of $\mathscr{P}$, one needs only to find a $b(s)$ such that $p(s)/b(s)$ is SPR for all $p(s) \in \mathscr{P}^*$. (Of course for the Kharitonov set, the required $\mathscr{P}^*$ can be even smaller being the four polynomials in (2)–(5).) Thus our initial data are a collection of Hurwitz polynomials $n_i(s) \in \mathscr{P}^*$ of the same degree l. We show that an integer M and a polynomial $d(s)$ of degree $l + M$ exists such that $n_i(s)(1 + s)^M/d(s)$ is SPR for all i, iff P is Hurwitz invariant. This condition is satisfied in the case of $\mathscr{P}^*$ derived from a Kharitonov set (1). Note that the original objective of [4] (which corre-

sponds to the special case $M = 0$) has been relaxed, so that greater freedom arises in the choice of $d(s)$.

A third contribution of the paper is to consider the continuous time problem with $n = 2, 3, 4$, and with $\mathscr{P}$ a Kharitonov set. We show that the sufficiency conditions of [4] for the existence of a polynomial $b(s)$ are always fulfilled.

II. DISCRETE-TIME SPR CONSTRUCTION

In this section, we shall prove the following main result:

Theorem 2.1: Let $p_i(z^{-1})$, $i = 1, 2, \cdots, r$ be a finite set of polynomials in z^{-1} with the stability property $p_i(z_0^{-1}) = 0$ implies $|z_0| < 1$. Then there exists $b(z^{-1})$, polynomial in z^{-1} and such that $b(z_0^{-1}) = 0$ implies $|z_0| < 1$, with

$$\text{Re}\left[\frac{p_i(e^{j\omega})}{b(e^{j\omega})}\right] > 0, \qquad \forall \omega \in [0, 2\pi], \forall i \qquad (6)$$

if and only if for all ω in $[0, 2\pi]$

$$\max_i \left[\arg p_i(e^{j\omega})\right] - \min_i \left[\arg p_i(e^{j\omega})\right] < \pi \qquad (7)$$

where it should be noted that the unwrapped phase rather than the phase $\mod 2\pi$ is computed.

Proof: (Only if): Equation (6), implies that for all i,

$$|\arg p_i(e^{j\omega}) - \arg b(e^{j\omega})| < \pi/2, \qquad \forall \omega \in [0, 2\pi].$$

Hence for any $i \neq k$,

$$|\arg p_i(e^{j\omega}) - \arg p_k(e^{j\omega})| < \pi, \qquad \forall \omega \in [0, 2\pi].$$

Then (7) is immediate.

(If): The proof will be by construction. Define

$$q_i(z^{-1}) = p_i(\rho z^{-1}) \qquad (8)$$

where $\rho > 1$ is selected to satisfy the two conditions shown in (9) and (10):

(a) $q_i(z_0^{-1}) = 0$ implies $|z_0| < 1$. $\qquad (9)$

Let ν be the maximum modulus of any z_1 such that $p_i(z_1^{-1}) = 0$ for some i. Notice that $\nu < 1$. Now $q_i(z_0^{-1}) = 0$ is equivalent to $p_i(\rho z_0^{-1}) = 0$, and evidently, $|z_0/\rho| \leqslant \nu$. So if ρ is chosen so that $\rho\nu < 1$, we ensure that (a) holds.

(b) $\max_i \left[\arg q_i(e^{j\omega})\right] - \min_i \left[\arg q_i(e^{j\omega})\right] < \pi,$

$$\forall \omega \in [0, 2\pi]. \quad (10)$$

Notice that $\arg q_i(e^{j\omega})$ depends continuously on ρ for ρ near 1. So, therefore, does

$$\max_i \left[\arg q_i(e^{j\omega})\right] - \min_i \left[\arg q_i(e^{j\omega})\right]. \qquad (11)$$

Since ω belongs to a finite interval, it is then clear that for some ρ^* and any $\rho \in (1, \rho^*)$, the condition shown in (10) holds because (7) holds.

Obviously, we take $\rho \in (1, \min(\nu^{-1}, \rho^*))$ to secure both conditions shown in (9) and (10).

