Systems & Control Letters 12 (1989) 95–101
North-Holland

Approximation and stabilization of distributed systems by lumped systems

M. VIDYASAGAR

Department of Electrical Engineering, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

B.D.O. ANDERSON

Department of Systems Engineering, Australian National University, Canberra, Australia 2601

Received 29 August 1988
Revised 29 October 1988

Abstract: In this note we study two questions regarding linear time-invariant distributed systems, namely: (1) When can the given system be approximated by a lumped system? (2) When can the given system be stabilized by a lumped system? In each case we obtain a complete answer to the question in terms of the behaviour of the purely atomic parts of the stable numerator and denominator of the plant.

Keywords: Distributed systems; lumped controllers; stable factorizations.

1. Introduction

In this note we study two questions regarding linear time-invariant distributed systems, namely: (1) When can the given system be approximated by a lumped system? (2) When can the given system be stabilized by a lumped system? The motivation behind the second question is obvious. One would like, if possible, to use only lumped controllers from the standpoint of implementation. Often, if one can show mathematically that a given system cannot be stabilized by a lumped controller, this is a comment on the validity of the mathematical model used to describe the plant. To motivate the first question, consider the problem of discretizing the partial differential equation describing the vibrations of a string, where the input is the force applied at one end and the output is the deflection of the other end. It is clear that the system is input–output controllable over every interval $[0, t]$ provided $t \geq T$ which is the time required for a signal to propagate down the string; but if $t < T$, the system is not controllable over the interval $[0, t]$. On the other hand, if we discretize the partial differential equations describing the string, the resulting system is controllable over *arbitarily short* time intervals. How can one explain this paradox? In this note, we obtain a complete answer to each of the questions posed above. The answers are couched in terms of the behaviour of the purely atomic parts of the stable numerator and denominator of the plant.

2. Preliminaries

In this section we recall some definitions and results from the theory of input–output stability, and introduce the notation used in the note.

Recall [1] the set $\mathscr{A}$, which consists of all distributions supported on $[0, \infty)$ of the form

$$f(t) = \sum_{i=0}^{\infty} f_i \delta(t - t_i) + f_{\mathrm{na}}(t), \tag{2.1}$$

where $0 \leq t_0 < t_1 < \cdots$, $\delta(\cdot)$ denotes the unit impulse distribution, and in addition

$$\| f \|_{\mathscr{A}} := \sum_{i=0}^{\infty} | f_i | + \int_0^{\infty} | f_{\mathrm{na}}(t) | \,\mathrm{d}t < \infty. \tag{2.2}$$

If the product of two distributions in $\mathscr{A}$ is defined as their convolution, then $\mathscr{A}$ is a Banach algebra with $\delta(\cdot)$ as the identity. If $f(\cdot)$ of the form (2.1) belongs to $\mathscr{A}$, then we refer to

$$f_{\mathrm{pa}}(t) = \sum_{i=0}^{\infty} f_i \delta(t - t_i) \tag{2.3}$$

as the *purely atomic part* of $f(\cdot)$, and to f_{na} as the *non-atomic part* of $f(\cdot)$. The terminology arises from viewing $f(\cdot)$ as a measure on the half-line. Let $\mathscr{P}$ denote the set of all purely atomic measures of bounded variation, i.e. all distributions of

0167-6911/89/$3.50

the form (2.3) where the sequence of weights $\{f_i\}$ belongs to l_1; let L_1 denote, as usual, the set of measurable, absolutely integrable functions. Then $\mathscr{A}$ is the direct sum of the subalgebras $\mathscr{P}$ and L_1. Moreover, L_1 is an *ideal* in $\mathscr{A}$; i.e., if $f \in L_1$ and $g \in \mathscr{A}$, then their convolution belongs to L_1. Thus, if f, $g \in \mathscr{A}$, then $(fg)_{pa} = f_{pa} g_{pa}$.

