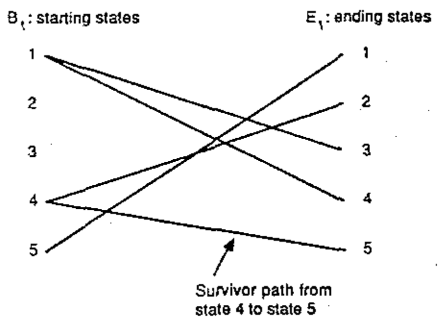

Fig. 3. An example of an FTB code to illustrate step 6) of Algorithm 2. Orders of B_t and E_t are both 5. According to step 6), states 1 and 4 should be deleted from B_t and E_t. Whether state 5 is deleted or not is arbitrary.

TABLE I
SIMULATION RESULTS

BSC p	E_b/N_0 dB	Optimum MLD BER	Optimum MLD VT	Bar-David VT	Two-Step VT
0.0449	4.588	8.71×10^{-3}	14.67	116.21	67.62
0.041	4.812	5.45×10^{-3}	13.41	72.1	60.17
0.0371	5.032	1.62×10^{-3}	5.76	67.65	58.24
0.0293	5.535	4.63×10^{-4}	2.3	62.25	54.27
0.0215	6.123	2.1×10^{-5}	1.48	56.79	47.11
0.0195	6.298	8×10^{-6}	1.44	47.32	41.32

Simulation results are summarized in Table I, along with some of the simulation results obtained by Ma and Wolf for a code of the same rate and memory length over the same channel and decoded by the Bar-David algorithm and the two-step algorithm [1].

From Table I, we can see that the MLD algorithm (Algorithm 2) is much more efficient than the other two suboptimum algorithms in terms of the number of VT's needed, and there is no sacrifice of BER performance to achieve this efficiency since we have proved the algorithm is an MLD algorithm. In fact, for this example, when $p = 2.15 \times 10^{-2}$, fewer than 1.5 VT's are needed on average, very close to the asymptotic situation for which only 1 VT is needed. In this case, the path highlighted in Fig. 2 is the most likely path because any other path requires at least 2 VT's. Note that as E_b/N_0 increases to infinity, the number of VT's will decrease from 1.44 to 1, which is the asymptotic limit. In general, the efficiency of the algorithm depends on both channel quality and the code used. Thus, for a properly designed code, the decoding algorithm should be as efficient as shown in the above example.

IV. Concluding Remarks

An efficient algorithm is presented for MLD of GTB codes including QC codes. The complexity of the algorithm is shown to be asymptotically equal to that of the Viterbi algorithm, which is verified by simulation results. Depending on the implementation of the algorithm, some simplifications may be possible and/or necessary, which may result in various suboptimum algorithms.

Just as with good convolutional codes for use with the Viterbi algorithm, GTB codes suitable for use with the proposed algorithm should have short total memory length. In construction of GTB codes, we have a parameter, i.e., the code length when they are considered as block codes, over which the optimization might be performed. Whether this parameter is substantially important in the construction of GTB codes is unknown. This and the development of suboptimum decoding algorithms for easier implementation are suggested for further work.

Acknowledgment

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When Has a Decision-Directed Equalizer Converged?

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Abstract—**Adaptive decision-directed equalizers can become locked for long periods onto incorrect equilibria. This note provides a test involving data available at the equalizer output for determining whether an equilibrium is correct or not, up to a fixed overall delay. The key result is the following. If an independent sequence of random variables taking values ±1 is the input to a finite impulse response filter, and the output of the filter is passed through a slicer, then the slicer output is uncorrelated if and only if the slicer output is a delayed version of the filter input. Determination of the actual delay parameter is a separate issue and not addressed. An analogous result for *M*-ary rather than binary data is outlined.**

I. Introduction

The behavior of decision-directed equalizers after adaptation, or with the adaptation mechanism switched off, can be described as follows (see Fig. 1). The input sequence $\{a_k\}$ to the channel is a sequence of independent random variables taking values ± 1 with equal probability. (Extension to the M-ary case will also be outlined.) The channel has impulse response $\{h_0, h_1, \cdots\}$ and we shall assume that $h_j = 0$ for $j > N_1$. (Extensions to the infinite impulse response case could also be considered.) The channel output sequence $\{y_k\}$ is

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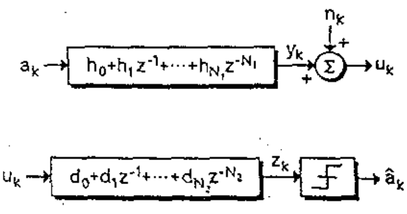

Fig. 1. Channel and decision-directed equalizer models.