Define now a function $\phi(\omega)$, $\omega \in [0, 2\pi]$, by

$$\phi(\omega) = \frac{\max_i \left[\arg q_i(e^{j\omega})\right] + \min_i \left[\arg q_i(e^{j\omega})\right]}{2} \qquad (12)$$

and observe that for all ω, in the light of the condition shown in (10), there holds

$$|\arg q_i(e^{j\omega}) - \phi(\omega)| < \frac{\pi}{2}, \qquad \forall i = 1, 2, \cdots, r. \quad (13)$$

Divide up the interval $[0, 2\pi]$ into subintervals $[0, \omega_1]$, $[\omega_1, \omega_2], [\omega_2, \omega_3], \cdots$, such that on each subinterval, $\arg q_i(e^{j\omega})$ is minimized by the same i through the interval, and maximized by the same i through the interval (the choice normally being different to that made for minimization). In (ω_i, ω_{i+1}), $\phi(\omega)$ is infinitely differentiable. Clearly on $[0, 2\pi]$, $\phi(\omega)$ has a derivative that is piecewise continuous.

Consequently, we can determine a Hilbert transform for $\phi(\omega)$ and indeed a function $v(z^{-1})$, analytic together with its inverse in $|z| > 1$, but not necessarily on $|z| = 1$, such that

$$\arg v(z^{-1}) = \phi(\omega). \quad (14)$$

Notice that the piecewise differentiability of $\phi(\omega)$ allows the possibility of approximating the Hilbert transform integral by a sum without necessarily incurring numerical problems, see e.g., some discussion in [7].

Now consider the transfer functions $q_i(z^{-1})/v(z^{-1})$. Because of (13) and (14), it follows that each of these transfer functions has positive real part for $z = e^{j\omega}$. Further, the analyticity properties of v ensure that $q_i(z^{-1})/v(z^{-1})$ is analytic in $|z| > 1$. It follows that $q_i(z^{-1})/v(z^{-1})$ is positive real. Now define $w(z^{-1})$ by

$$v(z^{-1}) = w(\rho z^{-1}) \quad (15)$$

or

$$w(z^{-1}) = v(\rho^{-1} z^{-1}). \quad (16)$$

Observe that

$$\frac{q_i(z^{-1})}{v(z^{-1})} = \frac{p_i(\rho z^{-1})}{w(\rho z^{-1})}. \quad (17)$$

It is a standard property that if $Z(z^{-1})$ is PR, then $Z(\rho^{-1}z^{-1})$ is SPR $\forall \rho > 1$. Clearly now, $p_i(z^{-1})/w(z^{-1})$ will be SPR, and free of singularities for $|z| > \rho^{-1}$. Also, $w(z^{-1})$ and $w^{-1}(z^{-1})$ will be free of singularities in $|z| > \rho^{-1}$, and so also on $|z| = 1$. This means that we can write down the Laurent series expansion for $w(z^{-1})$, viz.:

$$w(z^{-1}) = w_0 + w_1 z^{-1} + w_2 z^{-2} + \cdots + w_N z^{-N} + \cdots \quad (18)$$

with the property that given $\epsilon > 0$, we can choose N so that the truncation

$$w_N(z^{-1}) = w_0 + w_1 z^{-1} + \cdots + w_N z^{-N} \quad (19)$$

satisfies

$$\max_{\omega} |w(e^{j\omega}) - w_N(e^{j\omega})| < \epsilon. \quad (20)$$

By choosing ϵ sufficiently small, we can further ensure that $w_N(z_0^{-1}) = 0$ implies $|z_0| < 1$. To see this, recall that $w^{-1}(z^{-1})$ is free of singularities in $|z| \geq 1$, and so takes

the maximum value of its modulus on $|z| = 1$. Hence $|w(z)|$ takes its minimum value over the region $|z| \geq 1$ on the boundary $|z| = 1$. Call this value ϵ_1. Now $w(z^{-1}) - w_N(z^{-1})$ is free of singularities in $|z| \geq 1$ and so $|w(z^{-1}) - w_N(z^{-1})|$ takes its maximum values on $|z| = 1$; i.e., throughout $|z| \geq 1$, one has $|w(z^{-1}) - w_N(z^{-1})| < \epsilon$. Suppose ϵ satisfies $\epsilon < \epsilon_1/2$. Then throughout $|z| \geq 1$,

$$|w_N(z^{-1})| = |w_N(z^{-1}) - w(z^{-1}) + w(z^{-1})|$$
$$\geq |w(z^{-1})| - |w_N(z^{-1}) - w(z^{-1})|$$
$$\geq \epsilon_1 - \epsilon$$
$$> \epsilon_1/2. \quad (21)$$

