INT. J. CONTROL, 1989, VOL. 50, NO. 5, 1745–1762

Robust stability of polynomials with multilinear parameter dependence

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The problem is studied of testing for stability a class of real polynomials in which the coefficients depend on a number of variable parameters in a multilinear way. We show that the testing for real unstable roots can be achieved by examining the stability of a finite number of corner polynomials (obtained by setting parameters at their extreme values), while checking for unstable complex roots normally involves examining the real solutions of up to $m+1$ simultaneous polynomial equations, where m is the number of parameters. When $m=2$, this is an especially simple task.

1. Introduction

This paper is concerned with a robust stability problem. More specifically, we consider monic nth degree polynomials $f(s; \gamma_1, \ldots, \gamma_m)$ with real coefficients which depend in a multilinear fashion on the quantities γ_i. The parameters γ_i are contained in intervals $[\underline{\gamma}_i, \bar{\gamma}_i]$, and we seek a test for the stability of all $f(s)$, where by the term stability, we mean that $f(s)$ has all its roots in a prescribed region, e.g. Re $[s] < 0$, $|s| < 1$, etc. For the most part in this paper, we focus on the case Re $[s] < 0$; the ideas however with little variation will carry over to most other regions of interest. Stability inside the unit circle is easily covered for example by bilinear transformation.

To illustrate the occurrence of such problems we note that many physical systems described by linear differential equations in which parameters such as friction constants, mass, capacitance, etc. vary have associated transfer functions in which these variable parameters appear multilinearly in both numerator and denominator. Also, when a controller defined by a rational transfer function is connected, the characteristic polynomial of the closed-loop system is (apart from limited exceptions) necessarily multilinear in the parameters of the plant and controller transfer functions (see for example Section 9.17 of Zadeh and Desoer 1963, and Dasgupta and Anderson 1987).

In the following, two examples are given.

Example 1

Let us consider the electrical circuit depicted in Fig. 1. The transfer function from u_0 to u_c is given by

$$G(s) = \frac{1}{s^2LC + sRC + 1}$$

Received 15 February 1988. Revised 25 November 1988.

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0020-7179/89 $3.00

The parameters of the characteristic polynomial depend bilinearly on the physical parameters R, L, C which can vary slowly, for example because of temperature variations or ageing.

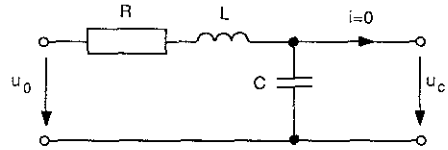

Figure 1. Electrical circuit.

Example 2

Assume that $G_1(s) = B_1(s)/A_1(s)$ and $G_2(s) = B_2(s)/A_2(s)$ in the SISO control system of Fig. 2, and that certain coefficients of A_i, B_i, $i = 1, 2$, depend linearly on some parameters, different for each of the four polynomials. Then the characteristic polynomial

$$N(s) = A_1(s)A_2(s) + B_1(s)B_2(s)$$

depends bilinearly on the coefficients of A_1 and A_2, and B_1 and B_2, respectively, and in turn bilinearly on the underlying parameters. This situation is significant in practical applications because we often need to build control systems from different parts. The parameters of the parts can differ from the nominal values. However, the stability of the closed loop should be preserved for all such parts.

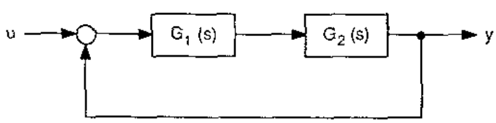

Figure 2. SISO control system.

Before proceeding further, we must define some terms. We work in three different spaces, i.e. parameter space, coefficient space and root space. The paramters γ_i are contained in closed intervals $[\underline{\gamma}_i, \bar{\gamma}_i]$. The endpoints of such intervals are denoted by $\{\underline{\gamma}_i, \bar{\gamma}_i\}$. Open parameter intervals are given by $(\underline{\gamma}_i, \bar{\gamma}_i)$. Corner points and corner polynomials in parameter and coefficient space are defined by taking $\gamma_i \in \{\underline{\gamma}_i, \bar{\gamma}_i\}$. Edges in parameter and coefficient space and edge polynomials are defined by taking $\gamma_i \in \{\underline{\gamma}_i, \bar{\gamma}_i\}$ for all but one value of i, say i_1, and $\gamma_{i1} \in [\underline{\gamma}_{i1}, \bar{\gamma}_{i1}]$. Notice that edges in both parameter and coefficient space are straight lines. Faces in parameter and coefficient space and face polynomials are defined by taking $\gamma_i \in \{\underline{\gamma}_i, \bar{\gamma}_i\}$ for all but two values of i, say i_1 and i_2, and $\gamma_{i1} \in [\underline{\gamma}_{i1}, \bar{\gamma}_{i1}]$ and $\gamma_{i2} \in [\underline{\gamma}_{i2}, \bar{\gamma}_{i2}]$.

In parameter space, faces are flat, while in coefficient space, faces are two-dimensional curved surfaces, but in general not planar. Coefficient space faces are however ruled surfaces, i.e. through every point on the face there pass in general two straight lines of the surface defined by $\gamma_{i1} = \text{constant}$ and $\gamma_{i2} = \text{constant}$.

