On the Problem of Stable Rational Interpolation

A. C. Antoulas*
Department of Electrical and Computer Engineering
Rice University
Houston, Texas 77251-1892
and
Mathematical System Theory
E.T.H. Zurich
CH-8092 Zurich, Switzerland

and

B. D. O. Anderson
Department of Systems Engineering
Australian National University
Canberra, A.C.T. 2601, Australia

Submitted by Jan C. Willems

ABSTRACT

Aspects of the stability issue in connection with rational interpolation are investigated. In particular, it is shown that *unconstrained* interpolation of a given set of points together with an associated mirror-image set of points yields a one-parameter family of *stable* interpolating functions. As an application, it is also shown that if a certain number of Markov parameters are given, by appropriate choice of the moments stable realizations are obtained.

1. INTRODUCTION

Recently the problem of parametrization of all scalar rational functions interpolating a given array of points has been solved. The parameter is the

*Most of this research was conducted while the first author was a Visiting Fellow at the Australian National University, May–June 1987. Support was also provided by N.S.F. through Grant ECS-05293.

655 Avenue of the Americas, New York, NY 10010 0024-3795/89/$3.50

complexity, usually defined in two different ways. The first, the *MacMillan degree*, is defined as the largest among the numerator and the denominator degrees. The second possibility is to define the complexity as the sum of the numerator and the denominator degrees of the interpolating function.

The main tool for the study of rational interpolation with the MacMillan degree as complexity is the so-called *Löwner matrix*. The Löwner matrix encodes the information about the minimal admissible complexity as a simple function of its rank and the rank of its submatrices. This approach leads to a generalization of the classical realization theory (see Kalman, Falb, and Arbib, 1969); recall that the latter can be considered as the special case of rational interpolation with all the data provided at a single point (infinity) of the complex plane. It can be shown that in such a case the Löwner matrix reduces to the familiar Hankel matrix. The details of this approach can be found in Antoulas and Anderson (1986).

The main tool for the study of the interpolation problem with the latter definition of complexity given above is the *Euclidean algorithm*. This approach leads to an understanding of the Cauchy and of the related Padé approximation problems. For details, see Antoulas (1988).

In this paper we will discuss further aspects of the former formulation in connection with *stability* of the interpolating functions. A central result in this regard is the following. The rational function of minimal MacMillan degree interpolating a given array of points, together with an associated so-called *mirror-image array* of points, is (automatically) *stable* in the following sense. If the points in the initially given array are inside (outside) the unit disc, stability is defined as boundedness inside (outside) the closed unit disc. This follows from a closer examination of the Nevanlinna-Pick algorithm. Thus, the Nevanlinna-Pick recursive algorithm can be replaced by the Löwner-matrix approach, which is nonrecursive. This new insight into the Nevanlinna-Pick algorithm actually provides a one-parameter family of stable functions. These results are worked out in the general case where the array considered is allowed to have multiplicities (see Section 4).

This is applied in the following section to the study of the stable realization problem. The key to this (as mentioned earlier) is to recognize that realization is a special case of interpolation, namely, interpolation at a single point. It is then shown that the mirror-image array of a set of *Markov parameters* is the array composed of the *moments* of the function. Therefore simultaneous (unconstrained) realization of a given array of Markov parameters together with the associated mirror-image moments yields stable solutions.

It readily follows from the general theory that the problem of rational interpolation of N pairs of points without a stability constraint can be solved *generically* with a function of degree $N/2$. If however the stability constraint is added to the problem, the solution will generically have degree $N-1$. There is therefore an obvious tradeoff between stability of the interpolating function and its complexity (MacMillan degree).

The paper is organized as follows. Sections 2 and 3 provide a review of the general rational-interpolation problem and of the Nevanlinna-Pick interpolation problem, respectively. In Section 4 we discuss a sufficient condition for the stability of the interpolating functions computed through the Löwner-matrix approach, and in Section 5 the application of this result to stable realization is developed.

2. THE GENERAL RATIONAL-INTERPOLATION PROBLEM

In this section we present a summary of the general rational-interpolation problem. For details, the reader is referred to Antoulas and Anderson (1986). Consider the array of distinct points $P := \{(x_i, y_i),\ i \in \underline{N}\}$, with $x_i \neq x_j$, $i \neq j$. A rational function

$$y(x) = \frac{n(x)}{d(x)}, \qquad n, d \text{ coprime}, \tag{2.1a}$$

is said to *interpolate* the above points iff

$$y(x_i) = y_i, \qquad i \in \underline{N}. \tag{2.1b}$$

The *complexity* or *MacMillan degree* of $y(x)$ is defined as

$$\deg y := \max(\deg n, \deg d). \tag{2.1c}$$

The rational interpolation problem is to parametrize all $y(x)$ of the form (2.1a) satisfying (2.1b); the parameter is defined by (2.1c). As it turns out, the main tool for studying this problem is the so-called Löwner matrix.

Consider the rational function $y(x)$ defined by the identity

$$\sum_{i=1}^{r+1} c_i \frac{y(x) - y_i}{x - x_i} = 0, \qquad c_i \neq 0. \tag{2.2a}$$

Generically, $\deg y = r$. If $r+1 = N$, then $y(x)$ as defined by (2.2a) satisfies (2.1b). If however $r+1 < N$, for arbitrary $c_i \neq 0$ it follows that $y(x)$ interpolates only the first $r+1$ points. Nevertheless, a choice of specific c_i's will allow interpolation of the remaining points, subject to satisfaction of a certain side condition described later (see Theorem 2.6). In particular, in order to interpolate the points indexed by $i=r+2,\ldots,N$, the coefficients c_i have to satisfy

$$Lc=0,$$

where $c:=(c_1\ldots c_{r+1})'$, and the (i,j)th element of L is

$$L_{ij}:=\frac{y_{r+1+i}-y_j}{x_{r+1+i}-x_j},\qquad i=1,2,\ldots,N-r-1,\quad j=1,\ldots,r+1. \tag{2.2b}$$

This is easily seen by inserting x_k for x and y_k for $y(x)$ in (2.2a), and letting k run from $r+2$ to N.

The $(N-r-1)\times(r+1)$ matrix L is called a *Löwner* or *divided-difference matrix*. It is shown in Antoulas and Anderson (1986) that L provides a key tool for studying the parametrization problem introduced above. Its main property is the following. Given a rational function $y(x)$, let the pairs (x_i,y_i), $i\in\underline{N}$, be obtained by sampling $y(x)$. If L is any $p\times q$ Löwner matrix formed from these pairs, and provided that $p,q\geqslant\deg y$, we have

$$\operatorname{rank}L=\deg y. \tag{2.3}$$

As a corollary, it follows that every square Löwner matrix of size $\deg y$ built from a subset of the above pairs of points is nonsingular.