accordingly given by

$$y_k = \sum_{i=0}^{N_1} h_i a_{k-i}. \tag{1.1}$$

In the absence of noise, this output drives the equalizer, which is a finite impulse response filter of response $\{d_0, \cdots, d_{N_2}\}$ followed by a slicer. (Note, in practice, one would choose N_2 sufficiently large, i.e., conservatively, on physical grounds.) Thus, the slicer input sequence $\{z_k\}$ is

$$z_k = \sum_{j=0}^{N_2} d_j y_{k-j} \tag{1.2}$$

and the equalizer output is

$$\hat{a}_k = \text{sgn}\ (z_k). \tag{1.3}$$

Correct operation of the equalizer is characterized by

$$\hat{a}_k = a_k \forall k \quad \text{or} \quad \hat{a}_k = -a_k \forall k \tag{1.4a}$$

or, for some fixed integer $\delta > 0$,

$$\hat{a}_k = a_{k-\delta} \forall k \quad \text{or} \quad \hat{a}_k = -a_{k-\delta} \forall k. \tag{1.4b}$$

We note that in practical operation, noise n_k must usually be assumed to add into (1.1) as in Fig. 1. However, a property of this form of equalizer is that it provides a certain margin against the noise due to the insensitivity of the signum function, i.e.,

$$\text{sgn}\left(\sum_{j=0}^{N_2} d_j y_{k-j}\right) = \text{sgn}\left(\sum_{j=0}^{N_2} d_j (y_{k-j} + n_{k-j})\right)$$

for many specific noise sequences. We defer consideration of noisy channels until Section III. Until then, we will take $n_k = 0$.

In decision-directed equalizers, the coefficients $\{d_j\}$ are adjusted by an adaptive algorithm [1]–[3], the details of which will not concern us here. One key property of these algorithms is, however, that convergence of the $\{d_k\}$ can occur to an undesirable setting; finite-step size effects in the adaptive algorithm mean that eventually such a setting would be left, but it would clearly be helpful to have some way of knowing whether the output of an equalizer which had appeared to converge was or was not correct.

This leads us to formulate the following problem. Let $\{l_0, \cdots, l_N\}$ where $N = N_1 + N_2$, be the convolution of $\{h_j\}$ and $\{d_j\}$. Given measurements $\{\hat{a}_k\}$ generated by

$$\hat{a}_k = \text{sgn}\left(\sum_{j=0}^{N} l_j a_{k-j}\right)$$

where the l_j are unknown but fixed, the $\{a_k\}$ are unmeasured, but known to be independent, and taking values ± 1 with equal probability, is there a test on the $\{\hat{a}_k\}$ which would determine whether or not (1.4) held?

Obviously, if (1.4) holds, then the $\{\hat{a}_k\}$ sequence is itself independent, so that independence of the measurements is a necessary condition for (1.4) to hold for some $\delta \in \{0, \cdots, N\}$. The main result of this paper is that this independence property is also sufficient for (1.4) to hold. This is established in Section II.

Notice that independence is a property that can, at least approximately, be readily checked. In this problem, as we wil see, it is a nontrivial fact that checking just a finite number of correlations $E[\hat{a}_k \hat{a}_{k-j}]$, $j \in \{1, \cdots, N\}$ implies full independence, thus considerably simplifying the testing.

We briefly expand the ideas to consider the effects of channel noise in Section III, and outline how the results generalize to M-ary transmission in Section IV.

II. Mathematical Statement and Proof of Main Result

The main result established here is the following.

Theorem 1: Let $\{a_k\}$ be an independent sequence of random variables taking values ± 1 with equal probability. Suppose that for some constants $\{l_0, \cdots, l_N\}$

$$\hat{a}_k = \text{sgn}\left(\sum_{i=0}^{N} l_i a_{k-i}\right) \tag{2.1}$$

and $E[\hat{a}_k \hat{a}_{k-j}] = 0$ for $j \in \{1, \cdots, N\}$. Suppose further than the l_i are such that the argument of the sign function in (2.1) can never take the value zero. Then for some $\delta \in \{0, \cdots, N\}$, there holds

$$\hat{a}_k = \text{sgn}\ (l_\delta) a_{k-\delta}\ \forall \{a_k\}. \tag{2.2}$$

Proof: Let us define integers, $I, J \in \{0, \cdots, N\}$ (which we can think of as labeling the first and last important impulse response coefficients) by the following requirements:

$$\hat{a}_k = \text{sgn}\left(\sum_{i=I}^{J} l_i a_{k-i}\right) = \text{sgn}\left(\sum_{i=0}^{N} l_i a_{k-i}\right) \quad \forall \{a_k\} \tag{2.3a}$$

while

$$\text{sgn}\left(\sum_{i=I+1}^{J} l_i a_{k-i}\right) \neq \text{sgn}\left(\sum_{i=I}^{J} l_i a_{k-i}\right) \quad \text{for some } \{a_k\} \tag{2.3b}$$

and

$$\text{sgn}\left(\sum_{i=I}^{J-1} l_i a_{k-i}\right) \neq \text{sgn}\left(\sum_{i=I}^{J} l_i a_{k-i}\right) \quad \text{for some } \{a_k\}. \tag{2.3c}$$

