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Almost disturbance decoupling with stabilization by measurement feedback *

A. LINNEMANN
University of Kassel, Department of Electrical Engineering, D-3500 Kassel, West Germany

I. POSTLETHWAITE
University of Leicester, Department of Engineering, Leicester, LE1 7RH, U.K.

B.D.O. ANDERSON
Australian National University, Research School of Physical Sciences, Department of Systems Engineering, Canberra, ACT 2601, Australia

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Abstract: Plants with two vector inputs and two vector outputs described by four rational transfer function matrices are considered. Necessary and sufficient conditions are given for the existence of a linear time invariant finite-dimensional controller which, when connected around one channel of the plant, leads to an internally stable closed loop system such that the frequency response of the remaining channel becomes arbitrarily small, as measured by the infinity-norm.

Keywords: Almost disturbance decoupling; disturbance rejection; dynamic output feedback; H-infinity optimization; system zeros.

1. Introduction

Consider the feedback control system shown in Figure 1, where u is the control input, d is the disturbance, z is the output to be controlled and y is the measured output. All signals are vector valued and the plant and the controller are linear time-invariant finite-dimensional continuous-time systems.

This paper gives necessary and sufficient conditions on the plant which guarantee the existence of a stabilizing controller such that the maximum norm of the frequency response matrix form d to z is arbitrarily small. In the notation of [1,2] we are dealing with an 'almost disturbance decoupling problem with stabilization by measurement feedback'.

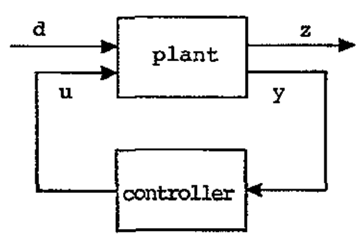

Fig. 1.

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Almost disturbance decoupling ideas have found application in various fields, such as robust control [3], decentralized control [4,5] noninteracting control [6], and reduced-order controller synthesis [7].

Variations and special cases of the above problem have already been solved: When there is no requirement on closed-loop stability, necessary and sufficient conditions are given in [2]. If the transfer function matrix from d to z is required to be identically zero, the necessary and sufficient conditions in, e.g., [8–12] can be used. For a static state feedback controller, sufficient solvability conditions are given in [1,3]. Sufficient conditions for the general output feedback problem are given in [4,13,14,7]. Necessary and sufficient conditions (not previously known) for the general problem, are the topic of this paper. A referee of this paper has informed us that a very similar but not identical problem is also treated in [15,16].

Until now, the almost disturbance decoupling problem has been analysed mainly in state space using almost invariant subspaces introduced in [1]. In contrast, we will take a purely frequency domain approach using some of the ideas which have already proven useful in H_∞ optimization [17,18]. Actually, the above disturbance decoupling problem can be viewed as a special H_∞ optimization problem. However, the usual assumptions which guarantee the existence of an optimal controller [18] are not satisfied so that results from H_∞ theory cannot be directly applied.

The paper is organized as follows. In Section 2, the problem is formally defined and simplified. Section 3 brings together some tools from H_∞ optimization which are required in the sequel. Necessary and sufficient solvability conditions for the almost disturbance decoupling problem are given in sections 4 and 5 for the scalar and multivariable cases, respectively. In Section 6, a method for the construction of controllers achieving almost disturbance decoupling is given along with a bound on the controller order. Section 7 contains two examples.

2. The problem

Consider a system given by

$$\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} d(s) \\ u(s) \end{bmatrix}, \tag{1}$$

where, for i, $j=1, 2$, $G_{ij}(s)$ is an $l_i \times m_j$ proper real-rational transfer function matrix.

Suppose the controller has the form

$$u(s) = K(s)y(s), \tag{2}$$

where $K(s)$ is an $m_2 \times l_2$ proper real-rational transfer function matrix. Then the closed-loop transfer function matrix from d to z is easily computed to be

$$L(s) = G_{11}(s) + G_{12}(s)K(s)(I - G_{22}(s)K(s))^{-1}G_{21}(s). \tag{3}$$

Obviously, the inverse of $I - G_{22}K$ in (3) has to exist. Therefore it is assumed that G_{22} is strictly proper, a condition which is satisfied in most applications.

