T-CAS/36/3//25756

Stability Testing of 2-D Recursive Digital Filters Based on a Circuit-Theoretic Approach

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Reprinted from
IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS
Vol. 36, No. 3, March 1989

Stability Testing of 2-D Recursive Digital Filters Based on a Circuit-Theoretic Approach

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Abstract —A new procedure for checking the stability of 2-D digital filters is proposed. The key part of the test involves construction of related all-pass sections of reduced order, stability of which is shown to be equivalent to the stability of the original function. This is a direct extension to a similar approach, developed recently in detail for the 1-D case [1]. The results are shown to be equivalent to some known stability testing methods in 2-D, but have the advantage of an additional insight gained from the circuit interpretation. Explicit formulas for checking the stability in terms of the filter coefficients are derived for some special cases, and numerical examples are included to clarify the application of the proposed approach.

I. Introduction

A MAJOR concern in the design of 2-D recursive digital filters is to ensure their stability. After the original work of Shanks [2], Huang [3] proposed a simplified criterion often used for checking the stability of 2-D digital filters. Huang's result, as modified in [4] is given by

Huang's Theorem: A causal filter with a 2-D transfer function $H(z_1, z_2) = P(z_1, z_2)/D(z_1, z_2)$, where P and D are polynomials in z_1 and z_2, is stable in the sense that there are no values of z_1 and z_2 for which $D(z_1, z_2) = 0$; $|z_1| \geqslant 1$ and $|z_2| \geqslant 1$ iff:

i) the map of $\partial d_2 \triangleq (z_2; |z_2| = 1)$ in the z_2 plane, according to $D(z_1, z_2) = 0$, lies strictly inside $d_1 \triangleq (z_1; |z_1| \leqslant 1)$; and

ii) no point in $d_2 \triangleq (z_2;\ |z_2| \geqslant 1)$ maps into the point $z_1 = \infty$ by the relation $z_1^{-M} D(z_1, z_2) = 0$, where M is the maximum degree of z_1 in $D(z_1, z_2)$.

To check the conditions of the theorem, a number of test procedures [3], [5]–[7] requiring different levels of computational efforts, have been proposed with each showing equivalent results on test examples.

In this paper we present a procedure for checking the stability conditions, which is believed to be conceptually simpler than the existing ones, as it provides a direct physical interpretation of each step. The procedure involves successive reduction of the order of certain all-pass sections, whose stability depends on the stability of the original filter. In addition it requires testing of functions of one variable for a strictly bounded real (SBR) property. In this paper the stability of the overall filter is assumed to depend on the denominator polynomial only and hence cases of reducible 2-D transfer functions or nonessential singularities of the second kind are not considered. The results are shown to be equivalent to those of some known methods, but have the advantage of additional insight gained from the circuit interpretation. The method can be considered as a direct extension to the 2-D case of the well developed theory for the 1-D case [1].

The organization of the paper is as follows. In Section II we outline the basic idea behind the method for the case of real filter coefficients. Possible circuit interpretation of the method is presented in Section III. In Section IV we explicitly derive, in terms of the filter coefficients, the conditions for stability of certain low order 2-D filters. In Section V the equivalence of this method to the modified Jury table method of Maria and Fahmy [4] is demonstrated. Some numerical examples of our approach are included in Section VI while Section VII contains concluding remarks.

Notations and Definitions

In the following derivation the variables z_1, z_2 denote the transform domain complex variables for the discrete-time systems under consideration. The explicit dependence on these variables is dropped for notational convenience in the absence of any ambiguity. The complex conjugate of a given quantity is denoted by the "*" superscript.