In terms of the problem statement of Section I, one can regard $\{l_0, \cdots, l_{I-1}\}$ as very small precursors and $\{l_{J+1}, \cdots, l_N\}$ as very small postcursors in the combined channel-equalizer impulse response.

Observe that if $I = J$, then

$$\hat{a}_k = \text{sgn}\left(\sum_{i=0}^{N} l_i a_{k-i}\right) = \text{sgn}\ (l_I a_{k-I}) = \text{sgn}\ (l_I) a_{k-I} \tag{2.4}$$

and we have the desired result. In the remainder of the proof, we shall show that $I < J$ contradicts the uncorrelatedness of the $\{\hat{a}_k\}$ sequence.

Notice that under (2.3)

$$\hat{a}_k = \text{sgn}\left(\sum_{i=I}^{J} l_i a_{k-i}\right) \tag{2.5}$$

which is the sign of an inner product of $[l_I \cdots l_J]'$ with the vector $X_k \triangleq [a_{k-I}a_{k-I-1} \cdots a_{k-J}]'$. Similarly,

$$\hat{a}_{k+J-I}=\operatorname{sgn}\left(\sum_{i=I}^{J} l_i a_{k-i+J-I}\right) \quad (2.6)$$

and this depends on the vector $X_{k+J-I} = [a_{k+J-2I}a_{k+J-2I-1} \cdots a_{k-I}]'$. The only entry in common between X_{k+J-I} and X_k is a_{k-I}; all other entries of the vectors are mutually independent. So, in a rough manner of speaking, any dependence between $\hat{a}_{k+J-I}$ and $\hat{a}_k$ can only arise though a_{k-I}. Let us now formalize this statement, and in the process show that uncorrelatedness for the $\{\hat{a}_k\}$ sequence is lost. We have

$$\begin{aligned}\Pr(\hat{a}_{k+J-I}=1|\hat{a}_k=1)=&\Pr(\hat{a}_{k+J-I}=1|a_{k-I}=1,\\ &\hat{a}_k=1)\Pr(a_{k-I}=1|\hat{a}_k=1)\\ &+\Pr(\hat{a}_{k+J-I}=1|a_{k-I}=-1,\\ &\hat{a}_k=1)\Pr(a_{k-I}=-1|\hat{a}_k=1).\end{aligned} \quad (2.7)$$

Now (2.5) and (2.6) at once show that some of the conditioning can be omitted because, for example,

$$\begin{aligned}&\Pr(\hat{a}_{k+J-I}=1|a_{k-1}=-1,\ \hat{a}_k=1)\\ &\equiv \Pr\left(\sum_{i=I}^{J} l_i a_{k+J-I-i}>0 \,\middle|\, a_{k-I}=-1,\right.\\ &\left.\sum_{i=I}^{J} l_i a_{k-i}>0\right).\end{aligned} \quad (2.8)$$

Therefore, (2.7) becomes

$$\begin{aligned}\Pr(\hat{a}_{k+J-I}=1|\hat{a}_k=1)=&\Pr(\hat{a}_{k+J-I}=1|a_{k-I}=1)\\ &\cdot\Pr(a_{k-I}=1|\hat{a}_k=1)\\ &+\Pr(\hat{a}_{k+J-I}=1|a_{k-I}=-1)\\ &\cdot\Pr(a_{k-I}=-1|\hat{a}_k=1).\end{aligned}$$

Then, Bayes' rule yields (because the input and output take values ± 1 with equal probability)

$$\begin{aligned}\Pr(\hat{a}_{k+J-I}=1|\hat{a}_k=1)=&\Pr(\hat{a}_{k+J-I}=1|a_{k-I}=1)\\ &\cdot\Pr(\hat{a}_k=1|a_{k-I}=1)\\ &+\Pr(\hat{a}_{k+J-I}=1|a_{k-I}=-1)\\ &\cdot\Pr(\hat{a}_k=1|a_{k-I}=-1).\end{aligned} \quad (2.9)$$