The following version of the almost disturbance decoupling problem with internal stability by measurement feedback is analysed in this paper:

(P) For all $\varepsilon > 0$ find a stabilizing controller of the form (2) for (1) such that

$$\| L(j\omega) \| \leq \varepsilon$$

for all $\omega \geq 0$.

Throughout this paper, $\|\cdot\|$ denotes a consistent (i.e. $\| A \cdot B \| \leq \| A \| \cdot \| B \|$) matrix norm.

A necessary condition for solvability of (P) is stabilizability of (1) by a controller of the form (2). Necessary and sufficient conditions for stabilizability can be found, for instance, in [18]. If $K_0(s)$ is a stabilizing controller matrix and a preliminary control $u = K_0 y + u_0$ is employed, then

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} G_{11}^0 & G_{12}^0 \\ G_{21}^0 & G_{22}^0 \end{bmatrix} \begin{bmatrix} d \\ u_0 \end{bmatrix}, \tag{4}$$

where $G_{ij}^0(s)$ is a stable proper real-rational transfer function matrix, i, $j = 1, 2$, and $G_{22}^0(s)$ is strictly proper. Moreover, (P) for (1) is solvable if and only if it is solvable for (4). Therefore there is no loss of generality in assuming that (1) is already stable. This will be done for the remainder of this paper.

3. Preliminaries

In this section some relevant tools from the theory of H_∞ optimization [17,18] will be brought together. Consider the partition of the complex plane into

$$\mathbf{C}_+ = \{s \in \mathbf{C}: \mathrm{Re}(s) > 0\}, \qquad \mathbf{C}_0 = \{\mathrm{j}\omega : \omega \in \mathbf{R}\}, \qquad \mathbf{C}_- = \{s \in \mathbf{C}: \mathrm{Re}(s) < 0\},$$

and let S be the ring of proper real-rational functions with no poles in $\mathbf{C}_+ \cup \mathbf{C}_0$. Moreover, let $S^{m \times l}$ be the set of $m \times l$ matrices with entries in S. Since S is a principal ideal domain, each matrix $M \in S^{m \times l}$ has a Smith-form; see [17]. This implies the following lemma which will be required in Section 5.

Lemma 3.1. *Let $M \in S^{m \times l}$. Then there exist matrices $U \in S^{m \times m}$, $V \in S^{l \times l}$ such that $U^{-1} \in S^{m \times m}$, $V^{-1} \in S^{l \times l}$,*

$$UMV = \begin{bmatrix} \mathrm{diag}(\sigma_1, \dots, \sigma_r) & 0 \\ 0 & 0 \end{bmatrix},$$

and $\sigma_1 \cdot \sigma_2 \cdot \cdots \sigma_i$ is a greatest common divisor of all $i \times i$ minors of M.

The following result is taken from [18].

Lemma 3.2. *Let the system* (1) *be stable (i.e. the transfer function matrix has entries in S). Then the set of all controllers of the form* (2) *which maintain stability is given by*

$$\{K = -Q(I - G_{22}Q)^{-1} = -(I - QG_{22})^{-1}Q : Q \in S^{m_2 \times l_2}\}. \tag{5}$$

With K as above, the closed-loop transfer function matrix from d to z is given by

$$L(s) = G_{11} - G_{12}QG_{21}. \tag{6}$$

Hence (P) is solvable if and only if

$$\inf\{\|L\|_\infty : Q \in S^{m_2 \times l_2}\} = 0 \tag{7}$$

where

$$\|L\|_\infty := \sup\{\|L(\mathrm{j}\omega)\| : \omega \geq 0\}.$$

Note that solvability of (P) does not depend on the matrix norm chosen, because in finite dimensional spaces all norms are equivalent.