Since the focus in this note is on approximating distributed systems by lumped ones, the latter notion should be defined precisely. We call a distribution (not necessarily in $\mathscr{A}$) *lumped* if it is Laplace-transformable, and its Laplace transform $\hat{f}(s)$ is a rational function of s. The set of all lumped distributions in $\mathscr{A}$ is denoted by $\mathscr{L}$. Since we can think of $\mathscr{A}$ as the set of all stable BIBO impulse responses, we can interpret $\mathscr{L}$ as the set of all *lumped* stable impulse responses. The question can now be asked: What is the (norm) closure of $\mathscr{L}$ in $\mathscr{A}$? Clearly every distribution in $\mathscr{L}$ is of the form

$$f(t) = f_0 \delta(t) + \sum_i \sum_j c_{ij} t^j \exp(-\lambda_i t), \tag{2.4}$$

where Re $\lambda_i > 0$ for all i and the sums are finite. Once it is realized [2] that every L_1 function can be approximated by a sum of exponentials, the answer to the question is immediate.

Fact 2.1. The closure $\bar{\mathscr{L}}$ is given by

$$\bar{\mathscr{L}} = \{ f \in \mathscr{A} : f(t) = f_0 \delta(t) + f_{na}(t) \}.$$
$$= \{ f : f(t) = f_0 \delta(t) + f_{na}(t),\ f_{na} \in L_1 \}. \tag{2.5}$$

In other words, a distribution f in $\mathscr{A}$ can be well-approximated by a lumped distribution if and only if the purely atomic part of f does not contain any delayed impulses. Note that $\bar{\mathscr{L}}$ consists precisely of those distributions in $\mathscr{A}$ whose Laplace transforms have well-defined limits at $j\infty$.

Now a word about our notation. Throughout the note, $\mathscr{M}(\mathscr{R})$ denotes the set of matrices with elements in the ring $\mathscr{R}$, of whatever order. Quantities with a '$\hat{\ }$' or '$\tilde{\ }$' denote Laplace transforms, whereas those without this symbol denote distributions.

3. Review of Bezout identities and stabilizing controllers

In this section we review the concepts of coprimeness, the parametrization of all solutions to the Bezout identity, and the parametrization of all stabilizing controllers.

Two matrices N and D in $\mathscr{M}(\mathscr{A})$ are *right-coprime* if there exist matrices X, Y in $\mathscr{M}(\mathscr{A})$ such that

$$XN + YD = \delta I \quad \text{in } \mathscr{A}, \tag{3.1a}$$

or equivalently

$$\hat{X}\hat{N} + \hat{Y}\hat{D} = I \quad \text{for all } s \in C_{+e}. \tag{3.1b}$$

For convenience, we use the phrases "N and D are right-coprime" and "$\hat{N}$ and $\hat{D}$ are right-coprime" interchangeably. Left-coprimeness is defined analogously. A pair $(\hat{N}, \hat{D})$ in $\mathscr{M}(\mathscr{A})$ is a *right-coprime factorization* (rcf) of a plant $\hat{P}$ if (i) $\hat{P} = \hat{N}\hat{D}^{-1}$, and (ii) $\hat{N}$, $\hat{D}$ are right-coprime. Again, it is sometimes convenient to say "(N, D) is an rcf of P" to mean "$(\hat{N}, \hat{D})$ is an rcf of $\hat{P}$." The symbol '$\tilde{\ }$' is used to denote quantities pertaining to left-coprime factorizations (lcf's).

Suppose two matrices $\hat{N}$ and $\hat{D}$ are right-coprime, and let $\hat{P} = \hat{N}\hat{D}^{-1}$. Then, as shown in [3, Theorem (8.1.68)], $\hat{P}$ also has an lcf $(\tilde{D}, \tilde{N})$. Since $\hat{N}$, $\hat{D}$ are right-coprime, there exist matrices $\hat{X}$, $\hat{Y}$ such that (3.1) holds. It is of interest to characterize *all* solutions to (3.1). The answer is easily deduced from [3, Lemma (4.1.32)].

Fact 3.1. With the notation of the preceding paragraph, the set of all solutions to

$$\hat{A}\hat{N} + \hat{B}\hat{D} = I \quad \text{for all } s \in C_{+e} \tag{3.2}$$

is given by

$$[\hat{A} \quad \hat{B}] = [I \quad \hat{R}] \begin{bmatrix} \hat{X} & \hat{Y} \\ \tilde{D} & -\tilde{N} \end{bmatrix}, \quad \hat{R} \in \mathscr{M}(\hat{\mathscr{A}}). \tag{3.3}$$

Fact 3.1 is the basis of the next result, which gives a simple formula for all controllers that stabilize a given plant [4,5].

