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Weighted q-Markov covariance equivalent realizations

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Reduced order models which preserve the first q Markov parameters and the first q covariances are called q-Markov covariance equivalent realizations (q-Markov COVERs). This paper extends these notions to the case where the input is coloured instead of white noise. We call this the 'weighted q-Markov COVER'.

1. Introduction

Let the linear stable system

$$\left.\begin{aligned} \dot{x}_1 &= A_1x_1 + D_1w, \quad w \in \mathbb{R}^{n_w}, \quad x_1 \in \mathbb{R}^{n_1} \\ y_1 &= C_1x, \quad y \in \mathbb{R}^{n_y} \end{aligned}\right\} \tag{1.1}$$

with $w(t)$ unit intensity white noise be reduced to

$$\left.\begin{aligned} \dot{x}_2 &= A_2x_2 + D_2w \quad x_2 \in \mathbb{R}^{n_2}, \quad n_2 < n_1 \\ y_2 &= C_2x_2, \quad y_R \in \mathbb{R}^{n_y} \end{aligned}\right\} \tag{1.2}$$

If the realization (A_2, D_2, C_2) has these two properties (1.3 *a*) and (1.3 *b*):

$$C_1A_1^iD_1 = C_2A_2^iD_2, \quad i = 0, 1, \ldots, q-1 \tag{1.3 a}$$

$$C_1A_1^iX_1C_1^* = C_2A_2^iX_2C_2^*, \quad i = 0, 1, \ldots, q-1 \tag{1.3 b}$$

where

$$\left.\begin{aligned} &0 = X_1A_1^* + A_1X_1 + D_1D_1^*, \quad 0 = X_2A_2^* + A_2X_2 + D_2D_2^* \\ &X_1 > 0, X_2 > 0 \end{aligned}\right\} \tag{1.4}$$

then the realization (A_2, D_2, C_2) is called a 'q-Markov COVariance equivalent realization' of (A_1, D_1, C_1) or simply a 'q-Markov COVER'.

For linear stable stationary stochastic processes the output autocorrelation matrix is

$$E[y(t+\tau)y^*(\tau)] = \sum_{i=0}^{\infty} R_it^i/i!$$

By a slight abuse of language we shall refer to the R_i sequence as the 'covariances' of

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the output y. More precisely

$$R_0 \triangleq Ey(t)y^*(t), \quad R_i \triangleq \frac{d^i}{dt^i}\{E[y(t+\tau)y^*(\tau)]\}_{t=0} = C_1 A_1^i X_1 C_1^*$$

Anderson (1969) described the class of reduced-order models which preserve the entire autocorrelation function and hence R_i, $i = 0, 1, \ldots, \infty$. This is accomplished by deleting the all-pass networks from the system. However, the Markov parameters are not preserved in Anderson (1969). Mullis and Roberts (1976) and Inouye (1983) find q-Markov COVERs using frequency domain techniques. The class of all q-Markov COVERs for multiple input–output processes is parameterized in Anderson and Skelton (1988) and this algorithm has been applied to the controller reduction problem (de Villemagne and Skelton 1987). However, the fundamental deficiency of these past procedures is their reliance upon *white* noise inputs to the system or subsystem being reduced. The purpose of this paper is to present a q-Markov COVER in the presence of coloured noise inputs. We call this the 'weighted q-Markov COVER'. This will widen the class of controller reduction problems that could be treated with q-Markov COVER ideas. Frequency weighted approximations of models exist based on other criteria for model reduction. Balanced realizations and the Hankel norm were developed for the weighted problem in Enns (1984), and have been also suggested for controller reduction in Latham and Anderson (1985).

The motivations for the weighted q-Markov COVER problem are three-fold. First, in the model reduction it may be of interest to dictate the range of frequencies over which the reduced model is required to be accurate. Indeed, the frequency 'range' may be quite detailed so as to prescribe a specific frequency *shaping* within this range. Since such methods can be characterized in the time domain the frequency weighting feature allows some degree of *unification* of the time and frequency domain methods since frequency domain specifications can influence the selection of state space models.

The second motivation derives from stability considerations in the controller reduction problem. Let the controller be reduced in order from $H_1(s)$ to $H_2(s)$. Then for some $\Delta H(s)$,

$$H_2(s) = H_1(s) + \Delta H(s) \tag{1.5}$$

Let the open-loop system $C_2(s)$, with its full-order controller $H_1(s)$ be closed-loop stable so that the closed-loop transfer matrix

$$[I + G_1(s)H_1(s)]^{-1}G_1(s) \tag{1.6}$$

has no poles in the right half-plane. Then the closed-loop system is stable with the reduced controller if (Enns, 1984; Latham and Anderson, 1985; Glover, 1984) $\Delta H(s)$ is stable and

$$\max_{\omega} \|\Delta H(j\omega)[I + G_1(j\omega)H_1(j\omega)]^{-1}G_1(j\omega)\| \leqslant 1 \tag{1.7}$$

This result suggests that $[I + G_1(j\omega)H_1(j\omega)]^{-1}G_1(j\omega)$ is an appropriate weighting to use in the model (controller) reduction problem. This frequency weighting idea has been treated in conjunction with *balancing* methods in Latham and Anderson (1985). To apply the idea of weighting to the q-Markov COVER model reduction methods is our primary goal.

Our last but not least motivation for studying the weighted q-Markov COVER problem is made manifest by the following result.

Let the controller

$$\left.\begin{aligned} u &= Gx_c + G_z z \\ \dot{x}_c &= A_c x_c + Fz \end{aligned}\right\} \tag{1.8 a}$$

driving the plant

$$\left.\begin{aligned} \dot{x} &= Ax + B(u + u_c) + Dw \\ y &= Cx \\ z &= Mx + v \end{aligned}\right\} \tag{1.8 b}$$

yield the closed-loop relation to u_c

$$\begin{bmatrix} y(s) \\ u(s) \end{bmatrix} = [Z(s)]u_c, \quad Z(s) = \sum_{i=0}^{\infty} \frac{\mathcal{C}\mathcal{A}^i\mathcal{D}}{s^{i+1}} \tag{1.8 c}$$

where

$$\mathcal{C} = \begin{bmatrix} C & 0 \\ G_z M & G \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix} = \begin{bmatrix} A + BG_z M & BG \\ FM & A_c \end{bmatrix}, \quad \mathcal{D} = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

Theorem 1

Any reduction of any linear controller which preserves the first $(q-1)$ Markov parameters (of the controller) will also preserve the first q Markov parameters of the closed-loop system between control inputs u_c and plant inputs u and outputs y.