In a search for necessary and sufficient conditions for stability, the general aim is naturally to avoid testing all possible values of the parameters, i.e. one wants theorems which establish stability for all values given that stability holds for some restricted set of values. A Kharitonov-like theorem (Kharitonov 1979) would be one which requires testing only at corner points, i.e. $\gamma_i \in \{\underline{\gamma}_i, \bar{\gamma}_i\}$. However, it is quickly seen that such a result is extremely unlikely; Kharitonov's theorem is valid for a region in coefficient space bounded by hyperplanes parallel to the coordinate axes, and only then for stability in the region $\text{Re}(s) < 0$ (counter-examples exist for the region $|s| < 1$, see Hollot and Bartlett 1986).

The next possibility is to examine stability at the corners and along the edges. Such an idea is suggested by the work of Bartlett *et al.* (1988); these authors show that if the coefficients of a set of polynomials depend in an *affine* way on a collection of parameters, each of which lies in an interval, such that in coefficient space the collection of polynomials under test is a polytope, then it suffices to check the edges for stability. More precisely, the authors prove the following:

(*a*) if s_0 is a real root of any polynomial in the set under test, it is a root of at least one edge polynomial;

(*b*) if s_0 is a complex root of any polynomial in the set under test, it is a root of at least one face polynomial;

(*c*) if s_0 is at the boundary of the set of roots of all face polynomials, then it is also a root of at least one edge polynomial;

(*d*) if D is a simply connected domain, then the roots of all polynomials lie in D if and only if the roots of all polynomials defined by all edges lie in D. This is a consequence of (*a*), (*b*) and (*c*).

When we seek to carry over these ideas to our problem, where the coefficients depend multilinearly on the parameters, it turns out that only (*a*) remains valid. The following counter-example to (*b*) was supplied to us by C. V. Hollot. The polynomial

$$f(s) = s^5 + (-\gamma_1)s^4 + (-\gamma_1 - \gamma_3 + 1)s^3 + \gamma_2 s^2 + (\gamma_2 + \gamma_1\gamma_2 + \gamma_2\gamma_3 + \gamma_3\gamma_1)s + (-\gamma_3 + \gamma_1\gamma_2\gamma_3)$$

with $|\gamma_i| \leqslant 1$ has the property that $\pm j$ is a root when $\gamma_1 = \gamma_2 = \gamma_3 = 0$. It is not, however, a root of any face polynomial.

What other approaches exist? In de Gaston and Safonov (1988), appeal is made to the fact that the set of all Nyquist diagrams of all polynomials in the set has a key property. For each $\omega, f(j\omega; \gamma_1, \ldots, \gamma_m)$ lies in the convex hull of the 2^m complex points obtained by setting the γ_i to their extreme value, a property pointed out by Zadeh and Desoer (1963) with the name 'Mapping theorem'. This idea is exploited to tackle the robust stability problem with a type of extension of Nyquist's theorem.

These ideas have something in common with those of Saeki (1986), who considers a roughly equivalent problem, but one in which the γ_i, in effect, are allowed to be complex. It turns out that in many ways, this simplifies the problem. Yet another possibility is to make special assumptions on the polynomials $f(s)$, which aims to make the problem equivalent to, or very like, the problem considered by Bartlett *et al.* (1988). For example, Panier *et al.* (1987) postulate uncoupled perturbations in the coefficients of even and odd powers of $f(s)$, while Djaferis and Hollot (1988) and Djaferis (1988) impose restrictions that ensure that the image for $f(j\omega; \gamma_1, \gamma_2, \ldots, \gamma_m)$ for each ω and variable γ_i is a polytopic set. This again allows an extension of Nyquist's theorem to be applied. The difficulty with this type of result is that it is highly non-generic.

Rather than working up from results such as Kharitonov's theorem and the edge theorem, another approach is to work down from the very general Tarski–Seidenberg decision algebra theorem described in textbooks such as Bose (1982) and Jacobson (1964). This theorem implies that the robust stability problem we have posed can always be solved using a finite number of rational calculations (in the sense that for a given polynomial dependent on $\gamma_1, \gamma_2, \ldots, \gamma_m$ a yes/no answer to the robust stability question can be obtained). The number of calculations may be prohibitive, and the real interest then lies in finding shortcuts so that the number of calculations becomes acceptable.

A variant on the Tarski–Seidenberg theorem was suggested by Anderson and Scott (1977), who showed that an alternative approach for any decision algebra problem could be found which involved the construction and solutions of q polynomial equations in q unknowns, q being an integer determined by the problem statement. When this procedure is followed, much of the interest lies in ensuring that q is as small as possible. This actually will be the approach followed in two later sections of the paper, where we shall have $q = m + 1$. Note that there exist systematic methods for solving such equations based on resultants (see Bose 1982). Also, software is increasingly becoming available (see Watson *et al.* 1987).

When the γ_i correspond to physical parameters, in many cases the value of m will be quite small, say 2, 3 or 4. Under these circumstances, there is a good possibility that the computational burdens will not prove excessive.

The layout of the paper is as follows. In the next section, we show that the set of real roots of all polynomials is identical with the set of real roots of the edge polynomials. In § 3 we study faces and explain a procedure whereby the faces may be checked for stability. In § 4 this is generalized to explain how stability inside the entire prescribed region of parameter space may be examined. (Several special cases yielding considerable simplifications are also covered in these sections.) In § 5, we show how differing necessary and sufficient conditions for robust stability can be derived, and we discuss how such conditions can be sharpened. Section 6 contains concluding remarks.

2. Significance of the edges for real roots

As mentioned before, we first establish the following result.

Theorem 2

Let $f(s; \gamma_1, \ldots, \gamma_m)$ be an nth degree monic polynomial with real coefficients dependent multilinearly on the γ_i, where γ_i is contained in an interval $\gamma_i \in [\underline{\gamma}_i, \bar{\gamma}_i]$, $i = 1, \ldots, m$. Let s_0 be a real root of some such polynomial. Then s_0 is also a real root of an edge polynomial.