Now (2.3c), applied at time $k+J-I$, says that $\hat{a}_{k+J-I}$ depends nontrivially on a_{k-I}. Noting that the other quantities on which $\hat{a}_{k+J-I}$ depends, viz. $a_{k-I+1}, \cdots, a_{k+J-2I}$, are independent of a_{k-I}, it follows that for some ϵ with $0<\epsilon\leq 1/2$,

$$\Pr(\hat{a}_{k+J-I}=1|a_{k-I}=1)=\frac{1}{2}+\epsilon\operatorname{sgn}(l_J) \quad (2.10a)$$

and

$$\Pr(\hat{a}_{k+J-I}=1|a_{k-I}=-1)=\frac{1}{2}-\epsilon\operatorname{sgn}(l_J). \quad (2.10b)$$

Actually, $\epsilon = 1/2$ would imply that $\hat{a}_{k+J-I} = \operatorname{sgn}(l_J)a_{k-I}$, a situation ruled out by (2.3b) and (2.3c) if $I<J$.

Similarly, for some η with $0<\eta<1/2$,

$$\Pr(\hat{a}_{k+J-I}=1|a_{k-I}=1)=\frac{1}{2}+\eta\operatorname{sgn}(l_J) \quad (2.11a)$$

and

$$\Pr(\hat{a}_{k+J-I}=1|a_{k-I}=-1)=\frac{1}{2}-\eta\operatorname{sgn}(l_J). \quad (2.11b)$$

Using this in (2.8) gives

$$\Pr(\hat{a}_{k+J-I}=1|\hat{a}_k=1)=\frac{1}{2}+2\epsilon\eta\operatorname{sgn}(l_I)\operatorname{sgn}(l_J).$$

Conditions (2.3b) and (2.3c) imply $l_I, l_J \neq 0$. Hence, for some $\mu \neq 0$,

$$\Pr(\hat{a}_{k+J-I}=1|\hat{a}_k=1)=\frac{1}{2}+\mu\neq\frac{1}{2}.$$

This causes violation of the uncorrelatedness assumption. More specifically,

$$\begin{aligned}&E[\hat{a}_{k+J-I}\hat{a}_k]\\ &=\Pr(\hat{a}_{k+J-I}=1|\hat{a}_k=1)\Pr(\hat{a}_k=1)\\ &\quad-\Pr(\hat{a}_{k+J-I}=1|\hat{a}_k=-1)\Pr(\hat{a}_k=-1)\\ &\quad-\Pr(\hat{a}_{k+J-I}=-1|\hat{a}_k=1)\Pr(\hat{a}_k=1)\\ &\quad+\Pr(\hat{a}_{k+J-I}=-1|\hat{a}_k=-1)\Pr(\hat{a}_k=-1)\\ &=\left(\frac{1}{2}+\mu\right)\frac{1}{2}-\left(\frac{1}{2}-\mu\right)\frac{1}{2}-\left(\frac{1}{2}-\mu\right)\frac{1}{2}+\left(\frac{1}{2}+\mu\right)\frac{1}{2}\\ &=2\mu.\end{aligned}$$

Hence, the condition $I<J$ is contradicted. This proves the result. □

Remarks:

1) The condition $\hat{a}_k = \operatorname{sgn}(l_\delta)a_{k-\delta}\ \forall k$ is assured if and only if

$$|l_\delta|>\sum_{\substack{i=0\\ i\neq\delta}}^{N}|l_i| \quad (2.12)$$

which is a form of eye condition.

2) We only require $\{\hat{a}_k\}$ to be uncorrelated to demonstrate $\hat{a}_k = \operatorname{sgn}(l_\delta)a_{k-\delta}\ \forall k$. (This then implies $\{\hat{a}_k\}$ is necessarily independent.) Hence, testing for convergence is far easier than checking for full independence.

3) Identification of the delay δ, based on output observations only, is, of course, not possible. To obtain such information would require explicit knowledge of the input signal.

III. Additive Noise Effects

We turn our attention now to the behavior of the decision-directed equalizer with additive channel noise. That is, we consider

$$\hat{a}_k=\operatorname{sgn}\left(\sum_{j=0}^{N} l_j a_{k-j}+n_k\right) \quad (3.1)$$

where $\{n_k\}$ is the extraneous noise.

Initially, we shall assume that $\{n_k\}$ and $\{a_k\}$ are individually independent sequences and mutually independent. More realistically, if n_k is correlated over time, as is the case for a moving average signal

$$n_k = \sum_{i=0}^{N_2} d_i \nu_{k-i}$$

with $\{\nu_k\}$ independent, then the independence of $\{\hat{a}_k\}$ will be violated by dint of this mechanism. We shall return to consideration of such noise signals later as they represent a typical class of noise a rising in the output of an equalized channel.