A proof appears in Yousuff (1983) and Skelton and Anderson (1986).

This theorem can also be proven for discrete-time systems. Theorem 1 provides strong motivation for using q-Markov COVER methods for controller reduction since this method preserves a specified number of Markov parameters of the *closed-loop* system. The introduction of weighting affects the covariance matching, but not the Markov parameter matching.

Following these motivational discussions we return now to state the problem of interest. Suppose the *coloured* noise $w(t)$ in (1.1), (1.2) has the Gauss–Markov model

$$\left.\begin{aligned} &w(t) = C_w x_w + E_w w', \quad w(s) = [C_w(sI - A_w)^{-1}D_w + E_w]w'(s) \\ &\dot{x}_w = A_w x_w + D_w w'(t), \quad w(s) = W(s)w'(s) \end{aligned}\right\} \tag{1.9}$$

where $w'(t)$ is a zero-mean white noise process with unit intensity. Then (1.1) becomes

$$\begin{bmatrix} \dot{x} \\ \dot{x}_w \end{bmatrix} = \begin{bmatrix} A & DC_w \\ 0 & A_w \end{bmatrix} \begin{bmatrix} x \\ x_w \end{bmatrix} + \begin{bmatrix} DE_w \\ D_w \end{bmatrix} w' = A_1 x_1 + D_1 w' \tag{1.10}$$

$$y = [C \quad 0] \begin{bmatrix} x \\ x_w \end{bmatrix} = C_1 x_1$$

and the problem is to find a realization (A_R, D_R, C_R) such that

$$\begin{bmatrix} \dot{x}_R \\ \dot{x}_w \end{bmatrix} = \begin{bmatrix} A_R & D_R C_w \\ 0 & A_w \end{bmatrix} \begin{bmatrix} x_R \\ x_w \end{bmatrix} + \begin{bmatrix} D_R E_w \\ D_w \end{bmatrix} w' \triangleq A_2 x_2 + D_2 w' \tag{1.11}$$

$$y_R = [C_R \quad 0] \begin{bmatrix} x_R \\ x_w \end{bmatrix} \triangleq C_2 x_2$$

is a q-Markov COVER of (1.10), where the vectors (x, x_R, x_w, y) have dimensions (n, r_q, n_w, n_y) respectively and $r_q \leqslant n$. This is equivalent to finding a realization of order $(r_q + n_w)$ with transfer function $G_2(s)$ so that

$$G_2(s) = G_R(s) W(s) \tag{1.12 a}$$

$$\left.\frac{d^i}{ds^i}[G(s)W(s)]_{s=\infty} = \frac{d^i}{ds^i}[G_R(s)W(s)]_{s=\infty}, \quad \begin{matrix} G_R(s) = C_R(sI - A_R)^{-1} D_R \\ W(s) = C_w(sI - A_w)^{-1} D_w + E_w \end{matrix} \right\} \tag{1.12 b}$$

for $i = 0, 1, 2, \dots, q-1$, where $G(s)$ is of order n and $G_R(s)$ is of order $r_q < n$. $W(s)$ is of order n_w. In addition, the reduced realization must match the covariance derivatives

$$\frac{d^i}{ds^i}[y(t+\tau)y^*(\tau)]_{t=0} = \frac{d^i}{ds^i}[y_R(t+\tau)y_R^*(\tau)]_{t=0}, \quad i = 0, 1, \dots, q-1 \tag{1.12 c}$$

This is also equivalent to matching the first q derivatives of the power spectral density at high frequencies $(s = j\infty)$.

Definition 1

The matrix triple (A_R, D_R, C_R) is said to be a weighted q-Markov COVER of (A, D, C) subject to the weight (A_w, D_w, C_w, E_w) if and only if (1.12) holds.

If the weighting (A_w, D_w, C_w, E_w) is understood, then we shall refer to (A_R, D_R, C_R) simply as a weighted q-Markov COVER of (A, D, C). This paper describes a weighted q-Markov COVER.

In § 2 we present an impossibility result to indicate that a solution to the problem cannot be found by the same sort of rational calculations used to accomplish the unweighted result (Anderson and Skelton 1988, Yousuff *et al.* 1985). Section 3 describes the structure of solutions to the weighted q-Markov COVER problem. Section 4 describes a special covariance *control problem* which arises out of the *model reduction* discussion of § 3. Section 5 presents the final algorithm for the construction of weighted q-Markov COVERs.

2. Impossibility result

To characterize the nature of the difficulties in the weighted q-Markov COVER problem, consider the following second-order system subject to a first-order weight.

$$\left.\begin{array}{ll}\text{Plant:} & G(s)=\dfrac{cs+d}{s^2+es+f} \\ \text{Weight:} & W(s)=\dfrac{s+\alpha}{s+\beta}\end{array}\right\} \tag{2.1}$$

A state space realization of this system

$$A_1=\begin{bmatrix}-e & 1 & c(\alpha-\beta) \\ -f & 0 & d(\alpha-\beta) \\ 0 & 0 & -\beta\end{bmatrix},\quad D_1=\begin{bmatrix}c \\ d \\ 1\end{bmatrix},\quad C_1=(1\quad 0\quad 0) \tag{2.2}$$

may be reduced to the 1-Markov COVER

$$A_2=\begin{bmatrix}-e-\gamma & c(\alpha-\beta) \\ 0 & -\beta\end{bmatrix},\quad D_2=\begin{bmatrix}c \\ 1\end{bmatrix},\quad C_2=(1\quad 0) \tag{2.3}$$

with the desired properties (1.12) where

$$G_R(s)=(1)(s+e+\gamma)^{-1}c \tag{2.4}$$

(The Markov parameter matching requirement forces the appearance of the parameter c in (2.2) and (2.3).) However, γ cannot be found by a rational calculation of the given data $(\alpha, \beta, c, d, e, f)$.