The discussion also heavily relies on the concept of "structural boundedness" [8], which we extend to the 2-D case. Thus, a stable digital filter transfer function $H(z_1, z_2)$, that is real valued for real z_1 and z_2, and satisfies $|H(e^{j\omega_1}, e^{j\omega_2})| \leqslant 1$ for $\forall\ \omega_1, \omega_2$, is called a "bounded real" (BR) function. BR functions, for which strict inequality holds for $\forall\ \omega_1, \omega_2$ are referred to as "strictly bounded real" (SBR) functions. A BR function, for which equality holds for $\forall\ \omega_1, \omega_2$ is called a "lossless bounded real" (LBR) function. Thus a stable all-pass function with real coefficients is LBR. Functions with complex coefficients, which satisfy above relations will be called "strictly bounded complex" (SBC), and "lossless bounded complex" (LBC), correspondingly.

Manuscript received October 16, 1987; revised August 30, 1988. This work was supported in part by the National Science Foundation under Grant MIP 85-08017. This paper was recommended by Associate Editor D. M. Goodman.

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IEEE Log Number 8825756.

II. The Generalized Schur–Cohn Test

Let $H(z_1, z_2) = P(z_1, z_2)/D(z_1, z_2)$ be the given 2-D transfer function with real coefficients. To check the stability of this function in the sense of Huang's theorem we first form the 2-D all-pass function:

$$S(z_1, z_2) = z_1^M z_2^N \frac{D(z_1^{-1}, z_2^{-1})}{D(z_1, z_2)} \tag{1}$$

where M and N are the maximum degrees of z_1 and z_2 in $D(z_1, z_2)$.

This function can be rewritten as a ratio of polynomials in one variable, with the coefficients being polynomials of the other variable, i.e.,

$$S(z_1, z_2) = \frac{\tilde{a}_M(z_2) + \cdots + \tilde{a}_0(z_2) z_1^M}{a_0(z_2) + \cdots + a_M(z_2) z_1^M} \tag{2}$$

where for $\forall\, i$, a_i and $\tilde{a}_i$ are polynomials in z_2, and the tilde sign denotes the operation of replacing z_2 by z_2^{-1} and multiplying by z_2^N. Note that in the case when the maximum degree of each $a_i(z_2)$ is equal to N, this operation is equivalent to taking the paraconjugate operation [8].

The function $S(z_1, z_2)$ can further be rewritten as

$$\begin{aligned} S(z_1, z_2) &= \frac{\tilde{a}_M(z_2)}{a_M(z_2)} \frac{1 + \tilde{n}_{M-1}(z_2) z_1 + \cdots + \tilde{n}_0(z_2) z_1^M}{n_0(z_2) + \cdots + n_{M-1}(z_2) z_1^{M-1} + z_1^M} \\ &= A_M(z_2) G_M(z_1, z_2) \end{aligned} \tag{3}$$

where

$$n_i(z_2) = \frac{a_i(z_2)}{a_M(z_2)};\ \tilde{n}_i(z_2) = n_i(z_2^{-1});\ A_M(z_2) = \frac{\tilde{a}_M(z_2)}{a_M(z_2)} \tag{4}$$

and

$$G_M(z_1, z_2) = \frac{1 + \tilde{n}_{M-1}(z_2) z_1 + \cdots + \tilde{n}_0(z_2) z_1^M}{n_0(z_2) + \cdots + n_{M-1}(z_2) z_1^{M-1} + z_1^M}. \tag{5}$$

For the overall stability of $S(z_1, z_2)$ the decomposition of (3) implies the necessary condition that $A_M(z_2)$ be a stable all-pass function in z_2, i.e., $|A_M(e^{j\omega_2})| = 1$ for $\forall\, \omega_2$, and all poles of $A_M(z_2)$ be inside the unit circle. This condition can easily be seen to correspond to the second condition of Huang's theorem. Assuming that the stability of $A_M(z_2)$ has been verified using any one of the known 1-D stability tests, we only have to show that the second term in (3), $G_M(z_1, z_2)$ is a stable all-pass function for $|z_2| = 1$.

The basic idea behind our method, in analogy with the 1-D case, is to successively reduce the order of G_M in the z_1 variable. As it is shown, the stability of the reduced order functions G_{M-i} formed in the method is directly related to the stability of G_M, so that after $M-1$ steps we can decide on the overall stability of the function from the coefficients of the remaining first-order all-pass function, which are functions of the other variable only. Thus at each step we have to deal with functions of one variable, a major improvement in terms of complexity reduction.