Before proceeding with the main result, we will show how to treat multiple points, i.e. points x_i at which information about not only the value of the function but about the value of a certain number of consecutive derivatives of the same function is available as well. The key is to define a *generalized* Löwner matrix L, which still has the property (2.3). The array in this case is as follows:

$$P:=\left\{(x_i;y_{i,j-1}):(i,j)\in I\right\},\qquad I:=\left\{(i,j):j\in\underline{\nu}_i,\,i\in\underline{\theta}\right\}; \tag{2.4a}$$

the number of points is $N=\nu_1+\cdots+\nu_\theta$; ν_i is the multiplicity of x_i, and $x_i\neq x_j$, $i\neq j$. The array is said to contain *distinct* pairs of points if $\nu_i=1$ for all i (for simplicity of notation in this case $y_i:=y_{i0}$); the array is said to contain a *single* point of multiplicity $\nu=N$ if $\theta=1$. In the sequel ν_i, θ are assumed to be finite.

A rational function $y(x)$ is said to *interpolate* the above pairs iff

$$D^{j-1}y(x_i) = y_{i,\,j-1}, \qquad (i,j)\in I, \tag{2.4b}$$

where D denotes derivation with respect to x. Then the array information is

$$P = \left\{\left(x_i;\, y(x_i),\, Dy(x_i),\, D^2(x_i),\dots,\, D^{\nu_i-1}(x_i)\right),\ i\in\underline{\theta}\right\}.$$

Let Q denote the set of x_i's, where each one is listed ν_i times; Q is partitioned arbitrarily into two nonempty subsets S, T called the *row set* and *column set* respectively. The sum of the numbers of times x_i occurs in S and T is equal to ν_i. The elements of S are ordered and denoted by s_i and those of T by t_j:

$$S := \left\{ s_i := x_{i'}: \text{for some } i'\in\underline{\theta},\ i\in\underline{N-r-1}\right\},$$

$$T := \left\{ t_j := x_{j'}: \text{for some } j'\in\underline{\theta},\ j\in\underline{r+1}\right\}.$$

To each such partitioning of Q, we associate a $(N-r-1)\times(r+1)$ matrix denoted by L and referred to as a *Löwner* or *generalized Löwner* matrix according as $\nu_i = 1$ for all i, or $\nu_i > 1$ for some i. To determine the (i,j)th element of L we need to know how many times s_i occurs in the subset $\{s_1,\dots,s_{i-1}\}$ of S, and how many times t_j occurs in the subset $\{t_1,\dots,t_{j-1}\}$ of T; let these two nonnegative integers be k, l respectively. Then if $s_i \neq t_j$,

$$L_{ij} := D_s^k D_t^l \left\{ \frac{y(s)-y(t)}{s-t} \right\}\Bigg|_{s=s_i,\,t=t_j}, \tag{2.5a}$$

where D_x^m denotes the mth derivative with respect to the variable x. If $s_i = t_j$, then

$$L_{ij} = \frac{k!\,l!}{(k+l+1)!} D_t^{(k+l+1)} y(t)\big|_{t=t_j} = \frac{k!\,l!}{(k+l+1)!} y_{i',\,k+l+1}. \tag{2.5b}$$

Notice that any submatrix of a Löwner matrix is a Löwner matrix. This property does not hold for generalized Löwner matrices; only certain submatrices are generalized Löwner matrices.

As mentioned earlier, in Antoulas and Anderson (1986) it is shown that (2.3) holds for generalized Löwner matrices as well. For use in the main theorem we also need to define the Löwner matrix L^* which is constructed from L by rearranging the row and column sets (actually reassigning the last element of the column set to be the last element of the row set) as follows:

$$S^* := S \cup \{t_{r+1}\} = \{s_1, s_2, \ldots, s_{N-r-1}, t_{r+1}\},$$

$$T^* := T - \{t_{r+1}\} = \{t_1, t_2, \ldots, t_r\}.$$

Thus if L is $(N-r-1)\times(r+1)$, it follows that L^* is $(N-r)\times r$.

We are now ready to state the main result.

THEOREM 2.6. *Given the pairs of points* $(x_i, y_{i,j-1})$, $j \in \nu_i$, $i \in \underline{\theta}$, *let* L *be an almost square generalized Löwner matrix* (*i.e.* $r =$ *integer part of* $N/2$). *We denote the rank of* L *by* q.

(a) *If* (a1) *either* N *is even and* $q < r$, *or* N *is odd, and if* (a2) *all* $q \times q$ *Löwner submatrices of* L *and* L^* *are nonsingular, then the minimal-MacMillan-degree rational function* $y^{\min}(x)$ *interpolating the given points satisfies*

$$\deg y^{\min} = q.$$

In this case $y^{\min}$ *is the unique interpolating function of degree* q, *and the degrees of all possible interpolating functions are* $q, N-q, N-q+1, \ldots$.

(b) *If the condition in* (a) *is not satisfied,*

$$\deg y^{\min} = N - q.$$

In this case $y^{\min}$ *is not unique, and the degrees of all possible interpolating functions are* $N-q, N-q+1, \ldots$.

For a proof of this result see Antoulas and Anderson [1986, Theorem (2.26)]. From part (a) of the above theorem follows furthermore

COROLLARY 2.7. *A rational function of degree less than* N *interpolating* $2N-1$ *points is necessarily unique.*

This corollary is also easy to prove from first principles. In order to parametrize all interpolating functions with degree up to a given (admissible) MacMillan degree r, we need to construct from the data an arbitrary Löwner matrix L of size $(N-r-1)\times(r+1)$; for details, see Theorem (2.26) of the abovementioned reference. Then, if

$$Lc=0, \tag{2.8a}$$

and if the inequalities given by (2.8b) are satisfied, in the case of distinct points, the corresponding $y(x)$ is given by Equation (2.2a), which implies

$$y(x)=\frac{n(x,c)}{d(x,c)}=\frac{\sum_{i=1}^{r+1} c_i y_i \prod_{j\neq i}(x-x_j)}{\sum_{i=1}^{r+1} c_i \prod_{j\neq i}(x-x_j)};$$

the dependence of the numerator and of the denominator of $y(x)$ on c is shown explicitly. This dependence turns out to be affine. The inequality condition is as follows. The coefficients c_i are such that the denominator satisfies

$$d(x_i,c)\neq 0, \qquad i\in \underline{N}. \tag{2.8b}$$

Notice that the main theorem guarantees the existence of c's satisfying (2.7a) and (2.7b), provided that r is one of the admissible degrees listed in the theorem. Similar but more involved formulae hold in the case of multiple points; they can be found in Antoulas and Anderson [1986, Equations (2.11a–e), Proposition (2.13)].