Proof

The proof of this result is by induction. Let s_0 be a real root of the polynomial f for some given $\tilde{\gamma}_i \in [\underline{\gamma}_i, \bar{\gamma}_i]$, $i = 1, \ldots, m$. Suppose that for s_0, $f(s_0; \hat{\gamma}_1, \ldots, \hat{\gamma}_r, \tilde{\gamma}_{r+1}, \ldots, \tilde{\gamma}_m) = 0$ for $\hat{\gamma}_1 \in \{\underline{\gamma}_1, \bar{\gamma}_1\}, \ldots, \hat{\gamma}_r \in \{\underline{\gamma}_r, \bar{\gamma}_r\}$ and to avoid trivial cases $\tilde{\gamma}_{r+1} \in (\underline{\gamma}_{r+1}, \bar{\gamma}_{r+1})$, $\ldots, \tilde{\gamma}_m \in (\underline{\gamma}_m, \bar{\gamma}_m)$, for some $r < m - 1$.

We shall show that we can adjust either $\tilde{\gamma}_{r+1}$ to $\hat{\gamma}_{r+1} \in \{\underline{\gamma}_{r+1}, \bar{\gamma}_{r+1}\}$ with $f(s_0, \hat{\gamma}_1, \ldots, \hat{\gamma}_r, \hat{\gamma}_{r+1}, \tilde{\gamma}_{r+2}, \ldots, \tilde{\gamma}_m) = 0$, or $\tilde{\gamma}_{r+2}$ to $\hat{\gamma}_{r+2} \in \{\underline{\gamma}_{r+2}, \bar{\gamma}_{r+2}\}$ with $f(s_0, \hat{\gamma}_1, \ldots, \hat{\gamma}_r, \tilde{\gamma}_{r+1}, \hat{\gamma}_{r+2}, \tilde{\gamma}_{r+3}, \ldots, \tilde{\gamma}_m) = 0$.

To verify this claim, set $\delta_{r+1} = \gamma_{r+1} - \tilde{\gamma}_{r+1}$, $\delta_{r+2} = \gamma_{r+2} - \tilde{\gamma}_{r+2}$; then we may write

$$\begin{aligned} f(s; \hat{\gamma}_1, \ldots, \hat{\gamma}_r, \gamma_{r+1}, \gamma_{r+2}, \tilde{\gamma}_{r+3}, \ldots, \tilde{\gamma}_m) &= f(s; \hat{\gamma}_1, \ldots, \hat{\gamma}_r, \tilde{\gamma}_{r+1} + \delta_{r+1}, \tilde{\gamma}_{r+2} + \delta_{r+2}, \tilde{\gamma}_{r+3}, \ldots, \tilde{\gamma}_m) \\ &= g_0(s; \hat{\gamma}_1, \ldots, \hat{\gamma}_r, \tilde{\gamma}_{r+1}, \tilde{\gamma}_{r+2}, \ldots, \tilde{\gamma}_m) \\ &\quad + \delta_{r+1} g_1(s; \hat{\gamma}_1, \ldots, \hat{\gamma}_r, \tilde{\gamma}_{r+2}, \ldots, \tilde{\gamma}_m) \\ &\quad + \delta_{r+2} g_2(s; \hat{\gamma}_1, \ldots, \hat{\gamma}_r, \tilde{\gamma}_{r+1}, \tilde{\gamma}_{r+3}, \ldots, \tilde{\gamma}_m) \\ &\quad + \delta_{r+1}\delta_{r+2} g_3(s; \hat{\gamma}_1, \ldots, \hat{\gamma}_r, \tilde{\gamma}_{r+3}, \ldots, \tilde{\gamma}_m) \end{aligned}$$

or, in abbreviated notation,

$$f(s) = g_0(s) + \delta_{r+1} g_1(s) + \delta_{r+2} g_2(s) + \delta_{r+1}\delta_{r+2} g_3(s)$$

The $g_i(s)$ are multilinear in the parameters on which they depend. Also, $g_0(s_0) = 0$. Now if $g_1(s_0) = 0$, set $\delta_{r+2} = 0$ and choose δ_{r+1} to correspond to an extreme value ($\delta_{r+1} = \bar{\gamma}_{r+1} - \tilde{\gamma}_{r+1}$ is the upper boundary for δ_{r+1}, for example). Thus, $\hat{\gamma}_{r+1} \in \{\underline{\gamma}_{r+1}, \bar{\gamma}_{r+1}\}$, and also $f(s_0; \hat{\gamma}_1, \ldots, \hat{\gamma}_r, \hat{\gamma}_{r+1}, \hat{\gamma}_{r+2}, \hat{\gamma}_{r+3}, \ldots, \hat{\gamma}_m) = 0$. Similarly, if $g_2(s_0) = 0$, set $\delta_{r+1} = 0$ and δ_{r+2} at an extreme value. If neither $g_1(s_0)$ nor $g_2(s_0)$ are zero, plot in the δ_{r+1}, δ_{r+2} plane the straight line or hyperbola defined by

$$\delta_{r+1} g_1(s_0) + \delta_{r+2} g_2(s_0) + \delta_{r+1}\delta_{r+2} g_3(s_0) = 0$$