To this end we consider the transfer function $G_{M-1}(z_1, z_2)$, derived as follows:

$$G_{M-1}(z_1, z_2) = z_1 \frac{G_M(z_1, z_2) - k_M(z_2)}{1 - \tilde{k}_M(z_2) G_M(z_1, z_2)} \tag{6}$$

where

$$k_M(z_2) \triangleq G_M(\infty, z_2) = \tilde{n}_0(z_2);\ \tilde{k}_M(z_2) = n_0(z_2). \tag{7}$$

With this definition, we can state the following important theorem.

Theorem: $G_M(z_1, z_2)$ is a stable all-pass function (and hence LBR) iff $G_{M-1}(z_1, z_2)$ is a stable all-pass function (LBR), and the function $\tilde{k}_M(z_2)$ is strictly bounded real (SBR).

Proof: Assume that $G_M(z_1, z_2)$ is a stable all-pass function and let $G_M(e^{j\omega_1}, e^{j\omega_2}) = e^{j\Theta}$. We first show that $G_{M-1}(z_1, z_2)$ is an all-pass function. Indeed from (6):

$$G_{M-1}(e^{j\omega_1}, e^{j\omega_2}) = e^{j\omega_1} e^{j\Theta} \frac{1 - k_M(e^{j\omega_2}) e^{-j\Theta}}{1 - [k_M(e^{j\omega_2}) e^{-j\Theta}]^*}. \tag{8}$$

Hence, as long as $|k_M(e^{j\omega_2})| \neq 1$ for $\forall\, \omega_2$, the all-pass property follows.

Also, using (3) and (6), and dropping for notational convenience the explicit dependence of $n_i, \tilde{n}_i$ on z_2 we have

$$\begin{aligned} G_{M-1}(z_1, z_2) &= z_1 \frac{\dfrac{1 + \cdots + \tilde{n}_0 z_1^M}{n_0 + \cdots + z_1^M} - \tilde{n}_0}{1 - n_0 \dfrac{1 + \cdots + \tilde{n}_0 z_1^M}{n_0 + \cdots + z_1^M}} \\ &= \frac{(1 - n_0 \tilde{n}_0) + \cdots + (\tilde{n}_1 - n_{M-1} \tilde{n}_0) z_1^{M-1}}{(n_1 - \tilde{n}_{M-1} n_0) + \cdots + (1 - n_0 \tilde{n}_0) z_1^{M-1}} \end{aligned} \tag{9}$$

which shows that the function $G_{M-1}(z_1, z_2)$ is an all-pass function of reduced order, $M-1$ in the z_1 variable.

Next we prove that if $G_M(z_1, z_2)$ is stable, so is $G_{M-1}(z_1, z_2)$. From (6) it follows that the singularities z_{10} of $G_{M-1}(z_1, z_2)$ are solutions of

$$G_M(z_{10}, z_2) = \frac{1}{\tilde{k}_M(z_2)}. \tag{10}$$

From the first condition of Huang's theorem we know that all the roots of $G_M(z_1, z_2)_{|z_2|=1} = 0$ are inside the unit circle and according to the maximum modulus theorem will attain values as [1]:

$$\begin{aligned} &|G_M(z_1, z_2)|_{|z_2|=1} < 1, && \text{for } |z_1| > 1 \\ &|G_M(z_1, z_2)|_{|z_2|=1} > 1, && \text{for } |z_1| < 1 \\ &|G_M(z_1, z_2)|_{|z_2|=1} = 1, && \text{for } |z_1| = 1. \end{aligned} \tag{11}$$

Since $k_M(z_2) \triangleq G_M(\infty, z_2)$, it follows from (11) that $|G_M(z_{10}, z_2)| > 1$ for $|z_2| = 1$. This implies that all singularities of $G_{M-1}(z_1, z_2)|_{|z_2|=1}$ are inside the unit circle in the z_1 plane and hence $G_{M-1}(z_1, z_2)|_{|z_2|=1}$ is a stable all-pass function in z_1.