3. THE NEVANLINNA-PICK INTERPOLATION PROBLEM

A review of the Nevanlinna-Pick interpolation problem is presented in this section. For details, see e.g. Walsh (1956) for more recent treatments, see Ball (1986), Georgiou (1987), Dym (1988), as well as the references therein. In particular, Chapter 6 of the last reference can be consulted for information on multiple-point Nevanlinna-Pick interpolation.

For $x\in\mathbf{C}$, x^* denotes its complex conjugate. Let

$$y(x)=a_0+a_1x+\cdots+a_nx^n+\cdots \tag{3.1a}$$

be analytic, say, in the closed unit disc. Then

$$y(x)_* := y^*(x^{-1}) := a_0^* + a_1^* x^{-1} + \cdots + a_n^* x^{-n} + \cdots \tag{3.1b}$$

is analytic in the complement of the closed unit disc.

Consider the array P of pairs of points given by (2.4a), and suppose that $|x_i| < 1$ for all i. The *Nevalinna-Pick* interpolation problem is to find a function $y(x)$ such that the interpolation conditions (2.4b) are satisfied, and in addition

$$|y(x)| \leqslant M \tag{3.2}$$

for all x in the closed unit disc, where M is a given positive constant.

The solution of this problem is divided into two parts. The first consists in finding the *admissible* M, i.e. the values of M for which a solution exists; the second consists of computing, for each admissible M, the corresponding solutions. To find the admissible M we set up the so-called *Nevanlinna-Pick matrix*, denoted by NP_M, which is due to Pick. Recall the definition of the set Q from the previous section, as the set of x_i's where each one is repeated ν_i times. We will write

$$Q = \{q_1, q_2, \dots, q_N\},$$

where it is understood that different q_j will take identical values in case any ν_i is greater than 1. Now NP_M is symmetric of size N, and its (i, j)th element is defined as follows:

$$(\mathrm{NP}_M)_{ij} := \mathrm{D}_s^k \mathrm{D}_t^l \left[\frac{M^2 - y(s) y^*(t)}{1 - st} \right] \Bigg|_{s = q_i,\, t = q_j^*}, \tag{3.3}$$

where k is the number of times the value assumed by q_i appears among the values assumed by the first $i-1$ elements of Q, and l is the number of times the value assumed by q_j appears among the values assumed by the first $j-1$ elements of Q. The result of Pick is that for a fixed $M = M_0$ the interpolation problem (2.4b) with (3.2) has a solution if and only if

$$\mathrm{NP}_{M_0} \geqslant 0. \tag{3.4a}$$

For sufficiently large M this condition will always be satisfied (a fact which may not be immediately obvious, but is not hard to prove); actually, there exists a constant M_* such that all admissible M are given by

$$M \geqslant M_* := \min\{M : \mathrm{NP}_M \geqslant 0\}. \tag{3.4b}$$

To construct the solutions for some given admissible M, the *Nevanlinna-Pick algorithm*, due to Nevanlinna, is used. It is a recursive algorithm which can be described as follows. With

$$y^k(x) = \frac{n^k(x)}{d^k(x)}, \qquad k = 0,1,\dots,N-1,N,$$

consider for $k = 1,2,\dots,N$ the relationship

$$\begin{bmatrix} n^{k-1}(x) \\ Md^{k-1}(x) \end{bmatrix} = \begin{bmatrix} 1 & \dfrac{y_k^{k-1}}{M} \\ \dfrac{[y_k^{k-1}]^*}{M} & 1 \end{bmatrix} \begin{bmatrix} x-q_k & 0 \\ 0 & 1-xq_k^* \end{bmatrix} \begin{bmatrix} n^k(x) \\ Md^k(x) \end{bmatrix}, \tag{3.5a}$$

or equivalently the relationship

$$\frac{y^{k-1}(x)}{M} = \frac{(x-q_k)\dfrac{y^k(x)}{M} + (1-xq_k^*)\dfrac{y_k^{k-1}}{M}}{(x-q_k)\dfrac{[y_k^{k-1}]^*}{M}\dfrac{y^k(x)}{M} + (1-xq_k^*)}, \tag{3.5b}$$

or still equivalently, the relationship

$$\frac{y^k(x)}{M} = \frac{1-xq_k^*}{x-q_k}\,\frac{\dfrac{y_k^{k-1}}{M} - \dfrac{y^{k-1}(x)}{M}}{\dfrac{[y_k^{k-1}]^*}{M}\dfrac{y^{k-1}(x)}{M} - 1}. \tag{3.5c}$$

Finally, denote the given values of the interpolating function by y_i^0.

The first part of the Nevanlinna-Pick algorithm consists of using these values, together with (3.5c), in order to construct recursively with k the sequence of values

$$y_m^k \qquad \text{for} \quad k=1,\dots,N-1 \text{ and } m=k+1,\dots,N, \tag{3.6a}$$

as follows:

$$y_m^k := D^\alpha y^k(x)\big|_{x=q_m}, \tag{3.6b}$$

where α is the number of times the value of q_m is repeated in the subset $\{q_{k+1},\dots,q_{m-1}\}$ of Q.

It is slightly nontrivial to see that the quantities y_m^k are recursively computable without the set of functions $y^0(x), y^1(x),\dots$ being known.

For the second part of the Nevanlinna-Pick algorithm, we are given an arbitrary but fixed rational function $y^N(x)$ such that

$$|y^N(x)| \leqslant M \qquad \text{for} \quad |x| \leqslant 1.$$

Making use of (3.5b) for $k=N, N-1,\dots,2,1$, together with the sequence of points y_m^k defined by (3.6a, b) we successively construct the functions

$$y^{N-1}(x), y^{N-2}(x),\dots, y^1(x), y^0(x).$$

Every solution $y(x)$ of the problem at hand is obtained as

$$y(x) := y^0(x), \tag{3.7}$$

using the above algorithm, for some choice of $y^N(x)$ subject to the norm constraint given above.