We can also choose ϵ sufficiently small to ensure that

$$\text{Re} \frac{p_i(e^{j\omega})}{w_N(e^{j\omega})} > 0, \qquad \text{for all } \omega \in [0, 2\pi] \text{ and all } i. \quad (22)$$

To see this, observe that

$$\text{Re} \frac{p_i}{w_N}$$
$$= 1/2 \frac{p_i w_N^* + p_i^* w_N}{|w d_N|^2}$$
$$= \frac{1}{2|w_N|^2}$$
$$\cdot [p_i w^* + p_i^* w + p_i(w_N^* - w^*)$$
$$+ p_i^*(w_N - w)]. \quad (23)$$

Now $p_i w^* + p_i^* w > 0$ for all ω on $[0, 2\pi]$ and all i by the SPR property for p_i/w. Let δ be the minimum value assumed over all ω and all i. Choose

$$\epsilon < \frac{\delta}{2 \max_{\omega, i} |p_i(e^{j\omega})|}. \quad (24)$$

This ensures that (22) holds. Together, (21), which is valid in $|z| \geq 1$, and (22) yield the SPR property for $p_i(z^{-1})/w_N(z^{-1})$. Taking $b(z^{-1}) \equiv w_N(z^{-1})$ completes the theorem proof.

Remark: The degree in z^{-1} of $b(z^{-1})$ may be much higher than the degree of any of the $p_i(z^{-1})$. The above arguments contain no information suggesting how this degree might be minimized.

We round off the main result above by noting the significance of the phase restriction (7) on the $p_i(z^{-1})$. From [6], we note that if the set $\mathscr{P}$, a convex polytope of polynomials, is such that for all $p(z^{-1}) \in \mathscr{P}$, $p(z_0^{-1}) = 0$ implies $|z_0| < 1$, then the members of the corner set obey (7). Of course, stability of all $p \in \mathscr{P}$ is also necessary for there to be a $b(z^{-1})$ such that $p(z^{-1})/b(z^{-1})$ is SPR for all $p \in \mathscr{P}$. Accordingly, we have established the following corollary.

Corollary 2.1: Consider a convex polytope $\mathscr{P}$ of polynomials in z^{-1}. Then there exists $b(z^{-1})$ such that $p(z^{-1})/b(z^{-1})$ (and $b(z^{-1})/p(z^{-1})$) is SPR for all $p(z^{-1}) \in \mathscr{P}$ if and only if for all $p(z^{-1}) \in \mathscr{P}$, $p(z_0^{-1}) = 0$ implies $|z_0| < 1$.

The results here demonstrate that a phase restriction among the members of a polynomial set $\mathcal{P}$ is necessary and sufficient for the existence of a single $b(z^{-1})$, whose ratio with every member of $\mathcal{P}$ is SPR. Recall that $1/b(z^{-1})$ represents a filter used on certain signals in the output error identification setting. Had this filter been linear but possibly time varying, e.g., $[b(z^{-1},k)]^{-1}$ (with obvious abuse of notation), then convergence would require that

$$Z(z^{-1},k) = [b(z^{-1},k)]^{-1}p(z^{-1}) \qquad (25)$$

be strictly passive [1], [2] (a definition of strictly passive systems appears below). Thus an additional question to address is if the members of $\mathcal{P}$ have a pointwise phase difference that somewhere exceeds 180° and somewhere is less than 180°, can one find a single linear operator $b(z^{-1},k)$, for which (25) is strictly passive for all members of $\mathcal{P}$? The answer unfortunately is no. This lack of advantage in the use of linear time varying systems over LTI systems is not confined to this context alone. Several results of this nature, with respect to robust stabilization of LTI, systems are known in the literature [8], [9].

A linear, possibly time varying system is strictly passive [2] if for some positive α_1 and all k and u_i

$$\sum_{i=0}^{k} u_i y_i \geq \alpha_1 \sum_{i=0}^{k} u_i^2 + K \qquad (26)$$

where K is a constant and u_i, y_i are the input/output sequences of the system. An LTI system is strictly passive iff its transfer function is SPR [2]. We remark that by selecting the initial conditions appropriately, K can be taken to be zero. We have the following theorem.

Theorem 2.2: Suppose two stable, real polynomials $p_1(z^{-1})$ and $p_2(z^{-1})$ are such that

$$|\arg p_1(e^{j\omega}) - \arg p_2(e^{j\omega})| < \pi \qquad (27)$$

holds for some but not all $\omega \in [0, 2\pi]$. Then there exists no linear operator $b(z^{-1},k)$, time varying or otherwise such that $b^{-1}(z^{-1},k)p_i(z^{-1})$ is strictly passive for $i = 1,2$.