The straight line is encountered precisely when $g_3(s_0) = 0$.) This hyperbola necessarily intersects at least one of the four lines in the δ_{r+1}, δ_{r+2} plane which define the boundaries of the allowed parameter values. We choose one of the intersection points. At this intersection point, one of the associated $\gamma_{r+1}, \gamma_{r+2}$ has an extreme value, say $\gamma_{r+2} = \hat{\gamma}_{r+2} \in \{\underline{\gamma}_{r+2}, \bar{\gamma}_{r+2}\}$. So we obviously have

$$f(s_0; \hat{\gamma}_1, \ldots, \hat{\gamma}_r, \tilde{\gamma}_{r+1}, \hat{\gamma}_{r+2}, \tilde{\gamma}_{r+3}, \ldots, \tilde{\gamma}_m) = 0$$

and therefore the induction step $r + 1$ is proved. This proves the theorem. □

Obviously, the theorem states that the set of all real roots of all polynomials is given by the set of all real roots of all edge polynomials. If one is interested in knowing whether or not there are unstable real roots, it is actually unnecessary to examine all edge polynomials, and it suffices, as we now argue, to consider corner polynomials only. We are indebted to J. Ackermann for this derivation. Suppose all corner polynomials are stable. This means that $f(s_0; \hat{\gamma}_1, \ldots, \hat{\gamma}_m) > 0$ for all real non-negative s_0 and $\gamma_i \in \{\underline{\gamma}_i, \bar{\gamma}_i\}$ for all i.

(When $s_0 \to \infty$ the monic character of f ensures that $f(s_0) \to \infty$ and if $f(s_0; \hat{\gamma}_1, \ldots, \hat{\gamma}_m) \leqslant 0$ for some non-negative s_0, then by continuity there would exist $\bar{s}_0 \geqslant 0$ such that $f(\bar{s}_0; \hat{\gamma}_1, \ldots, \hat{\gamma}_m) = 0$.)

Now the inequality $f(s_0; \hat{\gamma}_1, \ldots, \hat{\gamma}_m) > 0$ ensures that $f(s_0; \hat{\gamma}_1, \ldots, \hat{\gamma}_{i-1}, \gamma_i, \hat{\gamma}_{i+1}, \ldots, \hat{\gamma}_m) > 0$ for all $\gamma_i \in [\underline{\gamma}_i, \bar{\gamma}_i]$ since f is affine in γ_i alone, so that $f(s_0; \hat{\gamma}_1, \ldots, \hat{\gamma}_{i-1}, \gamma_i, \hat{\gamma}_{i+1}, \ldots, \hat{\gamma}_m)$ is a convex combination of the two values obtained by identifying γ_i with $\underline{\gamma}_i$ and $\bar{\gamma}_i$.

It is highly probable that one or more edges do need to be tested for stability, to rule out the possibility of either real or complex roots.

Edge tests are the most straightforward; basically, root locus procedures can be used. Actually, it is only necessary to use rational calculation. Suppose that the polynomial $f(s; \gamma_1) = g_0(s) + \gamma_1 g_1(s)$, $\gamma_1 \in (\underline{\gamma}_1, \bar{\gamma}_1)$. Stability is achieved by requiring all Hurwitz determinants to be positive; as functions of γ_1, these determinants are polynomials. So stability is equivalent to certain polynomials in γ_1 being positive for all $\gamma_1 \in [\underline{\gamma}_1, \bar{\gamma}_1]$. This can be checked by Sturm's theorem. Actually, two simplifications are possible. One can use the Liénard–Chipart form of stability conditions, and one only needs to check that all stability conditions are satisfied for one value of γ_1 and the $(n-1)$th Hurwitz determinant is positive for all $\gamma_1 \in [\underline{\gamma}_1, \bar{\gamma}_1]$. Alternatively, a result given by Ackermann and Barmish (1987) using Hurwitz matrices at the corners can be used to test the edges. Another method is given by Zeheb (1987), which requires evaluating the roots of a single polynomial.

3. Significance of the faces for complex roots

In the previous section we have shown that if s_0 is a real root of any real polynomial, it is a real root of an edge polynomial. Now even when $f(s; \gamma_1, \ldots, \gamma_m)$ is linear in the γ_i, the same result, with real replaced by complex, is not true. Rather, any point on the boundary of the complex root set of all polynomials is necessarily a root of an edge polynomial (Bartlett *et al.* 1988). It is thus natural to seek to extend this idea to the structures where $f(\cdot)$ is multilinear in the γ_i.

In general such an extension is impossible.

Example 3

Consider the polynomial

$$\begin{aligned} f(s; \gamma_1, \gamma_2) &= s^2 + (-\gamma_1 - \gamma_2)s + (\gamma_1\gamma_2 + 4) \\ &= s^2 + a_1(\gamma_1, \gamma_2)s + a_2(\gamma_1, \gamma_2) \end{aligned}$$

with $\gamma_1 \in [-1, 1]$ and $\gamma_2 \in [-3, 2]$. In Fig. 3, we have drawn the associated regions of parameter space and coefficient space. Points A_i and B_i correspond; the straight line A_5A_6 corresponds to the curve B_5B_6 (actually part of a parabola) and the two points A_7, A_7' both correspond to B_7. Notice that each of A_1A_2, A_2A_3, A_3A_4 and A_4A_1 maps into a straight line, but these straight lines do *not* bound the image of the rectangle $A_1A_2A_3A_4$.