The Nevanlinna-Pick algorithm has the following property. Let $\Gamma^k := (n^k(x) \;\; Md^k(x))'$; we have

$$\Delta^k(x,x^{-1}) := (\Gamma_*^k)' \operatorname{diag}(-1,1)\,\Gamma^k = M^2 d^k(x)d^k(x)_* - n^k(x)n^k(x)_*. \tag{3.8a}$$

It follows from (3.5a) that

$$\Delta^{k-1}(x, x^{-1}) = \det Y_k^{k-1}\{(x - x_k)(x - x_k)_*\}\, \Delta^k(x, x^{-1}), \qquad (3.8b)$$

where Y_k^{k-1} denotes the first matrix on the right-hand side of the equality sign in Equation (3.5a). Repeated application of the above relationship yields

PROPOSITION 3.9. *If at the last step of the Nevanlinna-Pick algorithm we choose $y^N(x) = 0$, it follows that*

$$\Delta^0(x, x^{-1}) = \gamma \prod_{i \in \underline{\theta - 1}} (x - x_i)^{\nu_i}(x - x_i)_*^{\nu_i}(x - x_\theta)^{\nu_\theta - 1}(x - x_\theta)_*^{\nu_\theta - 1},$$

where $\gamma := M^2 \prod_{i \in \underline{N}} \det Y_i^{i-1}$.

Proof. If $y^N(x) = 0$, it follows that $y^{N-1}(x) = y_N^{N-1}$. Substituting this in (3.5a) implies that

$$\Delta^{N-1}(x, x^{-1}) = M^2 \det Y_N^{N-1}.$$

Repeated application of (3.8b) yields the desired result. ■

The above proposition implies the following property of Δ^0, which will be used in the next section:

$$D_x^{j-1}\, \Delta^0(x, x^{-1})\big|_{x = x_i} = 0, \quad i \in \underline{\theta - 1}, \ j \in \underline{\nu_\theta}; \qquad \text{for } i = \theta, \quad j \in \underline{\nu_\theta - 1}, \qquad (3.10a)$$

from which, in turn, follows that

$$D_x^{j-1}\left[M^2 - y(x)y(x)_*\right]\big|_{x = x_i} = 0, \qquad (3.10b)$$

for the same range of the indices i, j as in (3.10a). This means that all pairs of points belonging to the array P and the corresponding derivatives, except for the last one, lie on the surface defined by

$$y(x)y(x)_* = M^2. \qquad (3.10c)$$

From the above proposition follows the

COROLLARY 3.11. *For the special case where $\theta = 1$, $\nu_1 = N$, and $x_1 = 0$, we have*

$$M^2 - y(x)y(x)_* = \frac{\gamma}{d^0(x)\,d^0(x)_*} = c_k x^{-k} + c_{k+1}x^{-k-1} + \cdots,$$

where $k = \deg d^0(x) < N$.

This corollary will be used in Section 5.3.

4. STABLE INTERPOLATION

Recall the definition of array P from (2.4a), and suppose as in Section 3 that $|x_i| < 1$ for all i. In the sequel we will make use of the *mirror-image array* P_*, attached to P, and defined as follows in terms of a positive constant M.

We first define the index set for P_* as

$$I_* := \left\{(i, j) : i \in \underline{\theta - 1},\ j \in \underline{\nu}_i;\ \text{for } i = \theta,\ j \in \underline{\nu_\theta - 1}\right\}. \tag{4.1a}$$

Assume that all x_i are nonzero. The case where some $x_i = 0$ for some i is discussed in Remark 4.9(b). The corresponding array P_* is

$$P_* := \left\{(x_{*i}, y_{*i,j-1}) : (i, j) \in I_*\right\}; \tag{4.1b}$$

it consists of $N - 1$ pairs of points, where

$$x_{*i} := 1/x_i^*, \tag{4.1c}$$

where $i \in \underline{\theta}$ if $\nu_\theta \neq 1$, and $i \in \underline{\theta - 1}$ otherwise; furthermore

$$y_{*i,j-1} := D_x^{j-1} \frac{M^2}{y^*(x^{-1})}\bigg|_{x = x_{*i}}. \tag{4.1d}$$

Notice that the array P_* is computable from the data in P. In the case where

P contains distinct points, the mirror-image array P_* is

$$P_* := \left\{ \left(\frac{1}{x_i^*}, \frac{M^2}{y_i^*} \right), i \in \underline{N-1} \right\}.$$

We are now ready to state

THEOREM 4.2. *There is a unique rational function $y^{\min}(x)$ of minimal MacMillan degree less than N, interpolating the array of $2N-1$ points*

$$P \cup P_*.$$

Let L be an almost square generalized Löwner matrix constructed using the $2N-1$ pairs of points in the array $P \cup P_*$. The rational function $y^{\min}(x)$ satisfies

$$\deg y^{\min} = \operatorname{rank} L \tag{4.3a}$$

and

$$|y^{\min}(x)| \leqslant M \tag{4.3b}$$

for all x in the closed unit disc and for all $M \geqslant M_*$, where M_* is given by (3.4b).

COROLLARY 4.4. *The function of minimal MacMillan degree interpolating a given array of pairs of points P together with its associated mirror image array P_* is stable for all sufficiently large M.*

Proof of Theorem 4.2. Let $y^{\min}(x)$ be a rational function of minimal MacMillan degree which interpolates the arrays P and P_*.

Recall the Nevanlinna-Pick algorithm discussed in Section 3. From Proposition 3.9 it follows that if we choose $y^N(x)=0$, the resulting function, denoted $y_{\mathrm{NP}}(x) = n_{\mathrm{NP}}(x)/d_{\mathrm{NP}}(x)$, satisfies

$$M^2 d_{\mathrm{NP}}(x)\, d_{\mathrm{NP}}(x)_* - n_{\mathrm{NP}}(x) n_{\mathrm{NP}}(x)_*$$

$$= \gamma \prod_{i \in \underline{\theta - 1}} (x - x_i)^{\nu_i} (x - x_i)_*^{\nu_i} (x - x_\theta)^{\nu_\theta - 1} (x - x_\theta)_*^{\nu_\theta - 1}.$$

By the Nevanlinna-Pick algorithm $y_{\mathrm{NP}}(x)$ is guaranteed to interpolate P. However, as noted at the end of the last section, at every interpolating point in P except the last, there holds

$$y_{\mathrm{NP}}(x)y_{\mathrm{NP}}(x)_* = M^2,$$

or derivatives of this equation, in case of multiple points. This means that the points of P_* are also interpolated by $y_{\mathrm{NP}}(x)$, i.e. $y_{\mathrm{NP}}(x)$ interpolates the points of P and of P_*. Furthermore, by construction

$$\deg y_{\mathrm{NP}}(x) < N.$$

Since by Corollary 2.7 a rational function of degree less than N interpolating $2N-1$ points is unique, we conclude that

$$y^{\min}(x) = y_{\mathrm{NP}}(x).$$

Furthermore, the degree property (4.3a) follows from Theorem 2.6a. This completes the proof. ■