Fact 3.2. Suppose a plant $\hat{P}$ has an rcf $(\hat{N}, \hat{D})$, and lcf $(\tilde{D}, \tilde{N})$, and select matrices $\hat{X}$, $\hat{Y}$, $\tilde{X}$, $\tilde{Y}$ in $\mathscr{M}(\hat{\mathscr{A}})$ such that

$$\hat{X}\hat{N} + \hat{Y}\hat{D} = I, \quad \tilde{N}\tilde{X} + \tilde{D}\tilde{Y} = I \quad \text{for all } s \in C_{+e}. \tag{3.4}$$

Then the set of all controllers that stabilize $\hat{P}$ is given by

$$\hat{S}(\hat{P}) = \{(\hat{Y} - \hat{R}\tilde{N})^{-1}(\hat{X} + \hat{R}\tilde{D}) \colon \hat{R} \in \mathcal{M}(\hat{\mathscr{A}})\}$$
$$= \{(\tilde{X} + \hat{D}\hat{R})(\tilde{Y} - \hat{N}\hat{R})^{-1} \colon \hat{R} \in \mathcal{M}(\hat{\mathscr{A}})\}. \tag{3.5}$$

In the above, it is *assumed* at the outset that the plant $\hat{P}$ has both an rcf and an lcf. But this is not a serious limitation, because if $\hat{P}$ doesn't have an rcf, then it cannot be stabilized by *any* controller that has an lcf [3, Lemma (8.3.2)]. In particular, it can neither be approximated nor stabilized by a lumped system. Hence, for the purposes of this note, it is assumed in the sequel that $\hat{P}$ has both an rcf and an lcf.

4. Approximation by lumped systems

In this section we study the problem of approximating a given distributed plant P by a lumped plant P_d for the purposes of controller design. In order to be meaningful in the context of feedback stabilization, this approximation should be in the graph topology [3, Ch. 7]. For the convenience of the reader, this topology is briefly summarized. Let P be a distribution with an rcf (N, D). Then a neighbourhood of P in the graph topology consists of all plants of the form $N_1^{-1}D_1^{-1}$ where (N_1, D_1) belongs to some ball centred at (N, D). Thus the question can be stated as follows: Given P, when is it the limit, in the graph topology, of a sequence $\{P_i\}$ of lumped plants? This is clearly equivalent to the question: When does P have a coprime factorization over the ring $\mathcal{M}(\bar{\mathscr{L}})$ and not just over $\mathcal{M}(\mathscr{A})$? Suppose (N, D) is an rcf of P over $\mathcal{M}(\mathscr{A})$. Then the set of *all* rcf's of $\hat{P}$ over $\mathcal{M}(\mathscr{A})$ is given by (NU, DU), as U ranges over all unit elements of the ring $\mathcal{M}(\mathscr{A})$ [3, p. 331]. Thus the question becomes: Given $N, D \in \mathcal{M}(\mathscr{A})$, right-coprime, when does there exist a unit matrix U in $\mathcal{M}(\mathscr{A})$ such that both NU and DU belong to $\mathcal{M}(\bar{\mathscr{L}})$? The answer is given next.

Theorem 4.1. *Suppose a plant P has dimensions $l \times m$, and has an* rcf (N, D), *where naturally $N \in \mathscr{A}^{l\times m}$ and $D \in \mathscr{A}^{m\times m}$. Then P can be approximated by a lumped plant in the graph topology if and only if there exists a constant matrix $M \in \mathbb{R}^{l\times(m+l)}$ of rank l such that*

$$MA_{\text{pa}} = 0, \quad \text{where } A = \begin{bmatrix} N \\ D \end{bmatrix}, \tag{4.1}$$

and $(\cdot)_{\text{pa}}$ denotes the purely atomic part.

Proof. 'Only if'. Suppose (A, B) is an rcf of P over $\mathcal{M}(\bar{\mathscr{L}})$. Then there is a unit matrix U such that

$$\begin{bmatrix} N \\ D \end{bmatrix} = \begin{bmatrix} S \\ T \end{bmatrix} U. \tag{4.2}$$

Since $S, T \in \mathcal{M}(\bar{\mathscr{L}})$, we have

$$\begin{bmatrix} S_{\text{pa}} \\ T_{\text{pa}} \end{bmatrix} = \begin{bmatrix} S_0 \\ T_0 \end{bmatrix} \delta(t). \tag{4.3}$$

Now, since L_1 is an ideal in $\mathscr{A}$, we see from (4.2) that

$$\begin{bmatrix} N_{\text{pa}} \\ D_{\text{pa}} \end{bmatrix} = \begin{bmatrix} S_0 \\ T_0 \end{bmatrix} U_{\text{pa}}. \tag{4.4}$$

Now select $M \in \mathbb{R}^{l\times(m+l)}$ to be of rank l and a left annihilator of the constant matrix $[S_0' \; T_0']'$. Then (4.1) holds.