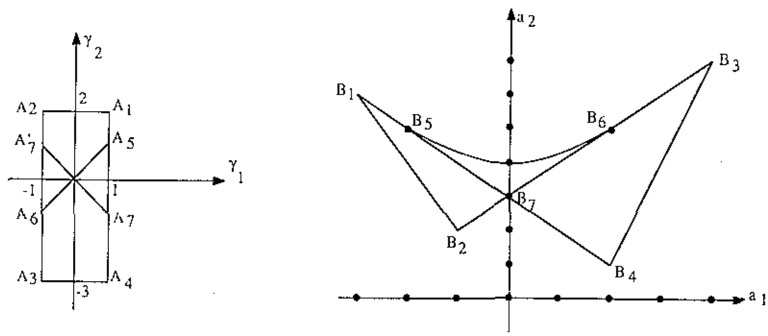

Figure 3. Parameter space and coefficient space.

Now consider the point $\gamma_1 = \gamma_2 = -1/2$. This corresponds to a point on the curve B_5B_6, viz. $a_1 = 1$, $a_2 = 17/4$. There do not exist variations $\Delta\gamma_1$, $\Delta\gamma_2$ around $\gamma_1 = \gamma_2 = -1/2$ which allow perturbations Δa_1, Δa_2 in an arbitrary direction—moving 'above' B_5B_6 is impossible. Consequently, since $\gamma_1 = \gamma_2 = -1/2$ corresponds to a point on the *boundary in coefficient space, and we are working with second-order polynomials, it* also corresponds to a point on the boundary in root space. Obviously, the root is complex. It is easy to see that there is no edge polynomial with the same complex root pair, for there is no point on any one of the straight lines B_1B_2, B_2B_3, B_3B_4, B_4B_1 (which define all the edge polynomials) that corresponds to the polynomial $s^2+s+17/4$.

This example illustrates a further point, which is that the boundary in coefficient space need not correspond with the boundary in parameter space; of course, for this two-dimensional example, this is almost the same statement as that concerning the roots. However, it is non-trivially different for higher degree polynomials.

In this example, the problem arises because within the region of parameter space of interest to us, the jacobian determinant

$$\frac{\partial(a_1, a_2)}{\delta(\gamma_1, \gamma_2)} = \gamma_2 - \gamma_1$$

can take zero values. Were this not the case, then the boundary of the parameter region would map into the boundary of the coefficient region. As we see below, the jacobian determinant is of critical importance in a more general treatment.

We now explain how stability on faces can be checked. This is equivalent to checking stability when there are only two variable parameters. Without loss of generality, let these two parameters be γ_1, γ_2, and let us suppress mention of the other parameters, if any.

The idea is as follows. Suppose it has been established that all edges are stable. Suppose also that for some γ_1, γ_2 there exist unstable roots of $f(s; \gamma_1, \gamma_2)$, then by continuity, there exists a value or values of γ_1, γ_2 for which $f(s; \gamma_1, \gamma_2)$ has a purely imaginary root, and indeed a purely imaginary root on the boundary of the root set. We shall show how such roots can be determined; if none exist, this means that $f(s; \gamma_1, \gamma_2)$ has no roots in $\mathrm{Re}\,(s) \geqslant 0$ over the entire face.

Let $\sigma + j\omega$ be a complex root of $f(s; \gamma_1, \gamma_2)$ and consider the jacobian determinant

$$\begin{aligned}\frac{\partial(\sigma, \omega)}{\partial(\gamma_1, \gamma_2)} &= \frac{\partial(\sigma, \omega)}{\partial(\mathrm{Re}\,f, \mathrm{Im}\,f)} \frac{\partial(\mathrm{Re}\,f, \mathrm{Im}\,f)}{\partial(\gamma_1, \gamma_2)} \\ &= \left[\frac{\partial(\mathrm{Re}\,f, \mathrm{Im}\,f)}{\partial(\sigma, \omega)}\right]^{-1} \frac{\partial(\mathrm{Re}\,f, \mathrm{Im}\,f)}{\partial(\gamma_1, \gamma_2)} \end{aligned} \tag{3.1}$$

Certainly,

$$\frac{\partial(\mathrm{Re}\,f, \mathrm{Im}\,f)}{\partial(\sigma, \omega)}$$

can never be infinite, being the 2×2 determinant of a matrix with entries polynomial in σ and ω. Hence,

$$\frac{\partial(\sigma, \omega)}{\partial(\gamma_1, \gamma_2)}$$

can only be zero if

$$\frac{\partial(\mathrm{Re}\,f, \mathrm{Im}\,f)}{\partial(\gamma_1, \gamma_2)}$$

is zero.

So we must recognize the possibility that boundary values in the root set of all roots of $f(s;\gamma_1,\gamma_2)$, $\gamma_1\in(\underline{\gamma}_1,\bar{\gamma}_1)$, $\gamma_2\in(\underline{\gamma}_2,\bar{\gamma}_2)$ *can* only be achieved where

$$f(s;\gamma_1,\gamma_2)=0 \tag{3.2 a}$$

and

$$\frac{\partial(\operatorname{Re} f,\operatorname{Im} f)}{\partial(\gamma_1,\gamma_2)}=0 \tag{3.2 b}$$

Now the root set of all polynomials $f(s;\gamma_1,\gamma_2)$ with $\gamma_1\in[\underline{\gamma}_1,\bar{\gamma}_1]$, $\gamma_2\in[\underline{\gamma}_2,\bar{\gamma}_2]$ is a union of a finite number of closed connected sets $R_1,\ldots,R_\alpha$, and is bounded. If all edge polynomials are stable, and if $f(s;\gamma_1,\gamma_2)$ has an unstable complex root pair for some γ_1,γ_2, it follows that the boundary of one of the sets R_i intersects the imaginary axis. Then we have proved the 'only if' part of the following proposition. The 'if' part is trivial. □

Proposition 3.1

Consider $f(s;\gamma_1,\gamma_2)$ with $\gamma_1\in[\underline{\gamma}_1,\bar{\gamma}_1]$, $\gamma_2\in[\underline{\gamma}_2,\bar{\gamma}_2]$, a monic nth degree polynomial bilinear in γ_1 and γ_2. Suppose that all edge polynomials are stable. Then at least one polynomial fails to be stable if and only if for some $d=j\omega$, ω real, (3.2 *a*) and (3.2 *b*) are satisfied.