Proof: Suppose that (27) holds at ω_1 and fails at ω_2. Then, by continuity of phase with ω, (27) implies at some $\omega' \in (\omega_1, \omega_2]$ and $\alpha_2 > 0$

$$p_1(e^{-j\omega^1}) = -\alpha_2 p_2(e^{-j\omega'}). \qquad (28)$$

Then with

$$u_k = \sin \omega' k \qquad (29)$$

any linear $b^{-1}(z^{-1},k)$ and appropriate initial conditions, the output in system $b^{-1}(z^{-1},k)p_1(z^{-1})$ is equal within a scaling factor to that in $b^{-1}(z^{-1},k)p_2(z^{-1})$ and has opposite sign. Thus these two systems cannot be simultaneously strictly passive.

We have in effect shown that if one cannot find an LTI $b(z^{-1})$ whose ratio with all members of $\mathcal{P}$ is strictly passive, then it is impossible to find a linear time varying $b(z^{-1},k)$ for which the operator in (25) is strictly passive for all $p \in \mathcal{P}$.

III. CONTINUOUS-TIME SPR CONSTRUCTION

In this section, we shall prove the following main result.

Theorem 3.1: Let $n_i(s)$, $i = 1, \cdots, r$ be a finite set of Hurwitz polynomials with equal degree l. Then there exists an integer M and a Hurwitz polynomial $B(s)$ of degree $M + l$ such that for all real ω

$$\text{Re}\left[\frac{n_i(j\omega)(l + j\omega)^M}{B(j\omega)}\right] > 0 \qquad (30)$$

if and only if, for all real ω

$$\max_i [\arg n_i(j\omega)] - \min_i [\arg n_i(j\omega)] < \pi. \qquad (31)$$

Proof: (Only if): The argument is virtually the same as for the discrete-time problem, Theorem 2.1.

(if): Define polynomials $p_i(z^{-1})$ in z^{-1} by

$$p_i(z^{-1}) = n_i\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)(1 + z^{-1})^l. \qquad (32)$$

Then $p_i(z_0^{-1}) = 0$ implies $|z_0| < 1$ because $n_i(s)$ is Hurwitz, and (31) implies a similar condition, viz. (7) on $p_i(e^{j\omega})$.

Hence Theorem 2.1 implies that there exists a $b(z^{-1})$ polynomial in z^{-1} and such that $b(z_0^{-1}) = 0$ implies $|z_0| < 1$, with

$$\text{Re}\left[\frac{p_i(e^{j\omega})}{b(e^{j\omega})}\right] > 0, \qquad \forall \omega \in [0, 2\pi], \qquad \forall i. \qquad (33)$$

Suppose the degree of $b(z^{-1})$ in z^{-1} is N. Define a polynomial $B(s)$ of degree N via $z^{-1} = (1 - s)(1 + s)^{-1}$ by

$$b(z^{-1}) = b\left(\frac{1 - s}{1 + s}\right) = \frac{B(s)}{(1 + s)^N}. \qquad (34)$$

Notice that

$$p_i(z^{-1}) = \frac{2n_i(s)}{(1 + s)^l}. \qquad (35)$$

Hence

$$\frac{p_i(z^{-1})}{b(z^{-1})} = \frac{2n_i(s)}{B(s)}(1 + s)^{N-l}. \qquad (36)$$

By identifying $M = N - l$, we obtain the result of the theorem.

Remark: The integer M above does not have to be non-negative. If non-negativity is desired and the procedure has led to $N < l$, one can proceed as follows. Modify $b(z^{-1})$ to $b(z^{-1}) + \epsilon z^{-l}$ where ϵ is a very small quantity. Then the stability property for $b(\cdot)$ and the PRness property (33) remain in force, while the degree of the new $b(z^{-1})$ in z^{-1} becomes l. Then $M = 0$.