The stability of $G_M(z_1, z_2)$, for $|z_2| = 1$, from (5) implies that $n_0 = \tilde{k}_M$ is SBR as asserted in the theorem, since it is the product of the roots of the denominator polynomial with respect to the z_1 variable.

Finally, we show that the second condition of Huang's theorem, which is clearly satisfied for the denominator of $G_M(z_1, z_2)$, still holds for the denominator of $G_{M-1}(z_1, z_2)$. Indeed, solving

$$z_1^{-(M-1)} D_{M-1}(z_1, z_2) = (1 - n_0 \tilde{n}_0) = 0 \tag{12}$$

where $D_{M-1}(z_1, z_2)$ is the denominator polynomial in (9), we observe that equality can hold only for values of z_2 with $|z_2| < 1$. This is implied by the relation $n_0(z_2) = \tilde{k}_M(z_2)$ and $\tilde{k}_M$ being SBR.

Thus if G_M is LBR, it follows that $\tilde{k}_M$ is SBR and that G_{M-1} is also LBR.

To prove the theorem in the other direction we assume that G_{M-1} is LBR, and $\tilde{k}_M$, as defined above, is SBR. Reversing the relation in (6) we arrive at:

$$G_M(z_1, z_2) = \frac{G_{M-1}(z_1, z_2) + z_1 k_M(z_2)}{z_1 + \tilde{k}_M(z_2) G_{M-1}(z_1, z_2)}. \tag{13}$$

If $G_{M-1}(z_1, z_2)$ is all pass, it is easy to see that so is $G_M(z_1, z_2)$. Next, if z_{10} is a pole of G_M, then

$$z_{10}^{-1} G_{M-1}(z_{10}, z_2) = -\frac{1}{\tilde{k}_M(z_2)}. \tag{14}$$

When evaluated at $|z_2| = 1$ this implies

$$\left| z_{10}^{-1} G_{M-1}(z_{10}, e^{j\omega_2}) \right| > 1. \tag{15}$$

Because of the LBR assumption on G_{M-1} we must then have $|z_{10}| < 1$ in order for (15) to hold, and hence the poles of $G_M(z_1, z_2)$ for $|z_2| = 1$ lie inside the unit circle in z_1, as required by Huang's first condition.

The second condition of Huang's theorem follows directly from the assumption that $\tilde{k}_M(z_2)$ is SBR.

Summarizing, a necessary and sufficient set of conditions for the all-pass function $G_M(z_1, z_2)$ to be stable are therefore: (a) the function $\tilde{k}_M(z_2)$ must be SBR, and (b) the all-pass function $G_{M-1}(z_1, z_2)$ be stable.

This result allows us, once the SBR nature of $\tilde{k}_M(z_2)$ is established, to simply check the stability of the lower order all-pass function G_{M-1}. The process can clearly be repeated, to generate a set of functions $k_i(z_2)$, and a set of all-pass functions $G_M, G_{M-1}, \cdots, G_0 = 1$. It then follows, that the overall function $G_M(z_1, z_2)$ is stable, iff $\tilde{k}_i(z_2)$ are SBR for all i. Thus we have reduced the problem to functions of one variable only, which is well understood and provides no difficulties for testing.

III. Circuit Interpretation of the Method

In this brief section we provide a circuit "implementation" of the function $G_M(z_1, z_2)$ in the form of a cascaded lattice structure as shown in Fig. 1. In Fig. 2 is shown the first building block of the lattice structure in cascade with the function $A_M(z_2)$ yielding the overall function $S_M(z_1, z_2)$. We can note, that this structure is not realizable, since by construction, the multiplier function k_M is unstable. This problem can still be resolved, as shown on Fig. 3, once we realize, that the same transfer function is obtained provided:

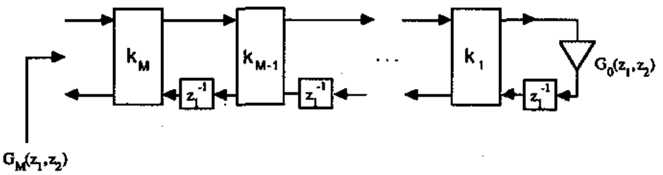

Fig. 1.