REMARK 4.5. Corollary 4.4 establishes a sufficient condition for the stability of interpolating functions. Actually the above approach yields a one-parameter family of interpolating functions. Clearly, the minimal-norm function interpolating the array P can also be obtained using the Löwner-matrix approach. Therefore, the Nevanlinna-Pick recursive algorithm described in the previous section with $y^N(x)=0$, can be replaced by the general interpolating algorithm described in Section 2, for $M \geqslant M_*$. It is also interesting to notice that the Nevanlinna-Pick algorithm is a special case of the general recursive interpolating algorithm given in Antoulas and Anderson (1986, Section 3). ■

EXAMPLE 4.6. Given an array P containing one point of multiplicity 2 and one of multiplicity 1:

$$P = \left\{\left(\tfrac{1}{2}; 1, 2\right), (0, 0)\right\}.$$

First we set up the Nevanlinna-Pick matrix according to (3.3). We have

$$\mathrm{NP}_M = M^2 \begin{bmatrix} \frac{4}{3}\left(1-\frac{1}{M^2}\right) & \frac{8}{9}\left(1-\frac{4}{M^2}\right) & 1 \\ \frac{8}{9}\left(1-\frac{4}{M^2}\right) & \frac{80}{27}\left(1-\frac{4}{M^2}\right) & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

It follows that $\mathrm{NP}_M \geqslant 0$ if and only if $M \geqslant M_* = 2$. From (4.1), the mirror image array of P is

$$P_* = \left\{ \left(2; M^2, M^2/2\right) \right\}.$$

The generalized Löwner matrix with row set $S = \{ s_1 = \frac{1}{2}; s_2 = 2\}$ and column set $T = \{ t_1 = \frac{1}{2}; t_2 = 2; t_3 = 0\}$ is

$$L = \begin{bmatrix} 2 & \frac{2}{3}(M^2-1) & 2 \\ \frac{2}{3}(M^2-1) & \frac{1}{2}M^2 & \frac{1}{2}M^2 \end{bmatrix}.$$

For $M = M_*$, we have $\deg y^{\min} = 1$. Otherwise, for $M > M_*$, we have $\deg y^{min} = 2$. Actually, in this case

$$y^{\min}(x) = \frac{-6M^2 x\left(x-\frac{5}{4}\right)}{\{(M-2)x-(2M-1)\}\{(M+2)x-(2M+1)\}}.$$

It is readily checked that the above rational function is a one-parameter family of stable functions interpolating the points in the array P. Actually, as predicted by Theorem 4.2, for each $M \geqslant M_*$, the norm of the function in the closed unit disc is bounded by M.

Remark 4.7. The Nevanlinna-Pick algorithm with $y^N(x) = 0$ can be modified to yield an $N-1$-parameter family of stable interpolating functions. To do this, we let M depend on k in the formulas (3.5a) through (3.5c). After the N steps $y^0(x)$ depends on the parameters

$$M_1, M_2, \ldots, M_{N-1}.$$

It readily follows that if these parameters are chosen so that

$$M_1 \geqslant M_2 \geqslant \cdots \geqslant M_{N-1} \geqslant M_*,$$

the interpolating function is stable and satisfies

$$|y^0(x)| \leqslant M_1.$$

EXAMPLE 4.8. Consider the array P containing three distinct points:

$$P = \{(-\tfrac{1}{2}, 1), (0, -1), (\tfrac{1}{2}, 1)\}.$$

The associated mirror image array is

$$P_* = \{(-2, M^2), (\infty, -M^2)\}.$$

The rational function of minimal MacMillan degree interpolating P and P_*, as discussed in Theorem 4.2, turns out to be $y(x) = n(x)/d(x)$, where

$$n(x) = n_2 x^2 + n_1 x + n_0, \qquad d(x) = d_2 x^2 + d_1 x + d_0,$$

$$d_2 = -8(M^2 - 1), \qquad d_1 = -\tfrac{1}{2}(65M^2 + 1)(63M^2 - 1), \qquad d_0 = M^4 - 1,$$

$$n_2 = -M^2 d_2, \qquad n_1 = d_1, \qquad n_0 = -d_0.$$

A sufficient condition for stability is

$$M \geqslant M_* = 4 + \sqrt{15}.$$

To generate the two-parameter family discussed in the remark above we apply (3.5a) for $k=1,2$. We obtain

$$n_2=\frac{8M_1^2}{M_1^2+1}-\left(\frac{4M_1^2}{M_2(M_1^2+1)}\right)^2, \qquad d_2=\frac{8}{M_1^2+1}-\left(\frac{4M_1^2}{M_2(M_1^2+1)}\right)^2,$$

$$n_1=d_1=\frac{1}{2}\left[1-\left(\frac{8M_1^2}{M_2(M_1^2+1)}\right)^2\right],$$

$$n_0=-d_0=-\frac{M_1^2-1}{M_1^2+1}.$$

The sufficient condition for stability as discussed above is

$$\frac{8M_1^2}{M_1^2+1}\leqslant M_2 \leqslant M_1.$$

This concludes the example.

REMARK 4.9.

(a) The family of stable interpolating functions obtained using the above parametrizations is a proper subset of the family of all stable interpolating functions. It is an open question, for example, how to obtain the stable interpolating function of minimal degree.

(b) If $x_i=0$ for some i, the interpolating function will necessarily be of the form

$$y(x)=y_i+x\bar{y}(x)$$

for an appropriate rational function $\bar{y}(x)$. An equivalent interpolation problem can then be set up in terms of this new function $\bar{y}(x)$. If, for example, the only x which is zero is the ith, and there are no multiplicities, the new interpolation conditions are

$$\bar{y}_j:=\frac{y_j-y_i}{x_j} \qquad \text{for} \quad j\neq i.$$

This means that interpolating points will include all but the ith. When x_i has multiplicity ν_i, then

$$y(x) = p(x) + x^{\nu_i}\bar{y}(x),$$

where $p(x)$ is a polynomial of degree $k-1$, and an interpolating problem in terms of $\bar{y}(x)$ can be formed.

(c) Let L be the Löwner matrix constructed using the points of the array P as the column set, and the points of the array P_* as the row set. There is a close connection between L and the first $N-1$ rows of the Nevanlinna-Pick matrix, which we will denote by $(\mathrm{NP}_M)'$, constructed using the array P. For the simple case of distinct points, this relationship is

$$(\mathrm{NP}_M)' = \Delta L, \tag{4.10a}$$

where

$$\Delta := \mathrm{diag}(y_1^*/x_1^*, \dots, y_{N-1}^*/x_{N-1}^*). \tag{4.10b}$$

Therefore, if all x_i, y_i are different from zero, any submatrix of $(\mathrm{NP}_M)'$ and the corresponding submatrix of L will have the same rank. This similarity between the Löwner and the Nevanlinna-Pick matrices has also been observed in Belevitch (1970, Section 9). A similar result holds in the case of multiple points.