4. The scalar case

In this section we study the special case $m_1 = m_2 = l_1 = l_2 = 1$, so that G_{11}, G_{12}, G_{21}, G_{22} and Q are elements of S. To illustrate the idea, we consider exact disturbance decoupling first. If

$$L = G_{11} - G_{12} Q G_{21} \equiv 0,$$

then

$$Q = G_{11}(G_{12}G_{21})^{-1}.$$

Since Q is constrained to be in S we have the following result, where $d_*(\cdot)$ denotes the relative degree.

Proposition 4.1. *There exists a stabilizing controller of the form* (2) *such that* $L(s) \equiv 0$ *if and only if the following two conditions are both satisfied:*

(i) *If* $\mu \in \mathbf{C}_+ \cup \mathbf{C}_0$ *is a zero of* $G_{12}G_{21}$ *with multiplicity* m, *then* μ *is also a zero of* G_{11} *with multiplicity at least* m.

(ii) $d_*(G_{12}) + d_*(G_{21}) \le d_*(G_{11})$.

Note that (ii) means that the infinite zeros of $G_{12}G_{21}$ are also zeros of G_{11}. Thus, for exact disturbance decoupling, infinite zeros play the same role as zeros in the closed right half plane. We will see (Theorem 4.1) that, for almost disturbance decoupling, infinite zeros play the same role as zeros on the imaginary axis.

For the solution of (P), a function $Q \notin S$ can be approximated by a sequence of functions in S. Thus the conditions on (finite and infinite) zeros on the imaginary axis can be relaxed, as shown by the following result.

Theorem 4.1. *Let* G_{11}, G_{12}, G_{21}, G_{22} *be elements of* S. *Then* (P) *is solvable if and only if the following conditions are all satisfied.*

(i) *If* $\mu \in \mathbf{C}_+$ *is a zero of* $G_{12}G_{21}$ *with multiplicity* m, *then* μ *is also a zero of* G_{11} *with multiplicity at least* m.

(ii) *If* $G_{12}(j\omega_0)G_{21}(j\omega_0) = 0$ *then also* $G_{11}(j\omega_0) = 0$ (*i.e. a zero of* $G_{12}G_{21}$ *in* $\mathbf{C}_0$ *is also a zero of* G_{11}, *but the multiplicities need not coincide*).

(iii) *If* $d_*(G_{12}G_{21}) > 0$ *then also* $d_*(G_{11}) > 0$ (*i.e. an infinite zero of* $G_{12}G_{21}$ *is also an infinite zero of* G_{11}, *but the multiplicities need not coincide*).

Proof. *Sufficiency*: Let $G_{12}G_{21} = (N_+(s)N_0(s)N_-(s))/D(s)$ be a coprime polynomial factorization such that the zeros of N_+, N_0, N_- are contained in $\mathbf{C}_+$, $\mathbf{C}_0$, $\mathbf{C}_-$, respectively, and N_0 is monic. For every $\varepsilon > 0$, choose a monic polynomial $N_\varepsilon(s)$ whose zeros are in $\mathbf{C}_-$ and convergence to those of $N_0(s)$, as $\varepsilon \to 0$. Define

$$Q_\varepsilon(s) = G_{11}(s)\frac{D(s)}{N_+(s)N_\varepsilon(s)N_-(s)}\frac{1}{(1+s\varepsilon)^{\gamma}},$$

where

$$\gamma = \max\{0,\ d_*(G_{12}G_{21}) - d_*(G_{11})\}.$$

By condition (i), $Q_\varepsilon(s) \in S$ for $\varepsilon > 0$. Using the stabilizing controller defined by Q_ε, the closed-loop transfer function becomes

$$L_\varepsilon(s) = G_{11}(s) - G_{12}(s)G_{21}(s)Q_\varepsilon(s) = G_{11}(s)R_\varepsilon(s),$$

where

$$R_\varepsilon(s) := = 1 - \frac{N_0(s)}{N_\varepsilon(s)(1+s\varepsilon)^{\gamma}}.$$