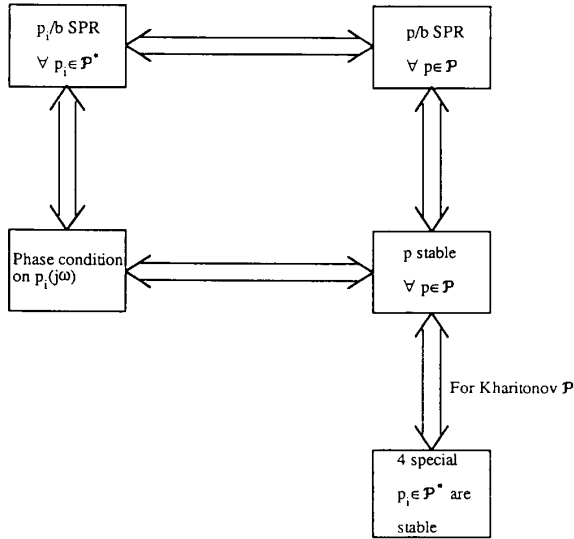Again, arguing as in Section II, the following corollary applies.

Fig. 1. Conditions for the existence of b producing SPR property; $\mathscr{P}$ is a convex polytope with corners $\mathscr{P}^*$.

Corollary 3.1: Consider a convex polytope $\mathscr{P}$ of l-th degree polynomials in s. There exists a non-negative integer M and a polynomial $B(s)$ of degree $l + M$ such that $p(s)(1 + s)^M/B(s)$ is SPR for all $p(s) \in \mathscr{P}$ if and only if $\mathscr{P}$ is Hurwitz invariant, i.e., for all $p(s) \in \mathscr{P}$, $p(s_0) = 0$ implies $\text{Re } s_0 < 0$.

The results obtained here and elsewhere are summed up in Fig. 1.

IV. SPECIAL CASES

In this section, we first consider polynomials in s of order 2, 3, and 4 with the prescribed set $n_i(s)$ being defined from a Kharitonov set. We show directly that denominator polynomials of order 2, 3, or 4 as appropriate can be found to secure SPR behavior.

Second-Order Polynomials: Consider the transfer function set

$$z(s) = \frac{s^2 + cs + d}{s^2 + as + b} \quad (37)$$

where c and d are variable, $c \in [c_1, c_2]$, $d \in [d_1, d_2]$, $c_1 > 0$, $d_1 > 0$, and a and b are to be found so that the ratio is SPR. Obviously, $a > 0$, $b > 0$ is required. Also,

$$\text{Re } z(j\omega) = \frac{1}{2} \frac{(-\omega^2 + cj\omega + d)(-\omega^2 - aj\omega + b) + (-\omega^2 - cj\omega + d)(-\omega^2 + aj\omega + b)}{(-\omega^2 + aj\omega + b)(-\omega^2 - aj\omega + b)}$$

$$= \frac{\omega^4 + (ac - b - d)\omega^2 + bd}{(-\omega^2 + b)^2 + a^2} \quad (38)$$

It is trivial to see that if

$$ac_1 > b + d_2 \quad (39)$$

and otherwise $a > 0$, $b > 0$, we obtain $\text{Re } z(j\omega) > 0$ for all ω, and all allowed c, d.

Third-Order Polynomials: Consider now

$$z(s) = \frac{s^3 + ds^2 + es + f}{s^3 + as^2 + bs + c} \quad (40)$$

with $d \in [d_1, d_2]$, $e \in [e_1, e_2]$, and $f \in [f_1, f_2]$, with $d_1 > 0$, $e_1 > 0$, $f > 0$. A sufficient condition of all the numerator polynomials to be Hurwitz is, see [10], that

$$d_1 e_1 - f_2 > 0. \quad (41)$$

A straightforward calculation yields

$\text{Re } z(j\omega)$

$$= \frac{\omega^6 + \omega^4(ad - b + e) + \omega^2(be - af - cd) + cf}{|(j\omega)^3 + a(j\omega)^2 + b(j\omega) + c|^2}. \quad (42)$$

So to secure positive realness it is sufficient to choose a, b, c all positive with

$$ad - b - e > 0 \quad (43)$$

$$be - af - cd > 0 \quad (44)$$

for all allowed d, e, f. (Hurwitzness of the denominator is automatic if the numerator is Hurwitz and $\text{Re}[z(j\omega)] > 0$; see, e.g., [4].) Now (43) will hold for all allowed d, e, f if

$$ad_1 - b - e_2 > 0. \quad (45)$$

The allowed region of a, b space is shown in Fig. 2(a). Also, (44) will hold for all d, e, f if

$$be_1 - af_2 - cd_2 > 0. \quad (46)$$

Now (41) implies $d_1 > f_2/e_1$ so that for arbitrary $c > 0$, the two regions depicted in Figs. 2(a) and (b) must have a common intersection. Any part in the common intersection yields a, b values such that $z(s)$ is SPR for all allowed d, e, f values.