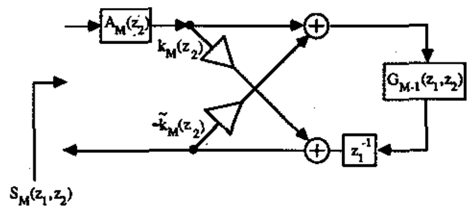

Fig. 2.

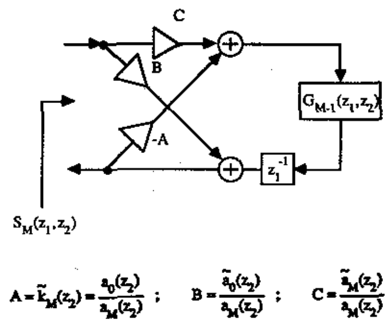

Fig. 3.

$$A = \tilde{k}_M(z_2) = \frac{a_0(z_2)}{a_M(z_2)};\; B = \frac{\tilde{a}_0(z_2)}{a_M(z_2)};\; C = \frac{\tilde{a}_M(z_2)}{a_M(z_2)}. \tag{16}$$

Note that the structure on Fig. 3 incorporates the function $A_M(z_2)$ as well.

IV. Explicit Coefficient Conditions for Some Low Order Transfer Functions

In this section we derive the explicit conditions on the filter coefficients for some low order cases, using the proposed method.

Example 1: Consider a first order transfer function with denominator polynomial

$$D(z_1, z_2) = c + bz_1 + az_2 + z_1 z_2.$$

Denoting $c + az_2 = a_0(z_2)$; $b + z_2 = a_1(z_2)$ we arrive at the associated all-pass function

$$H(z_1, z_2) = \frac{\tilde{a}_1(z_2)}{a_1(z_2)} \frac{1 + \tilde{n}_0(z_2) z_1}{n_0(z_2) + z_1} = A_1(z_2) G_1(z_1, z_2) \tag{17}$$

where

$$n_0(z_2) = \frac{a_0(z_2)}{a_1(z_2)}.$$

The function $A_1(z_2) = (1+bz_2)/(b+z_2)$ is a stable all-pass function in z_2 for $|b|<1$, which corresponds to the second condition of Huang's theorem.

The first condition holds if $\tilde{k}_1(z_2) = n_0(z_2)$ is SBR, i.e.,

$$|n_0(z_2)|^2\bigg|_{|z_2|=1} = \frac{(c+az_2)}{(b+z_2)}\frac{(c+az_2^*)}{(b+z_2^*)}\bigg|_{|z_2|=1} < 1. \quad (18)$$

Recognizing that on the unit circle $z_2^* = z_2^{-1}$, and denoting $z_2 + z_2^{-1} = 2x$ we get the inequality

$$c^2 + a^2 + ac(z_2 + z_2^{-1}) < 1 + b^2 + b(z_2 + z_2^{-1})$$

or equivalently the condition

$$1 + b^2 - a^2 - c^2 + 2(b - ac)x > 0, \qquad \text{for } -1 \leqslant x \leqslant 1 \quad (19)$$

which is derived in a number of papers [3], [5].