'If'. Suppose (4.1) holds. Since N and D are right-coprime, there exists a matrix $B \in \mathscr{A}^{m\times(m+l)}$ such that

$$BA = \delta I_m. \tag{4.5}$$

Taking purely atomic parts (and noting that L_1 is an ideal in $\mathscr{A}$) gives

$$B_{\text{pa}} A_{\text{pa}} = \delta I_m. \tag{4.6}$$

Now, by permuting rows and columns in (4.1) if necessary, we can rewrite (4.1) as

$$[I_l \quad K] \begin{bmatrix} A_{1\text{pa}} \\ A_{2\text{pa}} \end{bmatrix} = 0, \quad \text{or} \quad A_{1\text{pa}} = -KA_{2\text{pa}}. \tag{4.7}$$

Substituting from (4.7) in (4.6) and partitioning B commensurately with A shows that

$$(-B_{1\text{pa}}K + B_{2\text{pa}})A_{2\text{pa}} = \delta I_m. \tag{4.8}$$

Hence $A_{2\text{pa}}$ is a unit of $\mathcal{M}(\mathscr{A})$. (Actually it is also a unit of $\mathcal{M}(\mathscr{P})$, though we don't need this fact.) Let $U = A_{2\text{pa}}^{-1}$. Then

$$(AU)_{\text{pa}} = \begin{bmatrix} -K \\ I \end{bmatrix} \delta(\cdot). \tag{4.9}$$

This shows that $AU \in \mathscr{M}(\bar{\mathscr{L}})$, i.e. that AU can be approximated arbitrarily closely by a lumped system. Since U is a unit, the matrix AU also gives an rcf of the plant P. Hence P is the limit, in the graph topology, of a sequence of lumped plants.

If we add a mild assumption regarding the plant P, then the conditions of Theorem 4.1 can be made much more transparent and natural.

Theorem 4.2. *Suppose a plant P has an rcf (N, D) with $D_0 \neq 0$, where D_0 denotes the strength of the impulse at $t = 0$ of the denominator distribution D. Then P can be approximated by a lumped system if and only if the impulse response of P does not contain any delayed impulses.*

Remarks. The hypothesis regarding D_0 can be easily justified in terms of requiring P to be causal.

Proof. In view of Theorem 4.1, it is only necessary to show that the condition (4.1) is equivalent to the statement that the impulse response of P does not contain any delayed impulses. Accordingly, suppose first that the condition (4.1) holds. Then the assumption on D_0 shows that (4.7) is satisfied without permuting the rows of A. Now D_{pa} has an inverse in the extended space $\mathscr{P}_e$, i.e. the set of purely atomic measures which have a bounded variation over every finite interval. Moreover, by (4.7), it follows that $N_{pa}D_{pa}^{-1} = -K\delta(\cdot)$. Next, note that D has an inverse in the extended space $\mathscr{A}_e$, and that the extended space L_{1e} is an ideal of this algebra (which is not, however, a normed algebra). Hence the purely atomic part of $P = ND^{-1}$ is just $-K\delta(\cdot)$, i.e., P does not contain any delayed impulses. The reverse implication is proved by simply reversing the above reasoning.

In general, the conditions of Theorem 4.1 can be verified by examining only the purely atomic part of the part impulse response, call it P_{pa}. It is easy to see that if (N, D) is an rcf of P and if D has an inverse in the extended space $\mathscr{A}_e$ of measures with locally bounded variation, then (N_{pa}, D_{pa}) is an rcf of P_{pa}. Hence, in order to apply Theorem 4.1 to a given system, it is only necessary to find a coprime factorization of the purely atomic part P_{pa}.