To verify this, the 1-Markov COVER must match $R_0 \triangleq C_1 X_1 C_1^*$ where X_1 is the state covariance matrix satisfying

$$0=X_1 A_1^* + A_1 X_1 + D_1 D_1^* \tag{2.5}$$

In order to match the given $R_{1_0}=R_0$ the reduced-order system (2.3), which has output covariance

$$R_{2_0}=\frac{c^2}{2(\beta+e+\gamma)}\left[1+\frac{\alpha^2}{\beta(e+\gamma)}\right] \tag{2.6}$$

must satisfy, $R_{2_0}=R_{1_0}$, or

$$\frac{c^2}{2(\beta+a)}\left[1+\frac{\alpha^2}{\beta a}\right]=R_0,\quad a \triangleq e+\gamma \tag{2.7}$$

Solving for '*a*' leads to the equation:

$$[2\beta R_0]a^2+\beta[2\beta R_0-c^2]a-\alpha^2c^2=0 \tag{2.8}$$

which has the discriminant

$$\beta[4\beta^3R_0^2-4c^2\beta^2R_0+8R_0c^2\alpha^2+c^4\beta] \tag{2.9}$$

This is not a perfect square. Hence, the solution of (2.8) for '*a*' *necessarily* involves the non-rational square-root calculation. This is to be contrasted with the situation in the *unweighted* *q*-Markov COVER (Yousuff *et al.* 1985, Anderson and Skelton 1988) which involves only *rational* calculations. Hence the addition of a weight *completely* alters the character of the problem.

3. Structure of the weighted q-Markov COVER problem

This section illustrates the class of coordinate transformations and truncations which lead to weighted q-Markov COVERs, and relates this to the equivalent approach of oblique projections which will be ultimately used to derive our result.

The 'truncation' point of view is to transform the realization (1.10) described by (A_1, D_1, C_1, X_1) to new co-ordinates and delete some of the states in these co-ordinates. The truncation point of view is described by the mathematical steps

$$A_2 = LT^{-1}A_1TL^*, \quad D_2 = LT^{-1}D_1, \quad C_2 = C_1TL^*, \quad LL^* = I \tag{3.1}$$

where L is an $(r_q + n_w)x(n + n_w)$ matrix composed of only 0 or 1 entries with only one '1' entry in each row. Such a matrix L is called a 'truncation' matrix.

The 'projection' point of view in model reduction yields

$$A_2 = LA_1R, \quad A_2 = LD_1, \quad C_2 = C_1R, \quad LR = I \tag{3.2}$$

where the projection matrix $[RL]$ is idempotent, i.e. $[RL]^2 = [RL]$, due to the $LR = I$ property. Obviously (3.1) and (3.2) are equivalent if

$$L = LT^{-1}, \quad R = TL^* \tag{3.3}$$

We assume that the states to be deleted in (3.1) are the last states (bottom of the state vector x) associated with a possibly transformed (A, D, C) in (1.10). This can be accomplished by reordering states (imbedded in the choice of T). Hence, we may always choose

$$L = \begin{bmatrix} I_{r_q} & 0 & 0 \\ 0 & 0 & I_{n_w} \end{bmatrix} \tag{3.4}$$

For the discussion of the class of desired transformations, we shall concentrate initially on the projection approach, and we consider projections (3.2) described by

$$L = T_R^{-1}\begin{bmatrix} I_{r_q} & 0 & 0 \\ 0 & 0 & I_{n_w} \end{bmatrix}T_0^{-1} = T_R^{-1}LT_0^{-1} \tag{3.5 a}$$

$$R = T_0\begin{bmatrix} I_{r_q} & 0 \\ G & 0 \\ 0 & I_{n_w} \end{bmatrix}T_R = T_0[L^* + VGH]T_R \tag{3.5 b}$$

where L is described by (3.4) and

$$V \triangleq \begin{bmatrix} 0 \\ I_{n-r_q} \\ 0 \end{bmatrix}, \quad H = [I_{r_q} \quad 0] \tag{3.6}$$

where G is yet to be specified. Note that $LR = I$ as required. The matrix T_0 serves as a co-ordinate transformation on the original system

$$\hat{A}_1 = T_0^{-1}A_1T_0, \quad \hat{D}_1 = T_0^{-1}D_1, \quad \hat{C}_1 = C_1T_0 \tag{3.7}$$

and T_R serves as a co-ordinate transformation on the reduced model

$$\hat{A}_2 = T_RA_2T_R^{-1}, \quad \hat{D}_2 = T_RD_2, \quad \hat{C}_2 = C_2T_R^{-1} \tag{3.8}$$

We can make the connection with the truncation approach; let us define

$$T=T_0\begin{bmatrix} (T_R)_{11} & 0 & (T_R)_{12} \\ G(T_R)_{11} & K & G(T_R)_{12} \\ \underbrace{(T_R)_{21}}_{r_q} & \underbrace{0}_{n-r_q} & \underbrace{(T_R)_{22}}_{n_w} \end{bmatrix} \qquad (3.9\,a)$$

$$T^{-1}=\begin{bmatrix} (T_R^{-1})_{11} & 0 & (T_R^{-1}) \\ -K^{-1}G & K^{-1} & 0 \\ (T_R^{-1})_{21} & 0 & (T_R^{-1})_{22} \end{bmatrix} T_0^{-1} \qquad (3.9\,b)$$

where $|K|\neq 0$ and K is arbitrary. Then it is not hard to verify that (3.3) holds. Also, it may be verified that (3.9) can be expressed as

$$T=T_0\{[L^*+VGH]T_R L+VKV^*\} \qquad (3.10\,a)$$

$$T^{-1}=\{[L^*T_R^{-1}-VK^{-1}GH]L+VK^{-1}V^*\}T_0^{-1} \qquad (3.10\,b)$$

and it may be shown that

$$T^{-1}T=L^*L+VV^*=I \qquad (3.11\,a)$$

$$TT^{-1}=L^*L+VV^*=I \qquad (3.11\,b)$$

Note that an 'orthogonal' projection requires $RL=(RL)^*$, hence (3.3) is an orthogonal projection if T is unitary ($T^{-1}=T^*$). It is readily verified that (3.9) is generally not a unitary matrix. Hence (3.3) will usually be an oblique rather than an orthogonal projection.

Return now to the projection operator of (3.5), which is described by

$$A_2=LA_1R=T_R^{-1}LT_0^{-1}A_1T_0[L^*+VGH]T_R=T_R^{-1}\hat{A}_2T_R \qquad (3.12\,a)$$

$$D_2=LD_1=T_R^{-1}LT_0^{-1}D_1=T_R^{-1}\hat{D}_2 \qquad (3.12\,b)$$

$$C_2=C_1R=C_1T_0[L^*+VGH]T_R=\hat{C}_2T_R \qquad (3.12\,c)$$

with obvious definitions for $(\hat{A}_2, \hat{D}_2, \hat{C}_2)$. The transformation T_R will play essentially no role in what follows. Our focus will be on obtaining $\hat{A}_2, \hat{D}_2, \hat{C}_2$ from A_1, D_1, C_1 via two steps: transform with T_0, and then pre-post-multiply as appropriate by L and L^*+VGH. The transformed original system (1.10) is denoted by