(d) In Section 10 of Belevitch (1970), a preliminary version of Theorem 4.2 and Corollary 4.4 is given, with left-half-plane analyticity and a positivity property for the interpolating functions.

(e) A method for constructing the solutions to the Nevanlinna-Pick problem, which bears some similarities with the procedure given in this section, can be found in Krein and Nudelman (1977, Chapter V). The following important differences between the two constructions should be pointed out. When $\mathrm{NP}_M > 0$, the size of the system of linear equations which has to be solved in the approach described in the abovementioned reference is always equal to the number of interpolating points, i.e. N; in our approach the size is equal to the rank of the Löwner matrix, which is always less than N. When $\mathrm{NP}_M \geqslant 0$, in order to obtain the corresponding (unique) interpolating function of minimal degree, Krein and Nudelman make use of a different procedure from the one used when $\mathrm{NP}_M > 0$; in our framework both the definite and semidefinite cases are treated the same way.

5. APPLICATIONS

5.1. *The Two-Point Interpolation Problem*

Recall the notation established in Section 2. In this subsection we will investigate the special case of the general rational-interpolation problem where $\theta = 2$. This means that we are given two points of multiplicity $\nu_1 \leqslant \nu_2$ respectively. We set up the corresponding Löwner matrix L by choosing S so that it contains all ν_1 copies of the first point and as many copies of the second as necessary to make the number of elements of S and T approximately the same. For simplicity of notation, assume in the following that $\nu_1 = \nu_2$. In this case S contains ν_1 copies of the first point and T contains the same number of copies of the second point. It follows that

$$L_{ij} = D_s^{i-1} D_t^{j-1} \phi(s,t)\big|_{s=x_1,\, t=x_2}, \tag{5.1a}$$

where

$$\phi(s,t) := \frac{y(s) - y(t)}{s - t}, \tag{5.1b}$$

and

$$D_s^k y(s)\big|_{s=x_1} = y_{1,k}, \qquad D_t^l y(t)\big|_{t=x_2} = y_{2,l}. \tag{5.1c}$$

The Löwner matrix L defined above has the following property.

PROPOSITION 5.2.

$$L_{ij} = \frac{(j-1)L_{i,j-1} - (i-1)L_{i-1,j}}{s-t}.$$

Proof. The proof is by induction on i, j. For $i, j \leqslant 2$, the result follows by direct computation. Otherwise, suppose that

$$(s-t)L_{ij} = (j-1)L_{i,j-1} - (i-1)L_{i-1,j}.$$

Differentiating with respect to s and using (5.1a), we obtain

$$L_{ij} + (s-t)L_{i+1,j} = (j-1)L_{i+1,j-1} - (i-1)L_{ij}.$$

This implies

$$(s-t)L_{i+1,j}=(j-1)L_{i+1,j-1}-iL_{ij},$$

which proves the result with respect to the index i. Differentiating this relationship with respect to j, and using (5.1a) again, we obtain

$$-L_{i+1,j}+(s-t)L_{i+1,j+1}=(j-1)L_{i+1,j}+iL_{i,j+1},$$

which after solving for $L_{i+1,j+1}$ yields the desired result. ∎

Therefore, there exist triangular matrices Λ_1, Λ_2 such that

$$\Lambda_1 L \Lambda_2 = H, \tag{5.3}$$

where H is a Töplitz matrix (i.e. a Hankel matrix, up to appropriate row permutations), Λ_1 is upper triangular, and Λ_2 is lower triangular.

EXAMPLE 5.4. For $\nu_1=\nu_2=4$, H defined by (5.3) is given as follows.

The Löwner matrix of the two-point problem with row set $T=\{t,t,t,t\}$ and column set $S=\{s,s,s,s\}$ is, according to (5.1a),

$$L=\begin{bmatrix} \phi=\dfrac{y(s)-y(t)}{s-t} & \phi_s=\dfrac{\phi-y_s}{t-s} & \phi_{ss}=\dfrac{2\phi_s-y_{ss}}{t-s} & \phi_{sss}=\dfrac{3\phi_{ss}-y_{sss}}{t-s} \\ \phi_t=\dfrac{\phi-y_t}{s-t} & \phi_{ts}=\dfrac{\phi_t-\phi_s}{t-s} & \phi_{tss}=\dfrac{2\phi_{ts}-\phi_{ss}}{t-s} & \phi_{tsss}=\dfrac{3\phi_{tss}-\phi_{sss}}{t-s} \\ \phi_{tt}=\dfrac{2\phi_t-y_{tt}}{s-t} & \phi_{tts}=\dfrac{\phi_{tt}-2\phi_{ts}}{t-s} & \phi_{ttss}=\dfrac{2\phi_{tts}-2\phi_{tss}}{t-s} & \phi_{ttsss}=\dfrac{3\phi_{ttss}-2\phi_{tsss}}{t-s} \\ \phi_{ttt}=\dfrac{3\phi_{tt}-y_{ttt}}{s-t} & \phi_{ttts}=\dfrac{\phi_{ttt}-3\phi_{tts}}{t-s} & \phi_{tttss}=\dfrac{2\phi_{ttts}-3\phi_{ttss}}{t-s} & \phi_{tttsss}=\dfrac{3\phi_{tttss}-3\phi_{ttsss}}{t-s} \end{bmatrix}.$$

Here indices are used to denote derivatives with respect to the index variable. The transformation matrices are

$$\Lambda_1=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & t-s & 0 & 0 \\ 1 & 2(t-s) & \frac{1}{2}(t-s)^2 & 0 \\ 1 & 3(t-s) & \frac{3}{2}(t-s)^2 & \frac{1}{6}(t-s)^3 \end{bmatrix}$$

and

$$\Lambda_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & s-t & 2(s-t) & 3(s-t) \\ 0 & 0 & \frac{1}{2}(s-t)^2 & \frac{3}{2}(s-t)^2 \\ 0 & 0 & 0 & \frac{1}{6}(s-t)^3 \end{bmatrix}.$$

It follows that

$$\Lambda_1 L \Lambda_2 = H,$$

where

$$H = \begin{bmatrix} \phi & y_s & y_s + y_{ss}\dfrac{s-t}{2} & y_s + y_{ss}(s-t) + y_{sss}\dfrac{(s-t)^2}{6} \\ y_t & \phi & y_s & y_s + y_{ss}\dfrac{s-t}{2} \\ y_t + y_{tt}\dfrac{t-s}{2} & y_t & \phi & y_s \\ y_t + y_{tt}(t-s) + y_{ttt}\dfrac{(t-s)^2}{6} & y_t + y_{tt}\dfrac{t-s}{2} & y_t & \phi \end{bmatrix}$$

Notice that H has Töplitz structure.