We will show that $|L_\varepsilon(\mathrm{j}\omega)|$ uniformly converges to zero, i.e. for each $\delta > 0$ there exists $\varepsilon_0 > 0$ such that

$$|L_\varepsilon(\mathrm{j}\omega)| < \delta \tag{8}$$

for all $\varepsilon < \varepsilon_0$ and $\omega \in \mathbf{R}$. For this we use boundedness of $G_{11}(\mathrm{j}\omega)$ and $R_\varepsilon(\mathrm{j}\omega)$: there exist $K_1 > 0$, $K_2 > 0$, and $\varepsilon_2 > 0$ such that

$$|G_{11}(\mathrm{j}\omega)| \le K_1, \qquad |R_\varepsilon(\mathrm{j}\omega)| \le K_2 \tag{9}$$

for all $\omega \in \mathbf{R}$ and $0 < \varepsilon \le \varepsilon_2$.

Assume $d_*(G_{12}G_{21}) > 0$ so that $d_*(G_{11}) > 0$. Then there exists $\omega_\infty > 0$ such that

$$|G_{11}(\mathrm{j}\omega)| < \delta / K_2 \tag{10}$$

for $\omega \in \Omega_\infty := \{\omega : |\omega| > \omega_\infty\}$. Moreover, for each zero ω_i of $N_0(\mathrm{j}\omega)$ there exists an open neighbourhood Ω_i of ω_i such that

$$|G_{11}(\mathrm{j}\omega)| < \delta / K_2 \tag{11}$$

for $\omega \in \Omega_i$. Now let Ω be the union of all the sets Ω_i (including Ω_∞) and $\Omega^{\mathrm{c}} := \mathbf{R} \setminus \Omega$. Since Ω^{c} is compact and does not contain any zeros of $N_0(\mathrm{j}\omega)$, $|R_\varepsilon(\mathrm{j}\omega)|$ converges to zero uniformly in Ω^{c}. Hence there exists $\varepsilon_1 > 0$ such that

$$|R_\varepsilon(\mathrm{j}\omega)| \le \delta / K_1 \tag{12}$$

for all $\omega \in \Omega^{\mathrm{c}}$ and $\varepsilon < \varepsilon_1$. The relations (9)–(12) imply (8), as desired.

Now assume $d_*(G_{12}G_{21}) = 0$ so that $\gamma = 0$. It is straightforward to show that there exists $\omega_\infty > 0$ and $\varepsilon_\infty > 0$ such that

$$|R_\varepsilon(\mathrm{j}\omega)| \le \delta / K_1 \tag{13}$$

for all $\omega \in \Omega_\infty := = \{\omega : |\omega| > \omega_\infty\}$ and $\varepsilon < \varepsilon_\infty$. Uniform convergence of $L_\varepsilon(\mathrm{j}\omega)$ now follows from (9), (13), (11) and (12).

Necessity: Suppose that (i) is not satisfied. Let μ be a $\mathbf{C}_+$-zero of $G_{12}G_{21}$ of multiplicity $m > 0$. Also assume that μ is a zero of G_{11} of multiplicity k, $0 \le k < m$. Consider the all-pass

$$A(s) = \begin{cases} \left(\dfrac{s+\mu}{s-\mu}\right)^k & \text{if } \mu \text{ is real,} \\[2ex] \left(\dfrac{(s+\mu)(s+\mu^*)}{(s-\mu)(s-\mu^*)}\right)^k & \text{if } \mu \text{ is complex.} \end{cases}$$

Then, for all $Q \in S$, the rational function $A(s)L(s)$ is analytic in the closed right half plane. Moreover, for $s = \mu$ there holds

$$A(s)G_{11}(s)|_{s=\mu} \ne 0, \qquad A(s)G_{12}(s)G_{21}(s)|_{s=\mu} = 0.$$

Thus $A(s)L(s)|_{s=\mu}$ is nonzero and independent of Q. By the maximum modulus principle it follows that

$$\sup_{\omega \in \mathbf{R}} |L(\mathrm{j}\omega)| = \sup_{\omega \in \mathbf{R}} |A(\mathrm{j}\omega)L(\mathrm{j}\omega)| \ge |A(s)L(s)||_{s=\mu},$$

and thus (P) is not solvable.