Example 2: Our next example is of a general second-order polynomial. Its coefficients can conveniently be expressed in the matrix notation:

$$D_2(z_1, z_2) = \begin{bmatrix} 1 & z_1 & z_1^2 \end{bmatrix} \begin{bmatrix} f & g & d \\ h & c & a \\ e & b & 1 \end{bmatrix} \begin{bmatrix} 1 \\ z_2 \\ z_2^2 \end{bmatrix}. \quad (20)$$

The associated all-pass transfer function is then given by

$$H(z_1, z_2) = \frac{1 + az_1 + bz_2 + cz_1z_2 + dz_1^2 + ez_2^2 + hz_1z_2^2 + gz_1^2z_2 + fz_1^2z_2^2}{f + hz_1 + gz_2 + cz_1z_2 + ez_1^2 + dz_2^2 + az_1z_2^2 + bz_1^2z_2 + z_1^2z_2^2}$$
$$= \frac{(1 + bz_2 + ez_2^2) + (a + cz_2 + hz_2^2)z_1 + (d + gz_2 + fz_2^2)z_1^2}{(f + gz_2 + dz_2^2) + (h + cz_2 + az_2^2)z_1 + (e + bz_2 + z_2^2)z_1^2}$$

which can be rewritten as

$$H(z_1, z_2) = \frac{\tilde{a}_2(z_2)}{a_2(z_2)} \frac{1 + \tilde{n}_1(z_2)z_1 + \tilde{n}_0(z_2)z_1^2}{n_0(z_2) + n_1(z_2)z_1 + z_1^2} = A_2(z_2)G_2(z_1, z_2) \quad (21)$$

where

$$a_0(z_2) = f + gz_2 + dz_2^2;\; a_1(z_2) = h + cz_2 + az_2^2;$$
$$a_2(z_2) = e + bz_2 + z_2^2$$

and

$$n_0(z_2) = \frac{a_0(z_2)}{a_2(z_2)};\; n_1(z_2) = \frac{a_1(z_2)}{a_2(z_2)}.$$

We first check the stability of

$$A_2(z_2) = \frac{\tilde{a}_2(z_2)}{a_2(z_2)} = \frac{1 + bz_2 + ez_2^2}{e + bz_2 + z_2^2}$$

which corresponds to checking the second Huang's condition, i.e., we check the roots of

$$z_2^2 + bz_2 + e = 0. \quad (22)$$

If the test passes, i.e., $|z_2|_{1,2} < 1$, we proceed to check the first condition which requires that $\tilde{k}_2(z_2)$ be SBR.

Again recognizing that on the unit circle $z_2^{-1} = z_2^*$ and using (22) we can express the conditions on the coefficients in terms of the real variable $x = \frac{1}{2}(z_2 + z_2^{-1})$, which are

$$4(e - df)x^2 + 2(b + be - dg - fg)x + (1 + b^2 + e^2 - d^2 - g^2 - f^2 + 2df - 2e) > 0, \quad \text{for } |x| < 1. \quad (23)$$

If no roots of this quadratic equation are in $|x|<1$ we can proceed to form, using (9), the reduced order polynomial $G_1(z_1, z_2)$. The condition for $\tilde{k}_1(z_2)$ to be SBR, expressed in terms of the a_i's are thus as follows:

$$(\tilde{a}_1a_2 - a_1\tilde{a}_0)^2 < (a_2\tilde{a}_2 - a_0\tilde{a}_0)^2. \quad (24)$$

After some algebra, the polynomial in the real variable x to be checked for roots in $|x|<1$ becomes

$$16(F^2 - AE)x^4 + 8(2FG - AD - BE)x^3$$
$$+ 4(2FH + G^2 - 4F^2 - AC - BD - CE + 4AE)x^2$$
$$+ 2(2GH - 4FG - AB - BC - CD - DE + 3(AD + BE))x + H^2 - A^2 - B^2 - C^2 - D^2 - E^2$$
$$- 2(2FH - AC - BD - CE - AE) > 0 \quad (25)$$

where the quantities above are defined in terms of the denominator polynomial coefficients according to

$$A = ae - dh \qquad E = h - af$$
$$B = ab + ce - cd - gh \qquad F = e - df$$
$$C = a + bc + eh - fh - cg - ad \qquad G = b + be - dg - fg$$
$$D = c + bh - cf - ag \qquad H = 1 + b^2 + e^2 - d^2 - f^2 - g^2. \quad (26)$$

This fourth-order polynomial can further be checked for roots inside the $|x|<1$ by well-known methods, e.g., [5].