We shall now show that the question of whether (3.2 *a*) and (3.2 *b*) are satisfied for some $s=j\omega$ is an easily answered question, via a procedure which we now indicate.

Let us set

$$\operatorname{Re} f(j\omega)=g_0(\omega)+\gamma_1 g_1(\omega)+\gamma_2 g_2(\omega)+\gamma_1\gamma_2 g_3(\omega) \tag{3.3 a}$$

$$\operatorname{Im} f(j\omega)=h_0(\omega)+\gamma_1 h_1(\omega)+\gamma_2 h_2(\omega)+\gamma_1\gamma_2 h_3(\omega) \tag{3.3 b}$$

Each of the g_i, h_i takes real values for real ω. Observe then that

$$\frac{\partial(\operatorname{Re} f,\operatorname{Im} f)}{\partial(\gamma_1,\gamma_2)}=\det\begin{bmatrix} g_1(\omega)+\gamma_2 g_3(\omega) & g_2(\omega)+\gamma_1 g_3(\omega)\\ h_1(\omega)+\gamma_2 h_3(\omega) & h_2(\omega)+\gamma_1 h_3(\omega)\end{bmatrix}$$

$$=(g_1h_2-g_2h_1)+\gamma_1(g_1h_3-g_3h_1)+\gamma_2(g_3h_2-g_2h_3)=0 \tag{3.4 a}$$

Now (3.2 *a*) implies

$$g_0+\gamma_1 g_1+\gamma_2 g_2+\gamma_1\gamma_2 g_3=0 \tag{3.4 b}$$

$$h_0+\gamma_1 h_1+\gamma_2 h_2+\gamma_1\gamma_2 h_3=0 \tag{3.4 c}$$

From (3.4 *b*) and (3.4 *c*), there follows

$$(g_0h_3-h_0g_3)+\gamma_1(g_1h_3-g_3h_1)+\gamma_2(g_2h_3-g_3h_2)=0 \tag{3.5}$$

Together, (3.4 *a*) and (3.5) allow γ_1, γ_2 to be expressed in terms of the g_i, h_j. Their expressions can then be substituted into (3.4 *b*) to obtain a *single* polynomial equation in ω. It may have no non-zero real solution. If it does, each solution determines values for γ_1, γ_2, via (3.4 *a*) and (3.5). If these values lie inside the allowed region $[\underline{\gamma}_1,\bar{\gamma}_1]\times[\underline{\gamma}_2,\bar{\gamma}_2]$, then instability is proved.

Several other remarks should be made. First, in case $f(\cdot)$ is only linear rather than bilinear in γ_1,γ_2 (which is the situation considered by Bartlett *et al.* 1988), equations

(3.4 *a*), (3.4 *b*) and (3.4 *c*) become

$$g_1 h_2 - h_2 g_1 = 0 \tag{3.6 a}$$

$$g_0 + \gamma_1 g_1 + \gamma_2 g_2 = 0 \tag{3.6 b}$$

$$h_0 + \gamma_1 h_1 + \gamma_2 h_2 = 0 \tag{3.6 c}$$

If there exists a real ω for which (3.6 *a*) is zero, then (3.6 *b*) and (3.6 *c*) can only both be satisfied in cases where $g_0 h_2 - g_2 h_0$ is also zero at this frequency. In this case, the (γ_1, γ_2) pairs satisfying (3.6 *b*) and (3.6 *c*) lie on a straight line, and consequently, there exist edge values of either γ_1 or γ_2 which cause satisfaction for the same ω, i.e. there exists a pair satisfying (3.6) of one of the forms $(\bar{\gamma}_1, \gamma_2)$, $(\underline{\gamma}_1, \gamma_2)$, $(\gamma_1, \bar{\gamma}_2)$ or $(\gamma_1, \underline{\gamma}_2)$. Consequently, any root of a face polynomial on the boundary of the root set is a root of an edge polynomial. Then one never has explicitly to study face polynomials. This is the conclusion of Bartlett *et al.* (1988).

Secondly, decision algebra provides a tool for checking stability across a face which should not be too demanding (see Bose 1982). The Hurwitz determinants depend on two parameters γ_1, γ_2 and have to be checked for positivity inside a rectangle. Algorithms are available for this task, as set out by Bose (1982). These algorithms involve a finite number of rational calculations. The method we have suggested here, which introduces the need for polynomial factorization, is an example of a general approach to decision algebra problems involving the setting up of q polynomial equations in q unknowns.

Thirdly, bilinearity with respect to γ_1, γ_2 has not played a central role here, although it has played a helpful role. The derivation of a single equation in ω through the elimination of γ_1, γ_2 from (3.3) and (3.4 *a*) is more complicated when the dependence of f on γ_1, γ_2 is polynomial rather than multilinear.