Suppose that (ii) is not satisfied and let

$$G_{12}(\mathrm{j}\omega_0)G_{21}(\mathrm{j}\omega_0) = 0, \qquad G_{11}(\mathrm{j}\omega_0) \ne 0.$$

Then

$$L(\mathrm{j}\omega_0) = G_{11}(\mathrm{j}\omega_0) \ne 0$$

for all $Q \in S$, and (P) is not solvable.

Suppose that (iii) is not satisfied. Then, for $\omega \to \infty$, there holds

$$|L(j\omega)| \to \|G_{11}(s)\|_{s=\infty} \neq 0$$

for all $Q \in S$, i.e. (P) is not solvable. □

As mentioned in Section 2 there is no restriction in assuming (1) stable. For unstable plants a stabilizing controller can be constructed first, and Theorem 4.1 can then be applied to the stabilized system.

5. The multivariable case

In this section we consider the general case where m_1, m_2, l_1, l_2 are arbitrary positive integers, and G_{11}, G_{12}, G_{21}, G_{22} have entries in S. The multivariable case will be reduced to the scalar case using the Smith form of Lemma 3.1. Consider the Smith forms of G_{12} and G_{21}:

$$U_1 G_{12} V_1 = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \Sigma_1 = \operatorname{diag}(\sigma_{1,1}, \sigma_{1,2}, \ldots, \sigma_{1,r_1}),$$

$$U_2 G_{21} V_2 = \begin{bmatrix} \Sigma_2 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \Sigma_2 = \operatorname{diag}(\sigma_{2,1}, \sigma_{2,2}, \ldots, \sigma_{2,r_2}),$$

where U_1, U_1^{-1}, V_1, V_1^{-1}, U_2, U_2^{-1}, V_2, V_2^{-1} are matrices with entries in S. For the closed-loop transfer function from d to z we then have

$$L = U_1^{-1} \begin{bmatrix} T_{11} - \Sigma_1 Q_{11} \Sigma_2 & T_{12} \\ T_{21} & T_{22} \end{bmatrix} V_2^{-1},$$

where

$$U_1 G_{11} V_2 = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}, \qquad V_1^{-1} Q U_2^{-1} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix},$$

and T_{11}, Q_{11} each have dimensions $r_1 \times r_2$.

Since the transformation matrices are invertible over S and the matrix norm is assumed to be consistent, (P) is solvable if and only if $T_{12} = 0$, $T_{21} = 0$, $T_{22} = 0$ and

$$\inf\{ \|T_{11} - \Sigma_1 Q_{11} \Sigma_2\|_\infty : Q_{11} \in S^{r_1 \times r_2} \} = 0.$$

Now let t_{ij} be the (i, j)-th entry of T_{11}. With the above notation we can state:

Theorem 5.1. *Let G_{11}, G_{12}, G_{21}, G_{22} have entries in S. Then* (P) *is solvable if and only if*

(i) $T_{12} = 0$, $T_{21} = 0$, $T_{22} = 0$,

and the following conditions are satisfied for all $i = 1, 2, \ldots, r_1$ and $j = 1, 2, \ldots, r_2$.

(ii) *If $\mu \in \mathbb{C}_+$ is a zero of $\sigma_{1,i} \cdot \sigma_{2,j}$ with multiplicity m, then μ is also a zero of t_{ij} with multiplicity at least m.*

(iii) *If $\sigma_{1,i}(j\omega_0) \cdot \sigma_{2,j}(j\omega_0) = 0$ then also $t_{ij}(j\omega_0) = 0$ (i.e. a zero of $\sigma_{1i} \cdot \sigma_{2,j}$ in $\mathbb{C}_0$ is also a zero of t_{ij}, but the multiplicities need not coincide).*

(iv) *If $d_*(\sigma_{1,j} \cdot \sigma_{2,j}) > 0$ then also $d_*(t_{ij}) > 0$ (i.e. an infinite zero of $\sigma_{1,i} \cdot \sigma_{2,j}$ is also a zero of t_{ij}, but the multiplicities need not coincide).*

Conditions (ii), (iii) and (iv) amount to the scalar problem conditions of Theorem 4.1 applied to each entry of $T_{11} - \Sigma_1 Q_{11} \Sigma_2$. A similar approach has been taken in [11] for exact disturbance decoupling problems.