It is a common practice in filter design to impose some symmetry conditions on the coefficient matrix. In this case, assuming symmetry along the main diagonal of the coefficient matrix the computations can be somewhat simplified by using $a = b$, $d = e$, $g = h$. The polynomial to be checked then remains as in (25), however the quantities are redefined as

$$A = ad - dg \qquad E = g - af$$
$$B = a^2 - g^2 \qquad F = d - df$$
$$C = a + ac + dg - fg - cg - ad \qquad G = a + ad - dg - fg$$
$$D = c - cf \qquad H = 1 + a^2 - f^2 - g^2. \quad (27)$$

As it can be seen from above formulas, the advantage of this method is in the conceptually straightforward manner in which the equations are set up and solved. It should also be noted that derivation of the stability conditions for the general second-order polynomials using some other methods such as the direct implementation of Huang's method which involves bilinear transformations, may become very complicated. Note also that the circuit-theoretic approach proposed here can be extended relatively easily to higher dimensionalities as well.

Using a simple computer program it is easy to verify the conditions of (22), (23), (25)–(27), to determine the stability of a second-order 2-D filter. The remaining step is then to check the polynomials for having zeros in $|x|<1$, which can be solved either by using the methods outlined in [5] or, for low-order polynomials, with a root finding routine. This type of testing stability may be quite useful when a 2-D filter is designed using a computer-aided optimization approach where the stability needs to be checked at each iteration step.

V. Relation with the Maria–Fahmy Method

We now establish the equivalence between the above method and the Maria–Fahmy method [4]. The test procedure in [4] is based on rewriting the denominator polynomial in terms of one variable only:

$$D(z_1,z_2)=a_M(z_2)z_1^M+a_{M-1}(z_2)z_1^{M-1}+\cdots+a_0(z_2) \tag{28}$$

where

$$a_i(z_2)=\sum_{i=0}^{n} b_{ik}z_2^i.$$

The coefficients $a_i(z_2)$ are next arranged as first entries of the Jury table and are used to successively obtain the coefficient sets $b_i(z_2), c_i(z_2), d_i(z_2),\cdots,t_k(z_2)$ using the relations:

$$b_k(z_2)=\begin{vmatrix} a_0 & a_{n-k}\\ a_n^* & a_k^*\end{vmatrix},\ c_k(z_2)=\begin{vmatrix} b_0 & b_{n-1-k}\\ b_{n-1}^* & b_k^*\end{vmatrix},\cdots \tag{29}$$

The polynomials $b_0, c_0,\cdots,t_0$ can be shown to be functions of the real variable x. The modified Jury criterion, which is a test for Huang's first condition, then takes the form:

For $-1\leqslant x\leqslant 1$:

$$b_0(x)<0;\ c_0(x)>0;\ d_0(x)>0;\cdots;\ t_0(x)>0$$

or equivalently as

i) $b_0(0)<0$; $c_0(0)>0$; $d_0(0)>0\cdots$; $t_0(0)>0$, and

ii) the polynomials $b_0(x), c_0(x),\cdots,t_0(x)$ have no real roots for $-1\leqslant x\leqslant 1$.

The second Huang's condition is then checked by determining whether

$$\lim_{z_1\to\infty} z_1^{-M}D(z_1,z_2)=0 \tag{30}$$

has roots of magnitude greater than 1.

To show the equivalence of the two procedures we first observe that the test for the second condition corresponds directly to checking $A_M(z_2)$ for stability. Next, the condition on $b_0(x)=a_0^2-a_M^2<0$ corresponds to checking $\tilde{k}_M(z_2)$ for being SBR. Indeed from

$$\tilde{k}_M(e^{j\omega_2})\tilde{k}_M^*(e^{j\omega_2})=\frac{a_0}{a_M}\frac{a_0^*}{a_M^*}<1 \tag{31}$$

equality holds.