As an illustration of Theorem 4.1, consider the problem mentioned in the introduction, namely approximating a vibrating string by a finite-dimensional model. The control signal is applied at the near end of a string, while the output is taken as the position at the far end. To put it in more electrical terms, suppose the system is a uniform LC transmission line, the input is the voltage at one end, and the output is the voltage at the other end. Let us suppose that the time of propagation along the transmission line is T seconds. Then it is clear that, in order to steer the output of the system to a specified value, at least T seconds must elapse from the time when the control signal is initially applied. To put it another way, the system is controllable over the interval $[0, T]$, but not controllable over any shorter interval. On the other hand, suppose we approximate the partial differential equation describing the system (i.e. the wave equation) by a finite difference model discretized in space. Then the resulting system is linear, time-invariant, finite-dimensional and controllable. Thus, it is controllable over *arbitrarily short* time intervals. How can one explain this paradox?

In [7] Anderson and Parks explain this in terms of a notion called spatial aliasing, which results in the spatially quantized model exhibiting the same sort of loss of information that results from temporal sampling with the resulting aliasing of frequencies. But a more direct explanation is available in terms of Theorem 4.1. The transfer function of the system described above is

$$\hat{p}(s) = \frac{1}{\cosh s} = \frac{2\exp(-s)}{1+\exp(-2s)}. \tag{4.10}$$

Define

$$\hat{n}(s) = 2\exp(-s), \quad \hat{d}(s) = 1 + \exp(-2s). \tag{4.11}$$

These are coprime since

$$(-0.5\exp(-s))\hat{n}(s) + \hat{d}(s) = 1. \tag{4.12}$$

Now, using Theorem 4.1, we can show that $\hat{p}$ cannot be approximated in the sense of the graph topology by any lumped system. From (4.8) we see that $n_{pa} = n$, $d_{pa} = d$; moreover, one is not a *constant* multiple of the other. Hence no finite-dimensional approximation of $\hat{p}$ is possible.

5. Stabilization by lumped systems

In this section we consider the problem of stabilizing a given distributed system by a lumped system. Earlier work includes [8,9].

Theorem 5.1. *If P can be approximated by a lumped system, then it can also be stabilized by a lumped system.*

The proof of this theorem follows from the following result, which might be of independent interest.

Lemma 5.2. *Suppose* $N, D \in \mathcal{M}(\bar{\mathcal{L}})$ *are right-coprime in* $\mathcal{M}(\mathcal{A})$. *Then they are also right-coprime in* $\mathcal{M}(\bar{\mathcal{L}})$.

Remarks. The lemma states that, if $N, D \in \mathcal{M}(\bar{\mathcal{L}})$ and if it is possible to find X, Y in $\mathcal{M}(\mathcal{A})$ satisfying the identity (3.1), then in fact it is possible to choose such X, Y in the smaller set $\mathcal{M}(\bar{\mathcal{L}})$.

Proof of Lemma 5.2. Fact 2.1 shows that $\bar{\mathcal{L}}$ is just the direct sum of the complex field and the algebra L_1; i.e., $\bar{\mathcal{L}}$ is just L_1 with the identity added. Hence its maximal ideal space $\mathcal{I}$ is the one-point compactification of the closed RHP. In other words, $\mathcal{I} = C_{+e}$, where the complements of open bounded subsets of the RHP are defined to be compact neighborhoods of infinity. Hence two matrices N and D in $\mathcal{M}(\bar{\mathcal{L}})$ are right-coprime if and only if [3, Corollary (8.1.20)]

$$\operatorname{rank}\begin{bmatrix}\hat{N}(s)\\ \hat{D}(s)\end{bmatrix} = m \quad \text{for all } s \in C_{+e}, \tag{5.1}$$

where m is the number of columns of N and D. But this is implied by the coprimeness of N and D in $\mathcal{M}(\mathcal{A})$ [6, Theorem 2.1; 3, Lemma (8.1.34)].

Proof of Theorem 5.1. Suppose P can be approximated by a lumped plant. Then there exists an rcf (N, D) of P in $\mathcal{M}(\bar{\mathcal{L}})$. By Lemma 5.1, there exist X, Y in $\mathcal{M}(\bar{\mathcal{L}})$ (and not just in $\mathcal{M}(\mathcal{A})$) such that (3.1) is satisfied. Now the controller $C = Y^{-1}X$ stabilizes the plant P. Moreover, C is also the limit of a sequence of lumped plants, and since feedback stability is a robust property in the graph topology, it follows that P can be stabilized by a lumped controller.