We constrain our attention to the situation where occasional noise bursts are capable of altering the equalized signal, but this signal is more usually decoded correctly. To analyze this case, we repeat the arguments of the preceding section, suitably amended.

Denoted by $\mathfrak{F}_k$ the sigma algebra of events generated by $\{a_k, a_{k-1}, \cdots\}$ and define the integers I and J by the following conditional distributions:

$$\Pr\left(\operatorname{sgn}\left(\sum_{i=I}^{J} l_i a_{k-i}\right) = +1 \,\middle|\, \mathfrak{A}\right) = \Pr\left(\operatorname{sgn}\left(\sum_{i=0}^{N} l_i a_{k-i}\right) = +1 \,\middle|\, \mathfrak{A}\right) \quad \forall \mathfrak{A} \in \mathfrak{F}_k \tag{3.2a}$$

while

$$\Pr\left(\operatorname{sgn}\left(\sum_{i=I+1}^{J} l_i a_{k-i}\right) = +1 \,\middle|\, \mathfrak{B}\right) = \Pr\left(\operatorname{sgn}\left(\sum_{i=0}^{N} l_i a_{k-i}\right) = +1 \,\middle|\, \mathfrak{B}\right) \quad \text{for some } \mathfrak{B} \in \mathfrak{F}_k \tag{3.2b}$$

and

$$\Pr\left(\operatorname{sgn}\left(\sum_{i=I}^{J-1} l_i a_{k-i}\right) = +1 \,\middle|\, \mathfrak{C}\right) = \Pr\left(\operatorname{sgn}\left(\sum_{i=0}^{N} l_i a_{k-i}\right) = +1 \,\middle|\, \mathfrak{C}\right) \quad \text{for some } \mathfrak{C} \in \mathfrak{F}_k. \tag{3.2c}$$

To avoid a degenerate problem, we make the mild assumption

$$\Pr\left(\sum_{i=0}^{N} |l_i| > |n_k|\right) > 0. \tag{3.3}$$

Then, an identical argument to that of the previous section leads to the conclusion that the above definition of I and J, $I < J$ and $\{\hat{a}_k\}$ uncorrelated are incompatible. To recapitulate, we have the following.

Theorem 2: Let $\{a_k\}$ be an independent sequence of random variables taking the values ± 1 with equal probability. Let $\{n_k\}$ be a sequence of independent random variables independent from $\{a_k\}$. Suppose that for some constants $\{l_0, \cdots, l_N\}$

$$\hat{a}_k = \operatorname{sgn}\left(\sum_{j=0}^{N} l_j a_{k-j} + n_k\right). \tag{3.4}$$

and further that the distribution of $\{n_k\}$ satisfies (3.3) and is such that the occurrence of a zero argument of the sign function in (3.4) is a probability zero event.

Then, if $\{\hat{a}_k\}$ defined by (3.4) is itself a sequence of uncorrelated random variables, there holds for some $\delta \in \{0, \cdots, N\}$

$$\Pr(\hat{a}_k = +1 | \mathfrak{A}) = \Pr(l_\delta a_{k-\delta} + n_k > 0 | \mathfrak{A}) \qquad \forall \mathfrak{A} \in \mathfrak{F}_k. \tag{3.5}$$

This theorem demonstrates that the nonnoisy independence result carries over the case of additive independent noise, subject to a mild condition (3.3). That is, independence of $\{\hat{a}_k\}$ implies that its distribution is centered on sgn $(l_\delta)a_{k-\delta}$ for some δ.

Unfortunately, as remarked earlier, the more realistic assumption for an equalized channel is that $\{n_k\}$ is not independent, but a moving average (or m-dependent) process of order N_2, i.e., with a correlation length given by the number of taps in the equalizer component. In a well-designed system, the channel noise should be well below the signal level for a great proportion of the time. Thus, the probability of noise-induced decision error will be small. The moving average nature of n_k will, however, cause these noise-induced errors to persist for periods on the order of the equalizer length as the infrequent event of a large noise sample shifts through the tapped-delay line structure.

The net effect of small additive channel noise will be not to affect the decision, and therefore not to affect the correlation test for convergence except in infrequent intervals where bursts of errors, and therefore presumably correlated decisions, occur.

IV. M-ARY GENERALIZATION

In this section, we outline how the binary result, Theorem 1, generalizes to the M-ary case. To begin with, (2.1) becomes

$$\hat{a}_k = Q_M\left(\sum_{i=0}^{N} l_i a_{k-i}\right) \tag{4.1}$$

with $a_k \in \mathbb{S} \triangleq \{1-M, 3-M, \cdots, M-1\}$, M even and

$$Q_M(x) \triangleq \sum_{k=1-M/2}^{M/2-1} \operatorname{sgn}(x+2k). \tag{4.2}$$

Our first result, which is easy to prove, is what we call an eye condition. This is a generalization of (2.12) with zero noise. (A very short outline of the proof can be found in the Appendix.)