Fourth-Order Polynomials: Consider the transfer functions

$$z(s) = \frac{z^4 + es^3 + fs^2 + gs + h}{z^4 + as^3 + bs^2 + cs + d} \quad (47)$$

with $e \in [e_1, e_2], \cdots, h \in [h_1, h_2]$, all numerators stable. Therefore, $e_1 > 0$, $f_1 > 0$, $g_1 > 0$, $h_1 > 0$, and the Hurwitz determinant inequalities

$$ef - g > 0 \quad (48)$$

$$efg - e^2h - g^2 > 0 \quad (49)$$

hold for all e, f, g, h in the region of interest. It is established in [10] that the following conditions are necessary and sufficient for this:
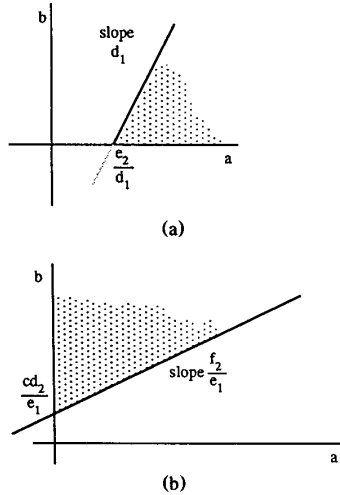
Fig. 2.   Restrictions on choice of a, b imposed by two inequalities.

and sufficient for this:

$$e_1 f_1 - g_2 > 0 \tag{50}$$

$$e_1 f_1 g_2 - e_1^2 h_2 - g_2^2 > 0 \tag{51}$$

$$e_2 f_1 - g_1 > 0 \tag{52}$$

$$e_2 f_1 g_1 - e_2^2 h_2 - g_1^2 > 0. \tag{53}$$

Now a straightforward calculation yields

$$\mathrm{Re}\, z(j\omega) = \frac{\omega^8 + (ae - b - f)\omega^6 + (d - ec + fb - ga + h)\omega^4 + (-fd + gc - h)\omega^2 + hd}{|(j\omega)^4 + a(j\omega)^3 + b(j\omega)^2 + c(j\omega) + d|^2}. \tag{54}$$

To secure $z(s)$ positive real, it is sufficient to choose a, b, c, d all positive and so that the numerator has all positive coefficients for all allowed e, f, g, h.

$$ae - b - f > 0 \tag{55}$$

$$d - ec + fb - ga + h > 0 \tag{56}$$

$$-fd + gc - bh > 0. \tag{57}$$

Arguing just as in [4], (55)–(57) hold for all allowed $e, f, g,$ and h if and only if a, b, c, d are such that

$$ae_1 - b - f_2 > 0 \tag{58}$$

$$d - e_2 c + f_1 b - g_2 a + h_1 > 0 \tag{59}$$

$$-f_2 d + g_1 c - h_2 b > 0. \tag{60}$$

We establish that a solution exists by setting up a dual linear programming problem (see the Appendix).

V. CONCLUSION

We have solved a long standing problem of adaptive system theory that almost certainly has a number of other applications. The key to generating a collection of SPR functions from a family of polynomials is either that the family be a convex set, with all elements stable (a fact which Kharitonov results may assist in checking), or that

the elements of the family have a restricted spread of phase at every frequency.

We have left open the question in the continuous time case as to whether there are families of polynomials or a Kharitonov set of polynomials (all of the same degree l) from which the SPR function can always be obtained through division of each element of the set by a single polynomial of degree l. In case $l \leq 4$, we have established an affirmative answer; the search then for a counterexample will be complicated by the number of parameters involved.

APPENDIX
SOLUTION OF LINEAR INEQUALITIES

Consider the inequalities (58)–(60), subject to (48) and (49) with e, f, g, h in (48) and (49) being either e_1 or e_2, f_1 or f_2, etc. Consider more particularly the linear programming (LP) problem minimize

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \tag{61}$$

subject to, for some small positive ϵ,

$$\begin{bmatrix} e_1 & -1 & 0 & 0 \\ -g_2 & f_1 & -e_2 & 1 \\ 0 & -h_2 & g_1 & -f_2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \geq \begin{bmatrix} f_2 + \epsilon \\ -h_1 + \epsilon \\ \epsilon \end{bmatrix} \tag{62}$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \geq 0. \tag{63}$$