The converse of Theorem 5.1 is false. Consider the plant

$$\hat{p}(s) = \tanh s = \frac{1-\exp(-2s)}{1+\exp(-2s)}. \tag{5.2}$$

Let $\hat{n}$, $\hat{d}$ denote respectively half the numerator and half the denominator in (5.2). Then clearly n and d are coprime, since $\hat{n} + \hat{d} = 1$ for all s, so that the identity (3.1) is satisfied with $\hat{x} = \hat{y} = 1$. Hence the plant p is stabilized by unit negative feedback $(= \hat{x}/\hat{y})$. However, p cannot be approximated by a lumped plant. To see this, note that n and d are already purely atomic, and they are not multiples of each other; now apply Theorem 4.1. The general solution to this problem is given next.

Theorem 5.3. *Suppose* $\hat{P}$ *has an* rcf $(\hat{N}, \hat{D})$, *and an* lcf $(\tilde{D}, \tilde{N})$. *Then* $\hat{P}$ *can be stabilized by a lumped controller if and only if there exists a constant matrix K such that* KA_{pa} *is a unit matrix, where A is defined in* (4.1).

Remark. Subject to a few technical assumptions, the condition of the theorem can be stated more transparently as: the system $\hat{P}_{pa} = \hat{N}_{pa}\hat{D}_{pa}^{-1}$ can be stabilized by a constant controller. Moreover, if all the delays in N_{pa}, D_{pa} are commensurate, then by mapping the RHP of the s-plane into the unit circle and using decision methods [10], it is conceptually straight-forward (though possibly computationally complex) to ascertain whether or not there exists a matrix K satisfying the hypotheses of the theorem.

Proof. 'Only if'. Suppose $\hat{C}$ is a lumped controller that stabilizes $\hat{P}$, and find a factorization $(\tilde{D}_c, \tilde{N}_c)$ for $\hat{C}$ over the set of stable *rational* matrices. Define

$$\hat{B} = [\tilde{N}_c \quad \tilde{D}_c]. \tag{5.3}$$

Then, by [3, Theorem (5.1.25)], the matrix BA is a unit of $\mathcal{M}(\mathcal{A})$. Taking the purely atomic part gives $B_{pa}A_{pa}$ = unit of $\mathcal{M}(\mathcal{A})$. The proof is completed by noting that B_{pa} is in fact a constant matrix.

'If'. Suppose K is a constant matrix such that $U = KA_{pa}$ is a unit. Then

$$KA_{pa}U^{-1} = I\delta(\cdot). \tag{5.4}$$

Now, since N and D are right-coprime, so are NU^{-1} and DU^{-1}. Hence there exists a matrix B, not necessarily in $\mathscr{M}(\bar{L})$, such that

$$BAU^{-1} = \delta I. \tag{5.5}$$

Taking purely atomic parts shows that

$$B_{\text{pa}} A_{\text{pa}} U^{-1} = \delta I. \tag{5.6}$$

(Note that U is already purely atomic.) Now

$$[\tilde{D} \quad -\tilde{N}]\hat{A} = 0. \tag{5.7}$$

Taking purely atomic parts again shows that

$$S_{\text{pa}} A_{\text{pa}} = 0, \tag{5.8}$$

where S is the inverse Laplace transform of $[\tilde{D} \; -\tilde{N}]$. In other words, S_{pa} is a left annihilator of A_{pa}. By Fact 3.1, there exists a matrix $R \in \mathscr{M}(\mathscr{P})$ such that

$$K = [I \quad R]\begin{bmatrix} B \\ S \end{bmatrix}_{\text{pa}}. \tag{5.9}$$

Now define

$$M = [I \quad R]\begin{bmatrix} B \\ S \end{bmatrix}, \tag{5.10}$$

and partition $\hat{M}$ as $[\tilde{N}_c \quad \tilde{D}_c]$. Then, by Fact 3.2, the controller $\hat{C} = \tilde{D}_c^{-1}\tilde{N}_c$ stabilizes the plant $\hat{P}$. Moreover, from (5.9), we see that $M_{\text{pa}} = K\delta(\cdot)$, which is an impulse supported only at $t = 0$. Hence, by Fact 2.1, $M \in \mathscr{M}(\bar{\mathscr{L}})$. This means that the controller C is the limit, in the graph topology, of a sequence of lumped controllers. However, since closed-loop stability is robust in the graph topology, it follows that $\hat{P}$ can be stabilized by a lumped controller.