Theorem 3: There exists precisely one $\delta \in \{0, \cdots, N\}$ with $\hat{a}_k = \operatorname{sgn}(l_\delta)a_{k-\delta}$, $\forall \{a_k\}$ if and only if

$$R(\delta) < |l_\delta| \quad \text{for } M=2 \tag{4.3a}$$

$$\frac{M-2+R(\delta)}{M-1} < |l_\delta| < \frac{M-2-R(\delta)}{M-3} \qquad \text{for } M = 4, 6, 8, \cdots \tag{4.3b}$$

where

$$R(\delta) \triangleq \max_{\{a_k\}} \{r_k(\delta)\} = (M-1) \sum_{\substack{i=0 \\ i\neq\delta}}^{N} |l_i|.$$

Remarks:

1) This gives necessary and sufficient conditions on the l_i for correct decision-directed equalizer performance.

2) Naturally, (4.3) cannot be used directly to test for convergence when we only have output measurements.

Before we move on to the M-ary generalization of Theorem 1 (which can be thought of as a restatement of Theorem 3), let us give an example to highlight the importance of the underlying hypotheses of the theorem.

Example (Mazo [1]): Suppose $\hat{a}_k = Q_4(1000\, a_k)$, implying $\hat{a}_k = 3 \operatorname{sgn}(a_k)$; then $\{\hat{a}_k\}$ is a sequence of independent random variables. However, $\Pr(\hat{a}_k = -1) = \Pr(\hat{a}_k = +1) = 0$ and $\Pr(\hat{a}_k = -3) = \Pr(\hat{a}_k = +3) = 0.5$, i.e., $\hat{a}_k$ has a nonuniform distribution.

This example tells us that, unlike the binary case, we need also to check that the output distribution matches the input (uniform in this present formulation). Our generalization of Theorem 1 to the M-ary case takes the following form. (Again, an outline of the proof may be found in the Appendix.)

Theorem 4: Let $\{a_k\}$ be an M-ary independent sequence taking values in the set $\mathcal{S} \triangleq \{1-M, 3-M, \cdots, M-1\}$ with equal probability. Let $\{a_k\}$ be the input to the system given by (4.1). Suppose that for all $j \in \{1, 2, \cdots, N\}$

$$\Pr(\hat{a}_{k+j}=\alpha_1,\, \hat{a}_k=\alpha_2)=\frac{1}{M^2}, \qquad \forall \alpha_1, \alpha_2 \in \mathcal{S} \tag{4.4}$$

and

$$\Pr(\hat{a}_k=\alpha_1)=\frac{1}{M}, \qquad \forall \alpha_1 \in \mathcal{S}. \tag{4.5}$$

Then for precisely one $\delta \in \{0, \cdots, N\}$, there holds

$$\hat{a}_k = \operatorname{sgn}(l_\delta) a_{k-\delta}, \qquad \forall \{a_k\}. \tag{4.6}$$

Remarks:

1) Condition (4.5) is automatically satisfied in the binary case.

2) Condition (4.4), which is a pairwise independence property, replaces the correlation test in Theorem 1.

3) Theorem 4 says both a correlation-type test and a distribution-type test need to be performed, unlike the binary which needed only a correlation test.

V. Conclusions and Discussion

We have shown that a decision-directed equalizer is decoding correctly, i.e., the output binary/M-ary sequence is a simple delay mapping of an independent input binary/M-ary sequence if and only if the output is uncorrelated/pairwise independent and uniformly distributed. This provides a simple test on the output sequence to show that the decision-directed equalizer has converged. We have also provided meaningful results in the case of small noise. Note that without explicit *a priori* knowledge of the input, the determination of the resulting delay is not possible using output measurements alone.

Our results have relied on the input data sequence being independent. When $\{a_k\}$ is purposely correlated, as for example to implement error correction, etc., an analogous result would be desirable. Also, when correlated noise of reasonable amplitude is present, the noise will tend to destroy the lack of correlation (or pairwise independence) of the output. This seems to be a fundamental limitation and the analysis required needs to be greatly extended. Further issues related to our results concern similar questions for decision feedback equalization. In this case, we have only been able to establish an analogous result for low order equalizers—the analytic demands being for more severe.