$$\hat{A}_1 \triangleq T_0^{-1}A_1T_0=T_0^{-1}\begin{bmatrix} A & DC_w \\ 0 & A_w \end{bmatrix}T_0=\begin{bmatrix} \hat{A} & \hat{D}\hat{C}_w \\ 0 & \hat{A}_w \end{bmatrix} \qquad (3.13\,a)$$

$$\hat{D}_1 \triangleq T_0^{-1}D_1=T_0^{-1}\begin{bmatrix} DE_w \\ D_w \end{bmatrix}=\begin{bmatrix} \hat{D}E_w \\ \hat{D}_w \end{bmatrix} \qquad (3.13\,b)$$

$$\hat{C}_1 \triangleq C_1T_0=[C \quad 0]T_0=[\hat{C} \quad 0] \qquad (3.13\,c)$$

which equations define $(\hat{A}, \hat{D}, \hat{C}, \hat{C}_w, \hat{D}_w, \hat{A}_w)$. Notice that the natural division of the state vector into two subvectors, one associated with the system being reduced, the other being associated with the system producing the coloured noise, is preserved after transformation by T_0, in view of the structure of $\hat{A}_1, \hat{D}_1, \hat{C}_1$. Moreover, we further select T_0 such that there is additional structure in the submatrices associated with the first entries of the state vector:

$$\hat{A}=\begin{bmatrix}\hat{A}_R & B\\ A_{TR} & A_T\end{bmatrix},\quad \hat{D}=\begin{bmatrix}\hat{D}_R\\ D_T\end{bmatrix} \tag{3.14 a}$$

$$\hat{C}=[\hat{C}_R \quad 0] \tag{3.14 b}$$

where the pair $(\hat{A}_R, \hat{C}_R)$ are in the normalized Hessenberg form (Anderson and Skelton 1988)

$$\hat{A}_R \triangleq \begin{bmatrix} A_{11} & A_{12} & 0 & \dots & 0\\ \vdots & & & & \vdots\\ & & & & 0\\ A_{q-1,1} & \dots & & & A_{q-1,q}\\ A_{q,1} & \dots & & & A_{q,q}\end{bmatrix},\quad B \triangleq \begin{bmatrix}0\\ \vdots\\ 0\\ 0\\ B_q\end{bmatrix} \tag{3.15 a}$$

$$\hat{C}_R \triangleq [C_{11} \quad 0 \quad 0 \quad \dots \quad 0] \tag{3.15 b}$$

where $n_1 = n_y$, and

$$\text{rank } A_{i,i+1} = n_{i+1} \leqslant n_i, \quad i=1,\dots,q-1 \tag{3.15 c}$$

$$A_{11} \in \mathbb{R}^{n_1 \times n_1}, \quad C_{11} \in \mathbb{R}^{n_1 \times n_1}, \quad |C_{11}| \neq 0 \tag{3.15 d}$$

$$B_q = [A_{q,q+1}, \dots, A_{q,n}], \quad \sum_{i=1}^{q} n_i = r_q$$

where $\hat{X}_1$ satisfies

$$0 = \hat{X}_1 \hat{A}_1^* + \hat{A}_1 \hat{X}_1 + \hat{D}_1 \hat{D}_1^* \tag{3.15 e}$$

and has the structure

$$\hat{X}_1 = \begin{bmatrix} I_n. & \hat{X}_{pw}\\ \hat{X}_{pw}^* & \hat{X}_w\end{bmatrix} \tag{3.15 f}$$

Further, the matrices C_{11} and $A_{i,i+1}$ for $i=1,\dots,q-1$ have a certain structure which is mentioned here. If rank $C = n_1$, a co-ordinate transformation T_0 always exists to take (A_1, D_1, C_1, X_1) to the structure (3.15). The algorithm for constructing T_0 is similar to that given in Anderson and Skelton (1988), except that the algorithm is stopped after q iterations. (Continuing further will yield $\hat{X}_1 = I$ and Hessenberg structure for the entire $\hat{A}_1$. This we do not require, so the algorithm is stopped when $\hat{A}_R$ is Hessenberg.) The definitions of $(\hat{A}_2, \hat{D}_2, \hat{C}_2)$ implied in (3.12) allow the expansions

$$\hat{A}_2 = L\hat{A}_1[L^* + VGH] = L\begin{bmatrix}\hat{A} & \hat{D}\hat{C}_w\\ 0 & \hat{A}_w\end{bmatrix}[L^* + VGH]$$

$$= \begin{bmatrix}\hat{A}_R + BG & \hat{D}_R\hat{C}_w\\ 0 & \hat{A}_w\end{bmatrix} = \begin{bmatrix}\hat{A}_R & \hat{D}_R\hat{C}_w\\ 0 & \hat{A}_w\end{bmatrix} + \begin{bmatrix}B\\ 0\end{bmatrix} G[I \quad 0] = \bar{A}_2 + \bar{B}_2 GH \tag{3.16 a}$$

$$\hat{D}_2 = L\hat{D}_1 = L\begin{bmatrix} \hat{D}_R E_w \\ D_T E_w \\ \hat{D}_w \end{bmatrix} = \begin{bmatrix} \hat{D}_R E_w \\ \hat{D}_w \end{bmatrix} \tag{3.16 b}$$

$$\hat{C}_2 = \hat{C}_1[L^* + VGH] = [\hat{C} \quad 0][L^* + VGH] = [\hat{C}_R \quad 0] \tag{3.16 c}$$

The structure of $\hat{C}_R$, $\hat{A}_R$, and B in (3.15 *a*), (3.15 *b*) easily yields

$$\hat{C}_R \hat{A}_R^i B = 0 \quad i = 0, 1, \ldots, q-1 \tag{3.17}$$

Now define

$$O_{1q} \triangleq \begin{bmatrix} \hat{C}_1 \\ \vdots \\ \hat{C}_1 \hat{A}_1^{q-1} \end{bmatrix}, \quad O_{2q} \triangleq \begin{bmatrix} \hat{C}_2 \\ \vdots \\ \hat{C}_2 \hat{A}_2^{q-1} \end{bmatrix} \tag{3.18}$$

and use the structure of the matrices (3.13) and (3.16) to see that

$$O_{1q} = O_{2q} L \tag{3.19 a}$$

$$O_{1q} = \begin{bmatrix} \hat{C}_R & 0 & 0 \\ & & \hat{C}_R \hat{D}_R \hat{C}_w \\ \vdots & \vdots & \vdots \\ \hat{C}_R \hat{A}_R^{q-1} & 0 & \sum_{i=1}^{q-2} \hat{C}_R \hat{A}_R^i \hat{D}_R \hat{C}_w \hat{A}_w^{q-2-i} \end{bmatrix} \tag{3.19 b}$$

and

$$O_{2q} = \begin{bmatrix} \hat{C}_R & 0 \\ & \hat{C}_R \hat{D}_R \hat{C}_w \\ \vdots & \vdots \\ \hat{C}_R \hat{A}_R^{q-1} & \sum_{i=0}^{q-2} \hat{C}_R \hat{A}_R^i \hat{D}_R \hat{C}_w \hat{A}_w^{q-2-i} \end{bmatrix} \tag{3.19 c}$$

which results rely on (3.17). Note from (3.19) that matrices O_{1q}, O_{2q} are independent of G. This proves the following.