As an application of Theorem 5.3, consider again the stabilization of the vibrating string. As shown in [11], the plant $\hat{p}$ of (4.9) cannot be stabilized by any constant gain, but the proof based on Theorem 5.3 is much shorter. Let $z = \exp(-s)$; then from (4.10) we have $\hat{n} = 2z$, $\hat{d} = 1 + z^2$. Now $k_1 n + k_2 d$ is a unit of $\mathscr{A}$ if and only if $2k_1 z + k_2(1 + z^2)$ has no zeros in the closed unit disk. If $k_2 = 0$, this is clearly not so, hence we may suppose that $k_2 \neq 0$, and let $\alpha = k_1/k_2$. The question now is whether the transfer function $2z/(1 + z^2)$ can be stabilized by a constant gain α. Elementary root locus arguments show that this is not possible.

Now consider the plant

$$\hat{q} = \frac{2\exp(-s) + \hat{a}_{\text{na}}(s)}{1 + \exp(-2s) + \hat{b}_{\text{na}}(s)}, \tag{5.11}$$

where a_{na}, b_{na} are non-atomic measures in $\mathscr{A}$. Let $\hat{a}$, $\hat{b}$ denote respectively the numerator and denominator in (5.11). Then there are two possibilities to consider:

(1) The ideal in $\mathscr{A}$ generated by a, b is not principal. In this case, from [3, Lemma (8.1.3)], it follows that $\hat{p}$ does not have a coprime factorization over $\mathscr{A}$, which in turn implies that $\hat{p}$ cannot be stabilized by any controller with a coprime factorization over $\mathscr{A}$, and that, in particular, no lumped controller can stabilize $\hat{p}$.

(2) The ideal in $\mathscr{A}$ generated by a and b is principal. Let m be the generator of this ideal, and let $n = a/m$, $d = b/m$. Since the purely atomic parts of a and b are already coprime, it is clear that $m_{pa} = \delta(\cdot)$, or some multiple thereof. Let us suppose without loss of generality that

$$\hat{n}_{\text{pa}} = 2\exp(-s), \quad \hat{d}_{\text{pa}} = 1 + \exp(-2s). \tag{5.12}$$

Then by [11] and earlier remarks, there is no constant matrix $K \in \mathbb{R}^{1\times 2}$ such that $k_1 n + k_2 d$ is a unit of $\mathscr{P}$. This shows that, once again, $\hat{q}$ cannot be stabilized by a lumped controller.

References

[1] B.D.O. Anderson, N.K. Bose and E.I. Jury, Output feedback stabilization and related problems – Solution via decision methods, *IEEE Trans. Automat. Control* **20** (1975) 53–66.

[2] B.D.O. Anderson and P.C. Parks, Lumped approximations of distributed systems and controllability questions, *IEE Proc. Part D* **132** (1985) 89–94.

[3] F.M Callier and C.A. Desoer, An algebra of transfer functions of distributed linear time-invariant systems, *IEEE Trans. Circuits and Systems* **25** (1978) 651–662.

[4] C.A. Desoer, R.-W. Liu, J. Murray and R. Saeks, Feedback system design: The fractional representation approach to analysis and synthesis, *IEEE Trans. Automat. Control* **25** (1980) 399–412.

[5] C.A. Desoer and M. Vidyasagar, *Feedback Systems: Input–output Properties* (Academic Press, New York, 1975).

[6] C.A. Jacobson and C.N. Nett, Linear state-space systems in infinite-dimensional space: The role and characterization of joint stabilizability/detectability, *IEEE Trans. Automat. Control* **33** (1988) 541–549.

[7] E.W. Kamen, P.P. Khargonekar and A. Tannenbaum, Stabilization of time-delay systems using finite-dimen-

sional compensators, *IEEE Trans. Automat. Control* **30** (1985) 75–78.

[8] D.W. Kammler, Approximation with sums of exponentials in $L_p[0, \infty)$, *J. Approximation Theory* **16** (1976) 384–408.

[9] M. Vidyasagar, Input–output stability of a broad class of linear time-invariant multivariable feedback systems, *SIAM J. Control* **10** (1972) 203–209.

[10] M. Vidyasagar, *Control System Synthesis: A Factorization Aproach* (M.I.T. Press, Cambridge, MA, 1985).

[11] D.C. Youla, H.A. Jabr and J.J. Bongiorno, Jr., Modern Wiener–Hopf design of optimal controllers, Part II: The multivariable case, *IEEE Trans. Automat. Control* **21** (1976) 319–338.