We conclude by presenting a further example of a nonlinear, m-dependent system generating $\hat{a}_k$ from a_k, viz.

$$\hat{a}_k = a_k a_{k-1}, \tag{5.1}$$

which is not of the DDE structure (2.1), satisfies all the other conditions of Theorem 1, but not the conclusions (2.2). The reason for including this simple system is to highlight the fact that the particular structure is important in establishing these results. General information theoretic or data-processing theorems appear to be unable to establish these conditions.

Appendix

Proofs of Theorems 3 and 4

The following proofs are given only in outline.

Proof (Theorem 3): For $M \geq 4$, suppose the RHS of (4.3b) were violated; then the input $\{a_{k-i} = (M-1)\operatorname{sgn}(l_i),\ i = 0, 1, \cdots, \delta-1, \delta+1, \cdots, N\}$ with $\{a_{k-\delta} = \operatorname{sgn}(l_\delta)(M-3)\}$ implies $\hat{a}_k = M - 1 \neq \operatorname{sgn}(l_\delta)a_{k-\delta}$. Similarly, if the LHS of (4.3b) were violated, then there exists a (nonzero probability) input where $a_{k-\delta} = \operatorname{sgn}(l_\delta)(M-1)$, leading to $\hat{a}_k = M - 3 \neq \operatorname{sgn}(l_\delta)a_{k-\delta}$. Note that (4.3b) is really just saying $|l_\delta| \approx 1$ and l_δ must be greater in magnitude than the sum of the absolute values of the remaining l_i. The omitted details in the proof are easily filled. □

Proof (Theorem 4): The proof mimics the binary case except for the following important differences. As in (2.3a), we define integers I (maximal) and J (minimal, but greater than I) such that

$$Q_M\left(\sum_{i=I}^{J} l_i a_{k-i}\right) = Q_M\left(\sum_{i=0}^{N} l_i a_{k-i}\right) \triangleq \hat{a}_k, \qquad \forall \{a_k\}. \tag{A1}$$

If $I = J$, we have

$$\hat{a}_k = Q_M(l_I a_{k-I}), \qquad \forall \{a_k\} \tag{A2}$$

which is not quite a delay system (unless $M = 2$). However, it is quite simple to show that (A2) coupled with condition (4.5) implies (4.6) with $\delta = I = J$. (One may use Theorem 3 for this demonstration.)

It remains to be shown that $I < J$ contradicts (4.4). We focus on an object which is intimately connected with (4.4):

$$\xi(s_1, s_2) \triangleq \Pr(\hat{a}_{k+J-I} \geq s_1 \mid \hat{a}_k \geq s_2) \tag{A3}$$

which generalizes (2.7). The manipulation involving Bayes' rule, mimicking (2.7) to (2.9), yields

$$\xi(s_1, s_2) = \frac{1}{f(s_2)} \sum_{m \in \mathcal{S}} \Pr(\hat{a}_{k+J-I} \geq s_1 \mid a_{k-I} = m) \cdot \Pr(\hat{a}_k \geq s_2 \mid a_{k-I} = m) \tag{A4}$$

where $f(\cdot): \mathcal{S} \to \mathbb{Z}_+$ is implicitly defined through

$$\Pr(\hat{a}_k \geq m) = \Pr(a_k \geq m) = \frac{f(m)}{M}, \qquad \forall m \in \mathcal{S}. \tag{A5}$$

Now we define functions $\epsilon(\cdot, \cdot)$, $\eta(\cdot, \cdot)$ whose domain is $\mathcal{S} \times \mathcal{S}$ implicitly through

$$\Pr(\hat{a}_{k+J-I} \geq s_1 \mid a_{k-I} = m) \triangleq \frac{f(s_1)}{M}(1 + \operatorname{sgn}(l_J)\epsilon(m, s_1)) \tag{A6a}$$

$$\Pr(\hat{a}_k \geq s_2 \mid a_{k-I} = m) \triangleq \frac{f(s_2)}{M}(1 + \operatorname{sgn}(l_I)\eta(m, s_2)) \tag{A6b}$$

noting that if the conditioning could be dropped in (A6), then $\epsilon(m, s_1) = 0$ and $\eta(m, s_2) = 0$.

With these definitions, we may rewrite (A4) as

$$\xi(s_1, s_2) = \frac{1}{f(s_2)} \sum_{m \in S} \frac{f(s_1)}{M} \frac{f(s_2)}{M} (1 + \text{sgn}(l_J)\epsilon(m, s_1)) \cdot (1 + \text{sgn}(l_I)\eta(m, s_2))$$

$$= \frac{f(s_1)}{M^2} \left\{ M + \text{sgn}(l_J) \sum_{m \in S} \epsilon(m, s_1) + \text{sgn}(l_I) \sum_{m \in S} \eta(m, s_2) + \text{sgn}(l_J l_I) \sum_{m \in S} \epsilon(m, s_1)\eta(m, s_2) \right\}. \quad \text{(A7)}$$

It then becomes critical to evaluate the expressions involving $\epsilon(m, s_1)$ and $\eta(m, s_2)$ and show that for at least some $s_1 = s_1^*$ and $s_2 = s_2^*$, the RHS of (A7) does not equal $f(s_1)/M$ (pairwise independence).