Lemma 1

Define O_{1q}, O_{2q} by (3.18), (3.13), (3.16), and L by (3.4). Then O_{2q} is independent of G and

(i) $$\hat{C}_R \hat{A}_R^i B = 0 \quad i = 0, 1, \ldots, q-1 \tag{3.17}$$

(ii) $$O_{1q} = O_{2q} L \tag{3.19 a}$$

Lemma 1 is useful in proving the following.

Lemma 2

The realization $(\hat{A}_2, \hat{D}_2, \hat{C}_2)$ defined by (3.16) matches the first q-Markov parameters of (A_1, D_1, C_1) for any choice of G.

Proof

The Markov parameters of (A_1, D_1, C_1) are the same as those of the transformed co-ordinates $(\hat{A}_1, \hat{D}_1, \hat{C}_1)$. Hence we only need to show that

$$O_{1q}\hat{D}_1 = O_{2q}\hat{D}_2 \tag{3.20}$$

But $\hat{D}_2 = L\hat{D}_1$ from (3.16 *b*). Hence

$$O_{1q}\hat{D}_1 = O_{2q}L\hat{D}_1 \tag{3.21}$$

and the proof of Lemma 2 follows immediately from (3.21) and (3.19 *a*). □

Note that the development of the q-Markov COVER requires the matching of q-Markov parameters and q-covariances. Lemma 2 promises the first of these objectives from the *structure* of co-ordinates (3.15), leaving G totally free to accomplish the second objective, which requires

$$O_{1q}\hat{X}_1\hat{C}_1^* = O_{2q}\hat{X}_2\hat{C}_2^* \tag{3.22}$$

where $\hat{X}_1$ satisfies (3.15 *e*) and $\hat{X}_2$ satisfies

$$0 = \hat{X}_2\hat{A}_2^* + \hat{A}_2\hat{X}_2 + \hat{D}_2\hat{D}_2^* \tag{3.23}$$

Use of (3.19 *a*) and (3.15 *b*) reduces the left-hand side of (3.22) to

$$O_{1q}\hat{X}_1\hat{C}_1^* = O_{2q}L\hat{X}_1\begin{bmatrix} C_{11}^* \\ 0 \end{bmatrix} = O_{2q}L(\hat{X}_1)_1C_{11}^*$$

where $(\hat{X}_1)_1$ denotes the first n_y columns of $\hat{X}_1$. Now the right-hand side of (3.22) reduces to

$$O_{2q}\hat{X}_2\hat{C}_2^* = O_{2q}\hat{X}_2\begin{bmatrix} C_{11}^* \\ 0 \end{bmatrix} = O_{2q}(\hat{X}_2)_1C_{11}^* \tag{3.24}$$

where $(\hat{X}_2)_1$ denotes the first n_y columns of $\hat{X}_2$. Now since C_{11} is non-singular from (3.15 *d*), (3.22) requires

$$O_{2q}L(\hat{X}_1)_1 = O_{2q}(\hat{X}_2)_1 \tag{3.25}$$

Note that $(O_{2q}, L, \hat{X}_1)$ are given *a priori* and that G represents the total freedom available to influence the yet-to-be-determined $\hat{X}_2$ so that (3.25) is satisfied. This leads to the following Lemma.

Lemma 3

Let $\hat{X}_2$ be the state covariance associated with the matrix pair $(\bar{A}_2 + \bar{B}_2GH, \hat{D}_2)$, where from (3.16),

$$\bar{A}_2 = \begin{bmatrix} \hat{A}_R & \hat{D}_R \hat{C}_w \\ 0 & \hat{A}_w \end{bmatrix}, \quad \bar{B}_2 = \begin{bmatrix} B \\ 0 \end{bmatrix} \tag{3.26}$$

$$\hat{D}_2 = \begin{bmatrix} \hat{D}_R E_w \\ \hat{D}_w \end{bmatrix}, \quad H = [I_{r_q} \quad 0]$$

where $(\hat{A}_R, \hat{D}_R, \hat{C}_R)$, B, $(\hat{A}_w, \hat{D}_w, E_w)$ are defined in (3.13)–(3.15). A weighted q-Markov COVER $(\hat{A}_R + BG, \hat{D}_R, \hat{C}_R)$ exists for (A, D, C) subject to the weight (A_w, D_w, C_w, E_w) if there exists a G which assigns $\hat{X}_2$ to satisfy (3.25).

Outline of proof

We have assumed in our definitions of (A, D, C) that A is stable and rank $C = n_y$. Truncations are described by (3.1), (3.4), and projections equivalent to this result are described by (3.5). This has led, without additional assumptions, to Lemmas 1 and 2 and the necessary condition (3.25).

4. Covariance control problem

This section solves the problem posed by Lemma 3, which suggests that weighted q-Markov COVERs exist if a certain *control* problem can be solved. However, it is not a standard type of control problem but a covariance assignment problem (Skelton 1988). The solution of (3.25) is first characterized.