If this problem has a solution with a, b, c, d, it is a solution of (58)–(60). If, say, $a = 0$, replace a by $a + \delta$ where δ is very small ($O(\epsilon)$) to secure a solution to (58)–(60). The above problem has a solution if and only if the dual problem has a solution [11]:

$$\mathrm{Maximize}\begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix} \begin{bmatrix} f_2 + \epsilon \\ -h_1 + \epsilon \\ \epsilon \end{bmatrix} \tag{64}$$

subject to

$$\begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix} \begin{bmatrix} e_1 & -1 & 0 & 0 \\ -g_2 & f_1 & -e_2 & 1 \\ 0 & -h_2 & g_1 & -f_2 \end{bmatrix}$$

$$\leq \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \tag{65}$$

$$\lambda_i \geq 0. \tag{66}$$

In turn, this problem has a solution if and only if we can
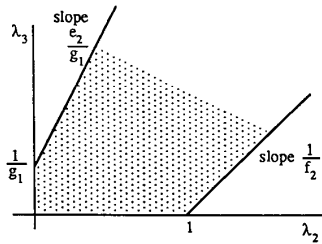
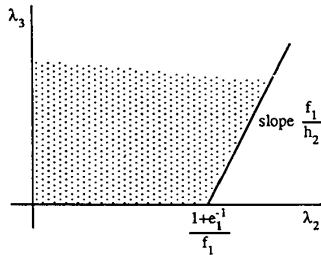Fig. 3.   One restriction on λ_2, λ_3 values.

Fig. 4.   Second restriction on λ_2, λ_3 values.

find any λ_i satisfying (66) and

$$[\lambda_1 \quad \lambda_2 \quad \lambda_3] \begin{bmatrix} e_1 & -1 & 0 & 0 \\ -g_2 & f_1 & -e_2 & 1 \\ 0 & -h_2 & g_1 & -f_2 \end{bmatrix}$$

$$\leqslant [1 \quad 1 \quad 1 \quad 1]. \quad (67)$$

We examine these constraints in the λ_2, λ_3 plane. The last two inequalities in (67) are

$$-e_2\lambda_2 + g_1\lambda_3 \leqslant 1 \quad (68)$$

$$\lambda_2 - f_2\lambda_3 \leqslant 1. \quad (69)$$

Together with the constraints $\lambda_2 \geqslant 0$, $\lambda_3 \geqslant 0$, the region so defined is depicted in Fig. 3. Notice that the stability conditions (48) force $f_2^{-1} < e_2 g_1^{-1}$ so that the region in Fig. 3 is not bounded.

Consider now the first two inequalities in (67) with λ_1 a parameter. These are

$$\lambda_2 \geqslant \frac{\lambda_1 e_1 - 1}{g_2} \quad (70)$$

$$\lambda_3 \geqslant \frac{\lambda_2 f_1}{h_2} - \frac{1 + \lambda_1}{h_2}. \quad (71)$$

Choose $\lambda_1 = e_1^{-1} > 0$. Then the region defined by these inequalities and $\lambda_i \geqslant 0$ is depicted in Fig. 4.

It is obvious that the region depicted in Figs. 3 and 4 have a common intersection, i.e., there exists λ_i satisfying (66) and (67).

REFERENCES

[1] Y. D. Landau, *Adaptive Control: The Model Reference Approach.* New York: Marcel Dekker, 1979.
[2] G. C. Goodwin and K. S. Sin, *Adaptive Filtering, Prediction and Control.* Englewood Cliffs, NJ: Prentice Hall, 1984.
[3] B. D. O. Anderson, R. R. Bitmead, C. R. Johnson, Jr., P. V. Kokotovic, R. L. Kosut, I. Mareels, L. Praly, and B. Riedle, *Stability of Adaptive Systems: Passivity and Averaging Analysis.* Cambridge, MA: MIT Press, 1987.
[4] S. Dasgupta and A. S. Bhagwat, "Conditions for designing strictly positive real transfer functions for adaptive output error identification," *IEEE Trans. Circuits Syst.*, vol. CAS-34, pp. 731–737, 1987.
[5] V. I. Kharitonov, "Asymptotic stability of an equilibrium position of a family of systems of linear differential equations," *Differential Equations*, pp. 1483–1485, Nov. 1985.
[6] S. Dasgupta, P. J. Parker, B. D. O. Anderson, F. J. Kraus, and M. Mansour, "Frequency domain conditions for the robust stability of linear and nonlinear dynamical systems," in *Proc. JACC 1988*, pp. 1863–1868.
[7] B. D. O. Anderson, "Identification of scalar errors-in-variables models with dynamics." *Automatica*, vol. 21, pp. 709–716, 1985.
[8] P. P. Khargonekar, T. T. Georgiou, and A. M. Pascoal, "On the robust stability of linear time invariant plants with unstructured uncertainty," *IEEE Trans. Automat. Contr.*, vol. AC-32, pp. 201–207, 1987.
[9] K. Poolla and T. Ting, "Nonlinear time-varying controllers for robust stabilization," *IEEE Trans. Automat. Contr.*, vol. AC-32, pp. 195–200, 1987.
[10] B. D. O. Anderson, E. I. Jury, and M. Mansour, "On robust Hurwitz polynomials," *IEEE Trans. Automat. Contr.*, vol. AC-32, pp. 909–913, 1987.
[11] D. G. Luenberger, *Introduction to Linear and Nonlinear Programming.* Reading, MA: Addison-Wesley, 1973.