According to Theorem 2.6, in order to determine the degree of the minimal interpolating functions, we have to look for appropriate submatrices of L and therefore H. Let

$$q = \operatorname{rank} H.$$

H_1 is defined to be the leading principal square submatrix of H of size q, i.e., it is composed of rows 1 through q and of columns 1 through q. Likewise, H_2 is the submatrix of H composed using rows 1 through q but columns 2 through $q+1$.

THEOREM 5.5. *We have*

$$\deg y^{\min}(x) = q,$$

provided that H_1, H_2 defined above are nonsingular. Otherwise

$$\deg y^{\min}(x) = N - q.$$

Proof. In order to check the nonsingularity condition of Theorem 2.6a in the special case where $\theta = 2$, we need to check at most two matrices. We will prove this result using the simplifying assumption $\nu_1 = \nu_2$ (cf. the beginning of this subsection).

Let rank $L = q$, and let $\bar{L}$ be a $q \times (q+1)$ generalized Löwner submatrix of L having full rank q. The submatrix $\bar{L}$ contains one $q \times q$ generalized Löwner submatrix, namely the one formed by the first q columns and rows of $\bar{L}$; it will be denoted by L_1. The matrix $\bar{L}^*$, derived from $\bar{L}$ as defined just before Theorem 2.6, furthermore contains one additional Löwner submatrix, namely the one composed of rows 1 through $q-1$ and $q+1$, and columns 1 through q; thus submatrix will be denoted by L_2.

By transforming the matrix L into H via (5.3), it is readily checked that L_1 is transformed into H_1. A transformation similar to (5.3) can also be applied to $\bar{L}^*$, which transforms L_2 into H_2. ■

5.2. *Realization of Markov Parameters and Moments*

Given a function $g(x)$, assumed *real* for simplicity, which is bounded at infinity, its *Markov parameters*, denoted by a_t, are defined by the Laurent expansion

$$g(x) = \sum_{t \geqslant 0} a_t x^{-t}, \qquad |x| > R_1 > 0,$$

valid outside a disc of radius R_1. The *moments* of $g(x)$, denoted by b_t, are defined by the Taylor series expansion

$$g(x) = \sum_{t \geqslant 0} b_t x^t, \qquad |x| < R_2,$$

valid inside a disc of radius R_2. The problem of *simultaneous* realization of Markov parameters and moments is the following. We are given $a_0, a_1, \dots, a_k$ and $b_0, b_1, \dots, b_m$. Parametrize all rational functions $y(x)$ matching the above sets of a_i's and b_j's. For realization-theory-type approaches to this problem, see Van Barel and Bultheel (1986) and Bitmead and Skelton (1987).

The approach we will follow here is to transform this problem to a two-point interpolation problem as follows. Consider a bilinear transformation, which transforms the variable x into $\bar{x}$:

$$\bar{x}(x) = \frac{\alpha x - \beta}{\gamma x - \delta}, \qquad \text{i.e.} \quad x = \frac{\delta \bar{x} - \beta}{\gamma \bar{x} - \alpha}, \tag{5.6a}$$

where $\gamma \neq 0$, $\delta \neq 0$, and

$$\Delta = \beta\gamma - \alpha\delta \neq 0. \tag{5.6b}$$

Let $\bar{y}(\bar{x})$ be defined as follows:

$$\bar{y}(\bar{x}(x)) := y(x). \tag{5.7}$$

It is readily checked that the Markov parameters are thus transformed into interpolation conditions on the function $\bar{y}(\bar{x})$ at the point $\bar{x} = \alpha/\gamma$, while the moments are transformed into interpolation conditions on the same function at the point $\bar{x} = \beta/\delta$. In particular:

$$\bar{y}(\alpha/\gamma) = a_0,$$

$$D\bar{y}(\alpha/\gamma) = -\frac{\gamma^2}{\Delta} a_1,$$

$$D^2\bar{y}(\alpha/\gamma) = \frac{2\gamma^3}{\Delta^2}(\gamma a_2 - \delta a_1),$$

$$D^3\bar{y}(\alpha/\gamma) = \frac{-6\gamma^4}{\Delta^3}\left(\gamma^2 a_3 - 2\delta\gamma a_2 + \delta^2 a_1\right);$$

in general,

$$D^k\bar{y}(\alpha/\gamma) = \frac{k!(-\gamma)^{k+1}}{\Delta^k}\left[c_{k-1}\gamma^{k-1}a_k - c_{k-2}\gamma^{k-2}\delta a_{k-1} + \cdots + (-1)^{k-1}c_0\delta^{k-1}a_1\right], \tag{5.8a}$$

where $c_i = (k-1)!/(k-i-1)!i!$, $i = 0, 1, \ldots, k-1$, are the binomial coefficients. Similarly

$$\bar{y}(\beta/\delta) = b_0,$$

$$D\bar{y}(\beta/\delta) = \frac{\delta^2}{\Delta} b_1,$$

$$D^2\bar{y}(\beta/\delta) = \frac{2\delta^3}{\Delta^2}(\delta b_2 - \gamma b_1),$$

$$D^3\bar{y}(\beta/\delta) = \frac{6\delta^4}{\Delta^3}(\delta^2 b_3 - 2\gamma\delta b_2 + \gamma^2 b_1);$$

in general,

$$D^k\bar{y}(\beta/\delta) = \frac{k!\delta^{k+1}}{\Delta^k}\left[c_{k-1}\delta^{k-1}b_k - c_{k-2}\delta^{k-2}\gamma b_{k-1} + \cdots + (-1)^{k-1}c_0\gamma^{k-1}b_1\right]. \tag{5.8b}$$

Substituting the above values in the matrix H defined by (5.3), we obtain the Töplitz matrix

$$\pi\overline{H} = \Delta_1 H \Delta_2,$$

where

$$\Delta_1 = k_1\Gamma(\alpha, \gamma) = k_1 \operatorname{diag}\left(1, \frac{\alpha}{\gamma}, \frac{\alpha^2}{\gamma^2}, \ldots\right),$$

$$\Delta_2 = k_2\Gamma(\beta, \delta), \quad \text{and} \quad k_1 k_2 = \frac{\alpha\gamma}{\Delta}.$$

$\overline{H}$ is an almost square Hankel matrix (π denotes an appropriate row permutation transformation); this Hankel matrix is composed of the sequence

$$(b_k, b_{k-1}, \ldots, b_1, b_0 - a_0, -a_1, \ldots, -a_l), \tag{5.9a}$$

that is,

$$\bar{H}=\begin{bmatrix} b_k & b_{k-1} & b_{k-2} & & & \\ b_{k-1} & b_{k-2} & & & & \\ b_{k-2} & & & & & \\ & & & \ddots & & \\ & & & & & -a_{l-2} \\ & & & & -a_{l-2} & -a_{l-1} \\ & & & -a_{l-2} & -a_{l-1} & -a_l \end{bmatrix}. \tag{5.9b}$$

It should be noticed that $\bar{H}$ can be written as

$$\bar{H}=\Gamma_b-\Gamma_a, \tag{5.9c}$$

where Γ_a, Γ_b are Hankel matrices of the same size as $\bar{H}$, which are constructed from the following two sequences;

$$(b_k, b_{k-1}, \ldots, b_0, 0, \ldots, 0) \quad \text{and} \quad (0, \ldots, 0, a_0, a_1, \ldots, a_l), \tag{5.9d}$$

respectively. To determine the degree of the minimal interpolating functions, we have to determine the rank of $\bar{H}$ and check the two minors defined in the previous subsection.