A weighted q-Markov COVER of (A, D, C) is subject to weight (A_w, D_w, C_w, E_w) if there exists a solution to (3.25). This is a linear equation in $(\hat{X}_2)$, and the necessary and sufficient condition for the existence of an $(\hat{X}_2)$, solving (3.25) is developed as follows. Let the singular value decomposition of O_{2q} be written

$$O_{2q} = [U_1 \quad U_2] \begin{bmatrix} \Sigma_2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix} = U_1 \Sigma_2 V_1^* \tag{4.1}$$

Then the Moore–Penrose inverse of O_{2q} is O_{2q}^+

$$O_{2q}^+ = V_1 \Sigma_2^{-1} U_1^* \tag{4.2}$$

and all solutions of (3.25) are described by

$$(\hat{X}_2)_1 = O_{2q}^+ O_{2q} L(\hat{X}_1)_1 + V_2 Z \tag{4.3}$$

for an arbitrary matrix Z. For the O_{2q} given in (3.18) the first n_1 rows of O_{2q} are given by $[C_{11} \quad 0]$, and it may be shown that the first n_1 rows of $\hat{O}_{2q}^+$ are $[C_{11}^{-1} \quad 0]$ and the first n_y rows of $[V_2 \quad Z]$ are zero. Hence (4.3) leads to the first n_y rows of $(\hat{X}_2)_1$ and $(\hat{X}_1)_1$

$$(\hat{X}_2)_{11} = (\hat{X}_1)_{11} = I_{n_1} \tag{4.4}$$

Using (4.2), (4.3) may be written

$$(\hat{X}_2)_1 \triangleq V_1 V_1^* L(\hat{X}_1)_1 + V_2 Z \tag{4.5}$$

One may conclude that $(\hat{X}_2)_1 = L(\hat{X}_1)_1$ is the *unique* solution of (3.25) or (4.5) only if $n_y q \geqslant (r_q + n_w)$, which is a necessary condition for O_{2q} to have linearly independent columns (in which case $V_2=0$). Thus the smallest q for which a weighted q-Markov COVER can have a unique state covariance must satisfy $q \geqslant (r_q+n_w)/n_y$. Since $n_{i+1} \leqslant n_i$ and $r_q=n_1+\ldots+n_q$, this condition for single output systems ($n_y=n_1=1$ because $q \geqslant q+n_w$) is impossible for any non-trivial weight ($n_w>0$). Hence, weighted q-Markov COVERs of single output systems cannot be unique.

Now describe the entire $\hat{X}_2$ matrix by

$$\hat{X}_2=\begin{bmatrix} I_{n_1} & X_{12} \\ X_{12}^* & F \end{bmatrix} \begin{matrix} \}\, n_1 \\ \}\, r_q+n_w-n_1 \end{matrix} \tag{4.6}$$

where $(\hat{X}_2)_1$ is given by (4.3) or (4.5), and $(\hat{X}_2)_{11}=I_{n_1}$ from (4.4). Hence, we define the partitioned matrices by

$$\begin{bmatrix} I \\ X_{12}^* \end{bmatrix} \triangleq (\hat{X}_2)_1 \tag{4.7}$$

Matrices F and X_{12} are free to be chosen subject to the constraint $\hat{X}_2>0$ and (4.5) or (4.3), and

$$0=\hat{X}_2[\bar{A}_2+\bar{B}_2GH]^*+[\bar{A}_2+\bar{B}_2GH]\hat{X}_2+\hat{D}_2\hat{D}_2^* \tag{4.8}$$

where $\hat{X}_2$ must have a preselected structure (4.6), (4.7).

We need the following theorem from Skelton (1988) which provides all solutions G to (4.8) for a prescribed $\hat{X}_2$. We shall need these definitions and the singular value decomposition of a given plant and weight data:

$$\bar{B}_2\bar{B}_2^+=[U_{B1} \quad U_{B2}]\begin{bmatrix} I_b & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} U_{B1}^* \\ U_{B2}^* \end{bmatrix} \quad (SVD) \tag{4.9 a}$$

$$H^+H=[U_{M1} \quad U_{M2}]\begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} U_{M1}^* \\ U_{M2}^* \end{bmatrix} \quad (SVD) \tag{4.9 b}$$

$$\begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \triangleq \begin{bmatrix} U_{B1}^* \\ U_{B2}^* \end{bmatrix} \hat{X}_2^{-1}U_{M2}U_{M2}^*\hat{X}_2U_B, \quad U_B \triangleq [U_{B1} \quad U_{B2}] \tag{4.9 c}$$

$$N_1N_1^+=[U_{N1}U_{N2}]\begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} U_{N1}^* \\ U_{N2}^* \end{bmatrix} \quad (SVD) \tag{4.9 d}$$

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix} \triangleq \begin{bmatrix} U_{B1}^* \\ U_{B2}^* \end{bmatrix}[\hat{X}_2\bar{A}_2^*+\bar{A}_2\hat{X}_2+\hat{D}_2\hat{D}_2^*][U_{B1} \quad U_{B2}] \tag{4.9 e}$$

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{12}^* & \Psi_{22} \end{bmatrix} \triangleq \begin{bmatrix} U_{N1}^* \\ U_{N2}^* \end{bmatrix}[Q_{12}N_2N_1^+ +(Q_{12}N_2N_1^+)^*+Q_{11}][U_{N1}, U_{N2}] \tag{4.9 f}$$

$$\begin{aligned} G_1 \triangleq -\bar{B}_2^+U_{B1}\{Q_{12}U_{B2}^*+[-Q_{12}N_2N_1^+ \\ +(U_{N1}\Psi_{12}+\tfrac{1}{2}U_{N2}\Psi_{22})U_{N2}^*]U_{B1}^*\}\hat{X}_2^{-1}H^+ \end{aligned} \tag{4.10 a}$$

$$G_2 \triangleq \bar{B}_2^+U_{B1}U_{N2} \tag{4.10 b}$$

$$G_3 \triangleq U_{N2}^*U_{B1}^*\hat{X}_2^{-1}H^+ \tag{4.10c}$$

Theorem 2 (adapted from Skelton 1988)

There exists a G which satisfies (4.8) if and only if the following three conditions hold:

$$Q_{22}=0 \tag{4.11 a}$$

$$\Psi_{11}=0 \tag{4.11 b}$$

$$Q_{12}N_2[I-N_1^+N_1]=0 \tag{4.11 c}$$

If these conditions are satisfied, then all G which satisfy (4.8) are given by

$$G=G_1+G_2SG_3, \quad S=-S^* \tag{4.12}$$

where S is an arbitrary skew-symmetric matrix.