✳

Brian D. O. Anderson (S'62–M'66–SM'74–F'75) received the B.S. degree in pure mathematics and electrical engineering from the University of Sydney, Sydney, Australia, and the Ph.D. degree in electrical engineering from Stanford University, Stanford, CA, in 1966.

He is currently professor and head of the Department of Systems Engineering at the Australian National University, Canberra, Australia. From 1967 through 1981, he was professor of electrical engineering at the University of Newcastle, New South Wales, Australia. He has also held appointments as visiting professor at a number of universities in the United States, Australia, and Europe. He is a coauthor of several books and his research interests are in control and signal processing.

Professor Anderson is a Fellow of the Australian Academy of Science, the Australian Academy of Technological Sciences, and the Royal Society, and an Honorary Fellow of the Institute of Engineers, Australia. Previously, he was Editor of *Automatica* and is currently President-elect of IFAC.

✳

Soura Dasgupta (M'87) received the B.E. degree in Electrical Engineering from the University of Queensland (Australia) in 1980 and the Ph.D. in Systems Engineering from the Australian National University.

Since 1985 he has been with the Department of Electrical and Computer Engineering, University of Iowa, where he is currently an assistant professor. He was a Junior Research Fellow at the Indian Statistical Institute, Calcutta (1881), and has held visiting appointments at the University of Notre Dame (1984–85) and Universite Catholique de Louvain-Le-Neuve, Belgium (1989).

Dr. Dasgupta is a member of Eta Kappa Nu, and serves as an Associate Editor of the IEEE TRANS. AUTOMAT. CONTR. His research interests are in adaptive signal processing and control, robust stability theory, and neural networks.

Frantisek J. Kraus (S'82–M'86); biography and photograph not available at time of publication.

✠

✠

Pramod P. Khargonekar (S'81–M'81) received the B.Tech. degree in electrical engineering from the Indian Institute of Technology, Bombay in 1977, and the M.S. degree in mathematics and the Ph.D. degree in electrical engineering from the University of Florida in 1980 and 1981, respectively.

From 1981 to 1984, he was with the Department of Electrical Engineering, University of Florida, and from 1984 to 1989 he was with the Department of Electrical Engineering, University of Minnesota. In September 1989, he joined the University of Michigan, where he is a professor of electrical engineering and computer science. His research interests include robust and H_2/H_∞ optimal control, adaptive control, distributed systems, algebraic system theory, and applications to aerospace control problems.

Dr. Khargonekar is the recipient of the Eckman Award, the NSF Presidential Young Investigator Award, and the George Taylor Award. He was an associate editor of the IEEE TRANS. AUTOMAT. CONTR. from 1987 to 1989. He is currently an associate editor of *Mathematics of Contr., Signals, and Syst.*, and *Syst. and Contr. Letters*.

Mohamed Mansour (SM'77–F'85) received the B.Sc. and M.Sc. degrees in electrical engineering from the University of Alexandria, Egypt, in 1951 and 1963, respectively, and the Dr. Sc.Techn. degree in electrical engineering from the ETH Zürich, Switzerland, in 1965.

In 1967–1968, he was with the Queen's University, Canada. Since 1968, he has been at ETH Zürich, where he is the director of the Institute of Automatic Control and Industrial Electronics. His fields of interest include control systems, stability theory and digital control, stability of power systems, and digital filters.

Dr. Mansour won the Silver Medal of ETH Zürich in 1965. In 1987, he was named an Honorary Professor of the Gnasu University of Technology and an Honorary Professor of the Guangxi University, both in China. In 1989, he received an Associate Fellowship of the Third World Academy of Science.