Proof. *Sufficiency*: Define, for $k=1, 2$ and $\varepsilon>0$, the matrices

$$\Sigma_{k,\varepsilon}=\operatorname{diag}\{\sigma_{k,1,\varepsilon}, \sigma_{k,2,\varepsilon},\dots,\sigma_{k,r_k,\varepsilon}\},$$

where

$$\sigma_{k,i,\varepsilon}:=\frac{N_+(\sigma_{k,i})N_-(\sigma_{k,i})\nu_{k,i,\varepsilon}(1+s\varepsilon)^{\gamma_{k,i}}}{D(\sigma_{k,i})}$$

and $D(\sigma_{k,i})$ is the denominator polynomial of $\sigma_{k,i}$, $N_-(\sigma_{k,i})$ is that part of the numerator polynomial of $\sigma_{k,i}$ with zeros only in $\mathbf{C}_-$, $N_+(\sigma_{k,i})$ is that part of the numerator polynomial of $\sigma_{k,i}$ with zeros only in $\mathbf{C}_+$, $\nu_{k,i,\varepsilon}$ is a stable monic polynomial approximating $N_0(\sigma_{k,i})$ which is that part of the numerator polynomial of $\sigma_{k,i}$ with zeros only in $\mathbf{C}_0$, and $\gamma_{k,i}$ is such that $d_*(\sigma_{k,i,\varepsilon})=0$. Let

$$Q_\varepsilon=\Sigma_{1,\varepsilon}^{-1}T_{11}\Sigma_{2,\varepsilon}^{-1}.$$

Then Q_ε has entries in S and the (i, j)-th entry of

$$L_\varepsilon=T_{11}-\Sigma_1 Q_\varepsilon\Sigma_2$$

is given by

$$l_{ij,\varepsilon}=t_{ij}-\sigma_{1,i}\sigma_{1,i,\varepsilon}^{-1}t_{ij}\sigma_{2,j}\sigma_{2,j,\varepsilon}^{-1}=t_{ij}\left(1-\frac{N_0(\sigma_{1,i})N_0(\sigma_{2,j})(1+s\varepsilon)^{\gamma_{1,i}}(1+s\varepsilon)^{\gamma_{2,j}}}{\nu_{1,i,\varepsilon}\nu_{2,j,\varepsilon}}\right)$$

As in the scalar case, it follows that $\|l_{ij,\varepsilon}\|_\infty$ converges to zero, as $\varepsilon\to 0$. Now, since matrix norms are continuous, it follows that $\|L_\varepsilon\|_\infty$ converges to zero, as $\varepsilon\to 0$.

Necessity: Suppose that one of the conditions (ii), (iii) or (iv) is not satisfied for some i_0, j_0. Then, as in the scalar case it follows that the infinity-norm of the (i_0, j_0)-th entry of $T_{11}-\Sigma_1 Q_{11}\Sigma_2$ cannot be made arbitrarily small. □

6. A sequence of controllers solving (P)

In this section it will be assumed that (P) is solvable for a possibly unstable plant (1). The proofs of Theorems 4.1 and 5.1 are constructive so that a family $\{K_i\}_{i\geq 0}$ of controllers solving (P) can be determined. The construction is however quite involved; for instance it requires the Smith form of the plant and approximation of (finite and infinite) zeros on the imaginary axis. The order of the controller (for stable plants) can be shown to be given by the order of the plant plus the number of (finite and infinite) zeros of G_{12} and G_{21} on the imaginary axis.

In this section we show that a sequence of controllers can alternatively be constructed by solving a sequence of H_∞ control problems [18]. This has the advantage that standard and reliable software can be used (e.g. [19]) and a controller order of at most n is obtained.