With respect to (A7), we claim two important properties:

$$1) \sum_{m \in \mathfrak{M}} \epsilon(m, s_1) = 0 \text{ and } \sum_{m \in S} \eta(m, s_2) = 0 \quad \forall s_1, s_2 \in S, \quad \text{(A8a)}$$

$$2) \sum_{m \in S} \epsilon(m, s_1^*)\eta(m, s_2^*) > 0 \quad \text{for some } s_1^*, s_2^* \in S. \quad \text{(A8b)}$$

These two properties imply, by (A7), that

$$\xi(s_1^*, s_2^*) \triangleq \Pr(\hat{a}_{k+J-I} \geq s_1^* | \hat{a}_k \geq s_2^*) \neq \frac{f(s_1^*)}{M} = \Pr(\hat{a}_{k+J-I} \geq s_1^*) \quad \text{(A9)}$$

which contradicts the pairwise independence of the output sequence. It follows that the hypothesis $I < J$ is flawed, and thus we must have $I = J$. We can reformulate (A9) into (4.4) by Bayes' rule.

What remains to be shown is the proof of properties (A8). The proof of property (A8a) from (A6) is a simple exercise again with Bayes' rule, and so is omitted. Property (A8b), on the other hand, is more subtle. First one establishes from (4.1) that there exists s_1^* such that $\epsilon(M - 1, s_1^*) > 0$, and there exists s_2^* such that $\eta(M - 1, s_2^*) > 0$. Then with s_1^* and s_2^* fixed, $\epsilon(\cdot, s_1^*)$ and $\eta(\cdot, s_2^*)$ are monotonic and, in fact, nonconstant [by property (A8a)]. Then property (A8b) follows by the theory of rearrangement inequalities [4]. □

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Comments on "Pulse Shape, Excess Bandwidth, and Timing Error Sensitivity in PRS Systems"

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Abstract—**Several nonminimum bandwidth PRS techniques were compared in the above paper.[1] One of the criteria used for the comparison was the robustness to the timing jitter. In this work, we find that according to this criterion the duobinary signaling technique is superior to dicode scheme at low values of the roll-off factor, unlike what is concluded in the above paper.[1]**

I. The Figure of Merit for Jitter Sensitivity

The PRS systems of interest receive a signal of the form (omitting the additive noise component)

$$r(t) = \sum_{k=-\infty}^{\infty} a_k h(t - kT) \quad (1)$$

where the data symbols a_k are assumed to be zero mean, unity variance, independent and identically distributed. For a nonminimum bandwidth PRS system with Fourier transform of $h(t)$ can be written as

$$H(f) = C(f)G(f) \quad (2)$$

where $G(f)$ corresponds to a pulse satisfying the Nyquist-*I* criterion: $\Sigma G(f - n/T)) = T$. It is assumed that $G(f)$ is real, symmetric and bandlimited to $(1 + \theta)/(2T)$ where θ is the roll-off factor $(0 \leq \theta \leq 1)$. $C(f)$ in (2) is given by

$$C(f) = c(D)|_{D = \exp(-j2\pi fT)} \quad (3)$$

where $c(D)$ is a polynomial describing the PRS shaping.

The figure of merit for the sensitivity to timing phase jitter used in the above paper[1] is given by

$$\beta(\theta) = \sum_{k=-\infty}^{\infty} h'(kT)^2 \quad (4)$$

where denotes the differentiation with respect to t. Thus, $\beta(\theta)$ is the energy of the differentiated pulse samples obtained at the optimum instants. The sum in (4) can also be written in terms of $H(f)$ to yield

$$\beta(\theta) = \frac{8\pi^2}{T} \int_0^{1/2T} |fH(f) - U[f - (1-\theta)/(2T)] \cdot (1/T - f)H(1/T - f)|^2 \, df \quad (5)$$

where $U(\cdot)$ is the unit step function.

II. Comparison of the PRS Systems

The same PRS systems used in the paper[1] is considered here, and their performances are compared using the measure given in the previous section. The three PRS schemes used are described by the polynomial $c(D)$: duobinary $(1 + D)$, dicode $(1 - D)$, and modified duobinary $(1 - D^2)$. The Nyquist-*I*

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