Fourthly, the paper of Djaferis (1988) is entirely concerned with the case when a so-called shaping condition is fulfilled, namely $g_3 h_2 - h_3 g_2 \equiv 0$. Clearly, this makes

$$\frac{\partial(\operatorname{Re} f, \operatorname{Im} f)}{\partial(\gamma_1, \gamma_2)}$$

independent of γ_2. It is also easy to check that when this condition holds and also

$$\frac{\partial(\operatorname{Re} f, \operatorname{Im} f)}{\partial(\gamma_1, \gamma_2)}$$

is zero, then $\operatorname{Re} f(j\omega)$ and $\operatorname{Im} f(j\omega)$ are independent of γ_2. Consequently, if $j\omega_0$ is on the boundary of the root set, $j\omega_0$ remains a root on the root set boundary when γ_2 varies. In particular, when γ_2 is set equal to an edge value $\underline{\gamma}_2, \bar{\gamma}_2$ it remains true that $j\omega_0$ is a root. Hence all purely imaginary roots on the root set boundary are roots of edge polynomials, which means that under the condition $g_3 h_2 - h_3 g_2 \equiv 0$ only edge polynomials need to be tested. Obviously, the same holds true if $g_1 h_3 - h_1 g_3 \equiv 0$.

Example 4 (Ackermann *et al.* 1988)

Consider $f(s;\ \gamma_1,\ \gamma_2) = s^3 + (\gamma_1 + \gamma_2 + 1)s^2 + (\gamma_1 + \gamma_2 + 3)s + (2\gamma_1\gamma_2 + 6\gamma_1 + 6\gamma_2 + 1{\cdot}25)$ with $\gamma_1 \in [0{\cdot}3;\ 2{\cdot}5]$ and $\gamma_2 \in [0;\ 1{\cdot}7]$. It turns out (and can be established with the aid of, for example, the Hurwitz test) that the parameter values giving unstable f are defined by the shaded regions in Fig. 4. The boundaries of the regions are given by

$2\gamma_1\gamma_2 + 6(\gamma_1 + \gamma_2) + 1{\cdot}25 = 0$ and $(\gamma_1 - 1)^2 + (\gamma_2 - 1)^2 - 0{\cdot}5^2 = 0$. These points are noted in order to allow comparison with the methods of this paper.

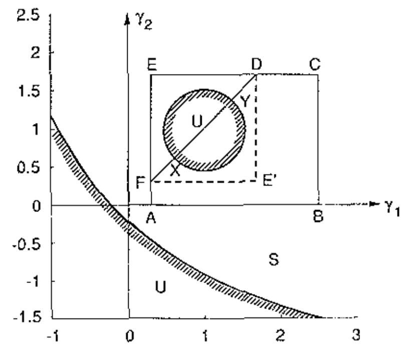

Figure 4. Parameter space and stability region: *s*, stable; *u*, unstable.

First, edge stability must be verified. Let us see how this can be done for one edge, say the edge $\gamma_2 = 0$. The Hurwitz conditions are

$$0 < \det \begin{bmatrix} \gamma_1 + 1 & 6\gamma_1 + 1{\cdot}25 \\ 1 & \gamma_1 + 3 \end{bmatrix} = \gamma_1^2 - 2\gamma_1 + 1{\cdot}75$$

and

$$\gamma_1 + 1 > 0$$
$$\gamma_1 + 3 > 0$$
$$6\gamma_1 + 1{\cdot}25 > 0$$

It is clear that these inequalities are all satisfied for $\gamma_1 \in [0{\cdot}3, 2{\cdot}5]$.

Next, we must look for points in the interior of the parameter region corresponding to purely imaginary roots on the boundary of the root set. These are determined from $\operatorname{Re} f(j\omega; \gamma_1, \gamma_2) = 0$, $\operatorname{Im} f(j\omega; \gamma_1, \gamma_2) = 0$ and

$$\frac{\partial(\operatorname{Re} f, \operatorname{Im} f)}{\partial(\gamma_1, \gamma_2)} = 0$$

The relevant equations are

$$-(\gamma_1 + \gamma_2 + 1)\omega^2 + (2\gamma_1\gamma_2 + 6\gamma_1 + 6\gamma_2 + 1{\cdot}25) = 0$$
$$-\omega^2 + (\gamma_1 + \gamma_2 + 3) = 0$$
$$2(\gamma_1 - \gamma_2) = 0$$

It is readily verified, that these equations are satisfied by

$$\gamma_1 = \gamma_2 = 1 \pm \sqrt{2}/4 \quad \omega = \sqrt{2\gamma_1 + 3} \cong \begin{cases} 2{\cdot}39 \\ 2{\cdot}07 \end{cases}$$

The corresponding points in parameter space are designated by *X*, *Y* in Fig. 4. The root set corresponding to all allowed γ_1, γ_2 is sketched in Fig. 5, and it will be observed

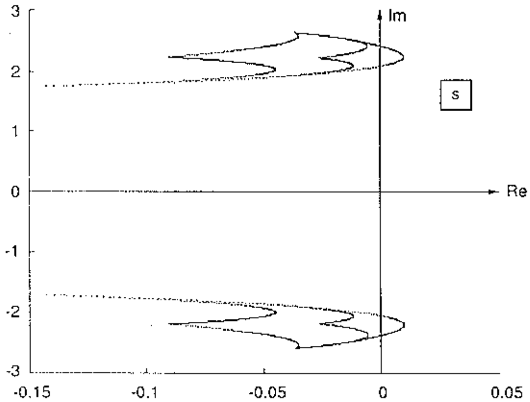

Figure 5. Roots boundary.

that the values for ω computed above define those boundary parts of the root set which lie on the imaginary axis.