The standard H_∞ optimization problem is to minimize the H_∞ norm of $L(s)$ in (3) given by

$$\|L\|_{H_\infty}:=\sup\{\|L(\mathrm{j}\omega)\|_2 : \omega\geq 0\},$$

where $\|\cdot\|_2$ denotes the largest singular value of a matrix. A sufficient condition for the existence of a solution (there exists a K attaining the minimum) is that the ranks of $G_{12}(\mathrm{j}\omega)$ and $G_{21}(\mathrm{j}\omega)$ are constant for all $0\leq\omega\leq\infty$. This condition is usually assumed, in theory [18], as well as in algorithms [19]. In the case of (P) this condition means that exact disturbance decoupling is solvable, and a controller can be directly

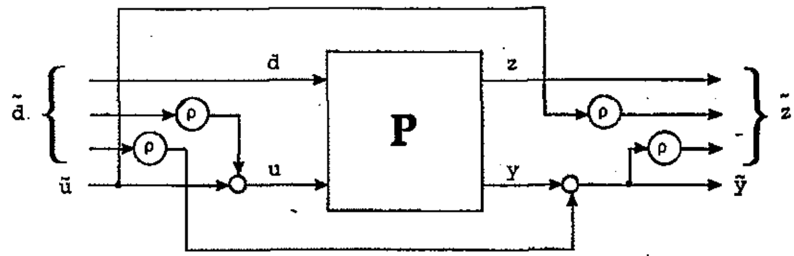

Fig. 2.

determined, see Section 4. In the case that exact disturbance decoupling is not possible we follow [20] and consider the H_∞ problem defined by

$$\begin{bmatrix} \tilde{z} \\ \tilde{y} \end{bmatrix} = \left[\begin{array}{ccc|c} G_{11} & \rho G_{12} & 0 & G_{12} \\ 0 & 0 & 0 & \rho I \\ \rho G_{21} & \rho^2 G_{22} & \rho^2 I & \rho G_{22} \\ \hline G_{21} & \rho G_{22} & \rho I & G_{22} \end{array}\right] \begin{bmatrix} \tilde{d} \\ \tilde{u} \end{bmatrix}, \tag{14}$$

$\rho \geq 0$, see Figure 2. It is obtained by including sensor and actuator noise as well as saturation constraints into the design, and can be shown to have a solution.

Proposition 6.1. *Suppose that* (P) *is solvable for the plant* (1). *Let* $\{\rho_i\}_{i\geq 0}$ *be a sequence of positive real numbers tending to zero, and let* K_i *be a controller solving the standard* H_∞ *optimization problem for* (14) *with* $\rho = \rho_i$. *Let* L_i *be the closed loop transfer function* (3) *from* d *to* z. *Then* $\| L_i \|_{H_\infty}$ *tends to zero, as* $i \to \infty$.

Proof. The closed loop transfer function $\tilde{L}_i(K)$ from $\tilde{d}$ to $\tilde{z}$ is given by

$$\begin{bmatrix} G_{11} + G_{12}R(K)G_{21} & \rho_i G_{12} + \rho_i G_{12}R(K)G_{22} & \rho_i G_{12}R(K) \\ \rho_i R(K)G_{21} & \rho_i^2 R(K)G_{22} & \rho_i^2 R(K) \\ \rho_i G_{21} + \rho_i G_{22}R(K)G_{21} & \rho_i^2 G_{22} + \rho_i^2 G_{22}R(K)G_{22} & \rho_i^2 I + \rho_i^2 G_{22}R(K) \end{bmatrix}$$

where $R(K) = K(I - G_{22}K)^{-1}$.

Let $\varepsilon > 0$ be given. Since (P) is solvable, there exists K such that

$$\| G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21} \|_{H_\infty} \leq \tfrac{1}{2}\varepsilon$$

and

$$\| K(I - G_{22}K)^{-1} \|_{H_\infty} < \infty.$$

Hence there exists $i_0 \geq 0$ such that $\| \tilde{L}_i(K) \|_{H_\infty} < \varepsilon$ for $i \geq i_0$. This implies $\| \tilde{L}_i(K_i) \|_{H_\infty} < \varepsilon$. Hence $\| \tilde{L}_i(K_i) \|_{H_\infty} \to 0$ and

$$\| G_{11} + G_{12}K_i(I - G_{22}K_i)^{-1}G_{21} \|_{H_\infty} \to 0. \quad \square$$