The condition $c_0(x)>0$ of Maria–Fahmy [4] is equivalent to $c_0=b_0^2-b_{M-1}^2>0$ or in terms of the a_i's:

$$\left(a_0^2-a_M^2\right)^2>\left(a_0a_{M-1}^*-a_1a_M^*\right)^2. \tag{32}$$

We next show that this is equivalent to require $\tilde{k}_{M-1}(z_2)$ be SBR. Indeed, from (24) it follows:

$$\tilde{k}_{M-1}(e^{j\omega_2})\tilde{k}_{M-1}^*(e^{j\omega_2})=\left(1-n_M^2\right)^2>\left(n_{M-1}^*-n_1n_M^*\right)^2.$$

Expressing this in terms of the a_i's we get

$$\left(a_0^2-a_M^2\right)^2>\left(a_0a_{M-1}^*-a_1a_M^*\right)^2 \tag{33}$$

which is equivalent to (32). Proceeding in the same way one can show that the conditions on $d_0,\cdots,t_0$ in [4] correspond to the functions $\tilde{k}_i(z_2)$ being SBR, when using our approach.

An example is next shown to illustrate the application of the method and its equivalence with the stability check procedures in [4].

VI. Illustrative Example

We next consider the same numerical example, solved by Huang, and Maria–Fahmy, to illustrate the application of our method. The transfer function considered is given by

$$H(z_1,z_2)=\frac{1}{1+\frac{1}{2}z_1^{-1}+\frac{1}{2}z_2^{-1}+\frac{1}{4}z_1^{-1}z_2^{-1}+\frac{1}{4}z_1^{-2}+\frac{1}{4}z_2^{-2}}.$$

We first form

$$D(z_1,z_2)=z_1^2z_2^2+\frac{1}{2}z_1z_2^2+\frac{1}{2}z_1^2z_2+\frac{1}{4}z_1z_2+\frac{1}{4}z_1^2+\frac{1}{4}z_2^2.$$

The coefficient matrix for this case is symmetric, hence we can directly use formulas (22), (23), (25) and (27) to arrive at the conditions:

1) $z_2^2+(1/2)z_2+1/4$ must have no roots outside the unit circle, which is clearly satisfied.

2) The polynomial $b_0(x)=x^2+(5/4)x+3/4$ should have no real roots in $|x|<1$, which is also satisfied.

3) For the computation of $c_0(x)$ we can use the symmetric case formulas (14), (16) which lead to

$$c_0(x)=x^4+\frac{9}{4}x^3+\frac{41}{16}x^2+\frac{3}{2}x+\frac{27}{64}.$$

The same polynomial was obtained in [4], and was shown to have no real roots in $|x|<1$. Hence the conclusion is that the original transfer function $H(z_1,z_2)$ is stable.

This example shows, that for general, second order filter one can directly use the formulas from Section IV to decide on the stability of the original polynomials. Extensions to higher orders are straightforward and lend themselves to easy programming.

VII. Concluding Remarks

While in this paper we concentrated on cases with polynomials with real coefficients, it is not difficult to show that only minor modifications of the procedure are necessary for the complex coefficient case. More specifically, the coefficients of the numerator polynomial have to be taken to be the complex conjugates of the corresponding coefficients of the denominator. Next, the set of functions $\tilde{k}_i(z_2)$ has to be checked for being strictly bounded complex (SBC). The function G_0 will in general also be complex, but of magnitude one.

Summarizing, a simple stability testing procedure for discrete-time 2-D systems was presented in an unified manner based on lossless network synthesis. Explicit formulas in terms of the filter coefficients were derived for the stability of general case first- and second-order 2-D polynomials and examples were solved by the proposed method and compared to existing solutions. It is to be understood that our test is best suitable for functions of strictly two variables, since simpler procedures have been established for the 1-D case.

Acknowledgment

The authors thank Dr. M. Bellanger, Dr. V. Ramachandran, and Dr. E. Zeheb, as well as the anonymous reviewers for their valuable comments.

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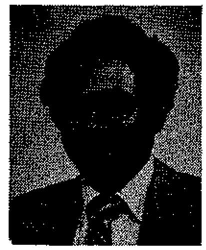

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