The matrix structures and dimensions are as follows:

$$b=\text{rank } B=\text{rank } B_q=\text{rank}\,[A_{qq+1} \quad \dots \quad A_{qn}] \tag{4.13}$$

from (3.15)

$$m=r_q \quad \text{since } H=[I_{r_q} \quad 0], \quad H^+=\begin{bmatrix} I_{r_q} \\ 0 \end{bmatrix}, \quad [U_{M1} \quad U_{M2}]=\begin{bmatrix} I_m & 0 \\ 0 & I_{n_w} \end{bmatrix} \tag{4.14}$$

$$U_{M2}^*\hat{X}_2=[\hat{X}_2]_{\text{last } n_w \text{ rows}} \tag{4.15}$$

$$\hat{X}_2^{-1}U_{M2}=\{\hat{X}_2^{-1}]_{\text{last } n_w \text{ cols}} \tag{4.16}$$

$$\hat{X}_2^{-1}H^+=\hat{X}_2^{-1}\begin{bmatrix} I_{r_q} \\ 0 \end{bmatrix}=[\hat{X}_2^{-1}]_{\text{first } r_q \text{ cols}} \tag{4.17}$$

We must now introduce the constraints and freedom as $\hat{X}_2$ into the above theorem. Note from (3.15*f*) that $\hat{X}_1$ has the structure

$$\hat{X}_1=\begin{bmatrix} I_{n_1} & 0 & 0 & 0 & X_{pw1} \\ 0 & I_r & 0 & 0 & X_{pw2} \\ 0 & 0 & I_{n_q} & 0 & X_{pw3} \\ 0 & 0 & 0 & I_{n-r_q} & X_{pwT} \\ X_{pw1}^* & X_{pw2}^* & X_{pw3}^* & X_{pwT}^* & \hat{X}_w \end{bmatrix} \tag{4.18}$$

It may be shown by construction that the structure

$$\hat{X}_2=\begin{bmatrix} I_{n_1} & 0 & 0 & X_{pw1} \\ 0 & I_r & 0 & X_{pw2} \\ 0 & 0 & F_{22} & X_{pw3} \\ X_{pw1}^* & X_{pw2}^* & X_{pw3}^* & \hat{X}_w \end{bmatrix} \tag{4.19}$$

satisfies (4.11 *a*) and (4.3) for any $F_{22}>0$, if we choose Z in (4.3) by

$$Z=V_2^+(\hat{X}_2)_1-O_{2q}+O_{2q}L(\hat{X}_1)_1] \tag{4.20}$$

where V_2 is computed from (4.1). Further, if $\lambda_{min}(F_{22})$ is sufficiently large, $\hat{X}_2$ is positive definite. (It is adequate to have $\hat{F}_{22} \geqslant I$ to secure $\hat{X}_2 > 0$.) These developments prove the following theorem.

Theorem 3

Let $(\hat{A}_1, \hat{D}_1, \hat{C}_1)$ have the weighted Hessenberg structure (3.13)–(3.15). Let Q_{12}, $\Psi_{11}, N_1, N_2, G_1, G_2, G_3, G$ be defined by the given data from calculations (4.9), (4.10), (4.12). Let $\bar{A}_2$, $\bar{B}_2$, $\hat{D}_2$, H be defined by (3.26), and define $\hat{A}_2$ by

$$\hat{A}_2 \triangleq \bar{A}_2 + \hat{B}_2 GH$$

Let $(\hat{A}_R, B, \hat{C}_R, \hat{D}_R)$ be defined by (3.14)–(3.15). Then $(\hat{A}_2, \hat{D}_2, \hat{C}_2)$ is a q-Markov COVER of $(\hat{A}_1, \hat{D}_1, \hat{C}_1)$ and $(\hat{A}_R + BG, \hat{D}_R, \hat{C}_R)$ is a weighted q-Markov COVER of $(\hat{A}, \hat{D}, \hat{C})$ subject to weight (A_w, D_w, C_w, E_w) if (4.11 *b*) and (4.11 *c*) hold for some choice of F_{22} in (4.31) such that $\hat{X}_2$ is positive definite.

Corollary

Suppose (4.11) does not hold so that G in (4.12) does not satisfy (4.8). Then (4.12) is the optimal choice of G in the sense of minimizing

$$\|\hat{X}_2[\bar{A}_2 + \bar{B}_2 GH]^* + [\bar{A}_2 + \bar{B}_2 GH]\bar{X}_2 + \bar{D}_2 \bar{D}_2^*\|$$

The proof of the corollary follows in a straightforward manner and is omitted.

5. Weighted q-Markov COVER algorithm

Step 0

Given the plant $C(sI - A)^{-1}D$ and the weight $C_w(sI - A_w)^{-1}D_w + E_w$, and q.

Step 1

Compute the weighted Hessenberg form of the plant $\{A, D, C\}$. First note the complete weighted system is described by

$$\{A_0, D_0, C_0, X_0\} = \left\{\begin{bmatrix} A & DC_w \\ O & A_w \end{bmatrix}, \begin{bmatrix} DE_w \\ D_w \end{bmatrix}, [C \quad O], \begin{bmatrix} X_p & X_{pw} \\ X_{pw}^* & X_w \end{bmatrix}\right\}$$

Solve $0 = X_0 A_0^* + A_0 X_0 + D_0 D_0^*$.

Steps 1.1–1.4 as follows are taken from Anderson and Skelton (1988) to get the Hessenberg form.

Step 1.1

Factor $X_p = T_p T_p^*$.

Step 1.2

Singular value decomposition

$$CT_p = U_c \Sigma_c V_c^*$$

Define

$$\{A_1, D_1, C_1\} \triangleq \{V_c^* T_p^{-1} A T_p V_c, V_c^* T_p^{-1} D, C T_p V_c\}$$

Step 1.3

Beginning with $i=1$, compute the singular value decomposition of the $r_i x(n_x-r_i)$ matrix $\left(r_i \triangleq \sum_{\alpha=1}^{i} n_\alpha\right)$

$$[A_i]_{12} \triangleq U_i \begin{bmatrix} \Sigma_i & 0 \\ 0 & 0 \end{bmatrix} V_i^*, \quad n_{i+1} \triangleq \dim \Sigma_i$$

$$T_i \triangleq \begin{bmatrix} I_i & 0 \\ 0 & V_i \end{bmatrix}, \quad I_i = \text{identity of dim } r_i$$

Step 1.4

$\{A_{i+1}, D_{i+1}, C_{i+1}\} = \{T_i^* A_i T_i, T_i^* D_i, C_i T_i\}$.