Proposition 6.2. *Suppose* (P) *is solvable for an n-th order plant* (1). *Then there exists a sequence of n-th order controllers solving (P).*

Proof. Proposition 6.1. shows that (P) can be reduced to a sequence of H_∞ problems. In [21] it is argued that these H_∞ problems can be solved by n-th order controllers. Note that the orders of (1) and (14) coincide. $\square$

7. Examples

Example 7.1. Let us consider the system shown in Figure 3, where B, A, U, V are polynomials. Here, B/A denotes the plant and U/V is a weight chosen by the designer to reflect those frequencies at which disturbance rejection should occur. Assuming the plant to be strictly proper and stable and the weight to be proper and stable, we have the situation of Theorem 4.1 with

$$G_{11} = G_{21} = U/V, \qquad G_{12} = G_{22} = B/A.$$

Theorem 4.1 gives the following necessary and sufficient conditions for solvability of (P):

(i) B has no zeros in $\mathbf{C}_+$.
(ii) If $B(j\omega_0) = 0$ then also $U(j\omega_0) = 0$.
(iii) $d_*(U/V) > 0$.

Sufficiency of the above conditions is shown in [17].

Example 7.2. Consider the state space system

$$\dot{x} = Ax + bu + gd, \qquad y = x, \qquad z = hx$$

given by the matrices

$$A = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad g = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}, \qquad h = [0 \quad 0 \quad 1].$$

Note that we allow full state feedback. The system is stable with characteristic polynomial $p(s) = (s+1)(s+2)(s+3)$ and (A, b) is completely controllable. To exclude trivial cases we assume $g \neq 0$. We have

$$G_{11} = \frac{g_1 + g_2 s + g_3 s^2}{p(s)}, \qquad G_{12} = \frac{s^2}{p(s)}.$$

Moreover, using the Jordan form of A, it can be seen that the Smith form of $G_{21} = (sI - A)^{-1}g$ is given by

$$U_2 G_{21} = \begin{bmatrix} \frac{1}{s+1} & 0 & 0 \end{bmatrix}^{\mathrm{T}}.$$

In the notation of Theorem 5.1, there holds $r_1 = r_2 = 1$ and

$$\sigma_{1,1} = \frac{s_2}{p(s)}, \qquad \sigma_{2,1} = \frac{1}{s+1}, \qquad t_{11} = G_{11}.$$

Hence (P) is solvable if and only if G_{11} has a zero at infinity and a zero at 0, both with multiplicity at least one. Thus (P) is solvable if and only if $g_1 = 0$.

It follows from [22] that solvability of the almost disturbance decoupling problem with stabilization using *dynamic* state feedback is equivalent to solvability of the corresponding problem using *static* state

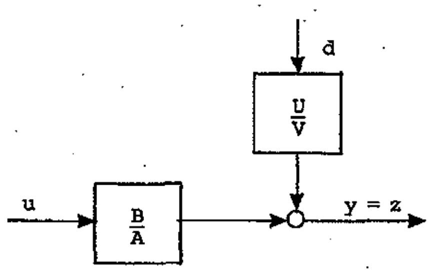

Fig. 3.

feedback. The sufficient solvability condition for the latter problem given in [1,3] requires $g_1 = g_2 = 0$. Moreover, in the notation of [1,3],

$$\mathbf{R}^*_{b,\ker h} = \operatorname{im}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{V}^*_{g,\ker h} = \{0\}, \qquad \mathbf{V}^*_{\bar{g},\ker h} = \operatorname{im}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

This shows that subspaces 'between' $\mathbf{R}^*_{b,\ker h} + \mathbf{V}^*_{g,\ker h}$ and $\mathbf{R}^*_{b,\ker h} + \mathbf{V}^*_{\bar{g},\ker h}$ are relevant for solvability of (P). These should perhaps be defined in terms of the eigenstructure of $A + bf$, where $(A + bf)\mathbf{V}^*_{\bar{g},\ker h} \subset \mathbf{V}^*_{\bar{g},\ker h}$.

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