Example 5.10. For the example of two points of multiplicity four each, discussed earlier in Example 5.4, $\bar{H}$ turns out to be

$$\bar{H}=\begin{bmatrix} b_3 & b_2 & b_1 & b_0-a_0 \\ b_2 & b_1 & b_0-a_0 & -a_1 \\ b_1 & b_0-a_0 & -a_1 & -a_2 \\ b_0-a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix}.$$

This concludes the example.

5.3. *Stable Realization*

Combining the results of the previous subsection together with the results of Section 4, we can derive a class of stable realizations. Given a set of Markov parameters, we will manufacture a set of moments. Then we will solve an *unconstrained* two-point interpolation problem. The result will yield a family of stable realizations of the original Markov parameters.

First notice that with $y(x), \bar{y}(\bar{x})$ as denoted in Section 5.2, in order to preserve boundedness properties the bilinear transformation (5.6a) must have the form

$$\bar{x}(x) = \frac{\alpha x - \gamma}{\gamma x - \alpha}. \tag{5.11a}$$

By Theorem 4.2 and Corollary 4.3, to assure stability, the set of points defined by (5.8b) must be the mirror image set of the set of points (5.8a), and in addition

$$\left|\frac{\alpha}{\gamma}\right| < 1. \tag{5.11b}$$

This implies

$$|\bar{y}(\bar{x})| \leqslant M \quad \text{for } |\bar{x}| \leqslant 1 \quad \text{iff} \quad |y(x)| \leqslant M \quad \text{for } |x| \geqslant 1. \tag{5.12}$$

From Corollary 3.11 follows

$$M^2 - y(x^{-1})y^*(x) = c_{N-1}x^{N-1} + c_N x^N + \cdots.$$

Therefore, for the set of moments $b_0, \dots, b_{N-2}$ to be the mirror image of the set of Markov parameters $a_0, \dots, a_{N-1}$, or equivalently, for the point set (5.8b) to be the mirror image of the point set (5.8a), the following conditions have to be satisfied for $M \geqslant M_* > 0$:

$$a_0 b_0 = M^2, \qquad \sum_{i+j=k} a_i b_j = 0, \quad k \in \underline{N-2}. \tag{5.13a}$$

Let Γ_b, Γ_a be two square Hankel matrices of size $N-1$, constructed from the data using (5.9c). The relationship (5.13a) can be expressed as follows:

$$\Gamma_b \Gamma_a = M^2 I, \tag{5.13b}$$

where I is the identity matrix of size $N-1$. We thus have the following result.

THEOREM 5.14. *The unconstrained minimal realization of the sequence* (5.9a), *where* $l = k+1 = N-1$, *and the* a*'s and* b*'s satisfy* (5.13b), *is necessarily stable, i.e. has zeros inside the unit disc, for* $M \geqslant M_*$.

To determine M_* we set up the corresponding Nevanlinna-Pick matrix NP_M at the point $\hat{x} = \alpha/\gamma$, according to (3.3). This turns out to be equal to

$$\mathrm{NP}_M = \Lambda^*\left(M^2 - \Gamma_a^2\right)\Lambda, \tag{5.15a}$$

where Λ is an upper triangular nonsingular matrix, and Λ^* is the complex conjugate transpose of Λ. Now M_* is the smallest value of M for which NP_M is positive semidefinite. It follows from the above relationship that

$$M_* = \left(\lambda_{\max}\left(\Gamma_a^2\right)\right)^{1/2}, \tag{5.15b}$$

where $\lambda_{\max}$ denotes the largest eigenvalue of the corresponding matrix. This result checks with Carathéodory-Féjer result, where instead of matching the first N coefficients of the power-series expansion of a certain function at *zero*, the same number of coefficients of the power-series expansion at *infinity* are matched (see Rosenblum and Rovnyak, 1985, Section 2.5). It is also interesting to notice that (5.15a) provides a connection between the Nevalinna-Pick matrix and the corresponding Hankel matrix, namely

$$\mathrm{NP}_M = \Lambda^* \overline{H} \Gamma_a \Lambda.$$

We conclude this section with the following illustrative

EXAMPLE 5.16. Consider the Markov parameters

$$a_0 = -2, \qquad a_1 = -3, \qquad a_2 = -6.$$

The corresponding minimal realization (computed without stability constraint) is $(x - \frac{1}{2})/(1 - \frac{1}{2}x)$, which is not bounded outside the closed unit disc. In order to find a realization $y(x)$ satisfying the boundedness property (5.12), we proceed as follows. From (5.13), the mirror-image moments of the above three Markov parameters have to be the two moments

$$b_0 = -\tfrac{1}{2}M^2, \qquad b_1 = \tfrac{3}{4}M^2.$$

From (5.15) it follows that

$$M_* = 8.$$

By (5.9a, b), the corresponding Hankel matrix is

$$\overline{H} = \begin{bmatrix} \frac{3}{4}M^2 & 2-\frac{1}{2}M^2 & 3 \\ 2-\frac{1}{2}M^2 & 3 & 6 \end{bmatrix}.$$

The resulting one-parameter family of stable functions (i.e. functions bounded in the complement of the unit disc) which realize the given three Markov parameters is

$$y(x) = -\frac{2(M^2-16)x^2+3M^2x+6M^2}{(M^2-16)x^2+24x+12}.$$

For all $M \geqslant 8$, this function is bounded by M in the complement of the closed unit disc, as predicted by the theory developed above. For $M = M_*$, $y(x)$ is *all-pass*. This is a well-known consequence of the Nevanlinna-Pick theory; see e.g. Rosenblum and Rovnyak (1985).

The authors wish to thank the reviewers for their helpful comments.

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Received August 1988; final manuscript accepted 21 November 1988