Setting $i=i+1$, return to Step 1.3. Repeat until $r_i = n_1+n_2+\ldots+n_q$. At this point the matrices $\{A_i, D_i, C_i, X_i\} = \{\hat{A}, \hat{D}, \hat{C}, I\}$, have the weighted Hessenberg structure down to the first $(n_1+n_2+\ldots+n_q)$ block of rows. Now the complete system is as described by (3.13)–(3.15), where $(\hat{A}_w, \hat{C}_w, \hat{D}_w) = (A_w, C_w, D_w)$ in this algorithm. (With a slight increase in computation one may change A_w, C_w, D_w, E_w to normalize $\hat{X}_w = I$, but we do not need to do so.)

$$\{\hat{A}_1, \hat{D}_1, \hat{C}_1, \hat{X}_1\} = \left\{ \begin{bmatrix} \hat{A} & \hat{D}\hat{C}_w \\ 0 & \hat{A}_w \end{bmatrix}, \begin{bmatrix} \hat{D}\hat{E}_w \\ \hat{D}_w \end{bmatrix}, [\hat{C} \quad 0], \begin{bmatrix} I & \hat{X}_{pw} \\ \hat{X}_{pw}^* & \hat{X}_w \end{bmatrix} \right\}$$

$$\hat{X}_{pw} = \ldots T_i^* \ldots T_2^* T_1^* V_c^* T_p^{-1} X_{pw}$$

Step 2

From (4.31), using $F_{22} = I$, define

$$\hat{X}_2 \triangleq \begin{bmatrix} I_{r_q} & \bar{X}_{pw} \\ \bar{X}_{pw}^* & \hat{X}_w \end{bmatrix} = \bar{X}_2$$

$$\bar{X}_{pw} \triangleq [I_{r_q} \quad 0] \hat{X}_{pw}, \quad r_q = n_1+n_2+\ldots+n_q$$

$$\hat{A} = \begin{bmatrix} \hat{A}_R & B \\ A_{TR} & A_T \end{bmatrix}, \quad \hat{A}_R \in \mathbb{R}^{r_q \times r_q}$$

$$\hat{D} = \begin{bmatrix} \hat{D}_R \\ D_T \end{bmatrix}, \quad \hat{C} = [\hat{C}_R \quad 0]$$

$$\bar{A}_2 = \begin{bmatrix} \hat{A}_R & \hat{D}_R \hat{C}_w \\ 0 & \hat{A}_w \end{bmatrix}, \quad \bar{B}_2 = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

$$H = [I_{r_q} \quad 0], \quad \hat{D}_2 = \begin{bmatrix} \hat{D}_R E_w \\ \hat{D}_w \end{bmatrix}$$

$$\bar{B}_2 \bar{B}_2^+ = [U_{B1} \quad U_{B2}] \begin{bmatrix} I_b & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_{B1}^* \\ U_{B2}^* \end{bmatrix}$$

$$[U_{M1}, U_{M2}] \triangleq \left[\begin{bmatrix} I_{r_q} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ I \end{bmatrix}\right] = I$$

$$\begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \triangleq \begin{bmatrix} U_{B1}^* \\ U_{B2}^* \end{bmatrix} [\bar{X}_2^{-1} U_{M2} U_{M2}^* \bar{X}_2 U_B]$$

$$N_1 N_1^+ = [U_{N1} U_{N2}] \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_{N1}^* \\ U_{N2}^* \end{bmatrix}$$

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix} \triangleq \begin{bmatrix} U_{B1}^* \\ U_{B2}^* \end{bmatrix} [\bar{X}_2 \bar{A}_2^* + \bar{A}_2 \bar{X}_2 + \hat{D}_2 \hat{D}_2^*][U_{B1} U_{B2}]$$

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{12}^* & \Psi_{22} \end{bmatrix} = \begin{bmatrix} U_{N1}^* \\ U_{N2}^* \end{bmatrix} [Q_{12} N_2 N_1^+ + (Q_{12} N_2 N_1^+)^* + Q_{11}][U_{N1} U_{N2}]$$

$$G_1 = -\bar{B}_2^+ U_{B1} \{Q_{12} U_{B2}^* + [-Q_{12} N_2 N_1^+ + (U_{N1} \Psi_{12} + \tfrac{1}{2} U_{N2} \Psi_{22}) U_{N2}^*] U_{B1}^*\} \bar{X}_2^{-1} H^*$$

$$G_2 = \bar{B}_2^+ U_{B1} U_{N2}$$

$$G_3 = U_{N2}^* U_{B1}^* \bar{X}_2^{-1} H^*$$

Step 3

If

$$Q_{22} = 0, \quad \Psi_{11} = 0, \quad Q_{12} N_2 [I - N_1^+ N_1] = 0$$

then $(\hat{A}_R + BG, \hat{D}_R, \hat{C}_R)$ is a weighted q-Markov COVER of (A, B, C), for any skew-symmetric S in $G = G_1 + G_2 S G_3$. Otherwise $(\hat{A}_R + BG, \hat{D}_R, \hat{C}_R)$ is a best approximation to a weighted q-Markov COVER (see the corollary).

Note. For convenience we have chosen $F_{22} = I$ in (4.31) in the above algorithm, although other $F_{22} > 0$ may be used so long as $\hat{X}_2$ is positive definite.

Example

For the example in § 2, let

$$G(s) = C(sI - A)^{-1} D = \frac{s+1}{s^2 + s + 1}$$

$$W(s) = C_w(sI - A_w)^{-1} D_w + E_w = \frac{s+1}{s+0{\cdot}1}$$

the above algorithm yields (for $F_{22} = 1$, $S = 0$),

$$\hat{A}_R + BG = -0{\cdot}8511 + 0{\cdot}3560(-8{\cdot}8852) = -4{\cdot}0144$$

$$\hat{C}_R = 2{\cdot}5202$$

$$\hat{D}_R = 0{\cdot}3968$$

which gives the weighted 1-Markov COVER

$$G_R(s)=\frac{1}{s+4{\cdot}0144}$$

The first Markov and covariance parameters match as promised

$$M_0=1=[C\quad 0]\begin{bmatrix} DE_w \\ D_w \end{bmatrix}=[\hat{C}_R\quad 0]\begin{bmatrix} \hat{D}_R E_w \\ D_w \end{bmatrix}.$$

$$R_0=6{\cdot}35=[C\quad 0]\begin{bmatrix} X_p & X_{pw} \\ X_{pw}^* & X_w \end{bmatrix}\begin{bmatrix} C^* \\ 0 \end{bmatrix}=[\hat{C}_R\quad 0]\begin{bmatrix} 1 & \bar{X}_{pw} \\ \bar{X}_{pw} & X_w \end{bmatrix}\begin{bmatrix} \hat{C}_R \\ 0 \end{bmatrix}$$

5. Conclusion

This paper presents a theory and an algorithm for constructing a weighted q-Markov COVER. A large class (but not the whole class) of parameterizations is found. To satisfy the existence conditions the freedom in our parameterization is an $n_q \times n_q$ positive definite matrix F_{22}. Also, a free skew symmetric matrix S is found that will not alter the matching of the first q-Markov and covariance parameters.

These results allow models to be reduced in order while preserving the first q-Markov and covariance parameters from input to output, when the input is not white noise.

Two issues need further attention. The present theory does not generate the entire class of weighted q-Markov COVERs. Secondly, the algorithm presented is quite complex. Simpler algorithms would be desirable.

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