In this example, it is also possible exactly to determine the root boundary. Candidates for this boundary are, besides all edge polynomials, also points in the interior of the parameter region $ABCE$ with

$$f(\sigma + j\omega; \gamma_1, \gamma_2) = 0$$

$$\frac{\partial(\operatorname{Re} f, \operatorname{Im} f)}{\partial(\gamma_1, \gamma_2)} = 0$$

The relevant equations are

$$2\gamma_1\gamma_2 + (\gamma_1 + \gamma_2)(6 + \sigma + \sigma^2 - \omega^2) + (1{\cdot}25 + 3\sigma + \sigma^2 + \sigma^3 - \omega^2 - 3\sigma\omega^2) = 0$$

$$(\gamma_1 + \gamma_2)(1 + 2\sigma) + (3 + 2\sigma + 3\sigma^2 - \omega^2) = 0$$

$$2\omega(1 + 2\sigma)(\gamma_1 - \gamma_2) = 0$$

For the complex root boundary, two different cases are distinguished:

Case 1: $\sigma = -0{\cdot}5$

Then

$$\omega_0^2 = 2{\cdot}75 \quad \text{and} \quad \gamma_2 = -(3\gamma_1 + 1{\cdot}25)/(2\gamma_1 + 3)$$

This is an isolated singular point. For variations along the given γ_1, γ_2 hyperbola the root pair $-0{\cdot}5 \pm j\omega_0$ does not change.

Case 2: $\gamma_1 = \gamma_2$

Because of the symmetry of $f(\,\cdot\,)$ with respect to γ_1, γ_2 it is obvious that for γ_1, γ_2 from the triangle DEF the same roots result as from the triangle $DE'F$. Therefore, the root boundaries for γ_1, γ_2 from $ABCE$ and from $ABCDF$ are the same. For γ_1, γ_2 from DF a part of the root boundary is built. This is for γ_1, γ_2 between X and Y unstable.

4. Stability testing in parameter region interior

We have already described how testing of edges and faces may proceed. In an m-dimensional parameter space ($m > 2$) it is necessary to look successively at three-dimensional boundaries (all but three of the γ_i take extreme values), four-dimensional boundaries, ... the interior of the entire m-dimensional region. In each case, we seek to identify frequencies ω such that $j\omega$ is on the boundary of the root set of all polynomials. When looking at say four-dimensional regions, this is done by setting up five simultaneous equations in five unknowns, viz. ω and the four-variable γ_i, and seeking solutions which are real in ω and the γ_i, with each γ_i in the prescribed interval $[\underline{\gamma}_i, \bar{\gamma}_i]$. In the absence of such solutions, it is known that the entire four-dimensional region defines stable polynomials if the three-dimensional regions bounding it are known to define stable polynomials.

We shall explain the idea in more detail for the case when three parameters vary. It is a generalization of the two-variable parameter case considered in the previous section; the generalization to more than three-variable parameters is straightforward.

Let $\sigma + j\omega$ be a complex root of $f(s; \gamma_1, \gamma_2, \gamma_3)$ for $\gamma_i \in (\underline{\gamma}_i, \bar{\gamma}_i)$. Consider the effect of changing the γ_i on the root. In particular, let $\Delta\gamma_i$, $i = 1, 2, 3$, denote very small changes in the γ_i, and let $\Delta\sigma$, $\Delta\omega$ denote the corresponding changes in the root. Then, neglecting second-order terms,

$$\begin{aligned}\begin{bmatrix}\Delta\sigma\\ \Delta\omega\end{bmatrix} &= \begin{bmatrix}\dfrac{\partial\sigma}{\partial\gamma_1} & \dfrac{\partial\sigma}{\partial\gamma_2} & \dfrac{\partial\sigma}{\partial\gamma_3}\\[2ex] \dfrac{\partial\omega}{\partial\gamma_1} & \dfrac{\partial\omega}{\partial\gamma_2} & \dfrac{\partial\omega}{\partial\gamma_3}\end{bmatrix}\begin{bmatrix}\Delta\gamma_1\\ \Delta\gamma_2\\ \Delta\gamma_3\end{bmatrix}\\ &= \begin{bmatrix}\dfrac{\partial\sigma}{\partial\,\mathrm{Re}\,f} & \dfrac{\partial\sigma}{\partial\,\mathrm{Im}\,f}\\[2ex] \dfrac{\partial\omega}{\partial\,\mathrm{Re}\,f} & \dfrac{\partial\omega}{\partial\,\mathrm{Im}\,f}\end{bmatrix}\begin{bmatrix}\dfrac{\partial\,\mathrm{Re}\,f}{\partial\gamma_1} & \dfrac{\partial\,\mathrm{Re}\,f}{\partial\gamma_2} & \dfrac{\partial\,\mathrm{Re}\,f}{\partial\gamma_3}\\[2ex] \dfrac{\partial\,\mathrm{Im}\,f}{\partial\gamma_1} & \dfrac{\partial\,\mathrm{Im}\,f}{\partial\gamma_2} & \dfrac{\partial\,\mathrm{Im}\,f}{\partial\gamma_3}\end{bmatrix}\begin{bmatrix}\Delta\gamma_1\\ \Delta\gamma_2\\ \Delta\gamma_3\end{bmatrix}\\ &= \begin{bmatrix}\dfrac{\partial\,\mathrm{Re}\,f}{\partial\sigma} & \dfrac{\partial\,\mathrm{Re}\,f}{\partial\omega}\\[2ex] \dfrac{\partial\,\mathrm{Im}\,f}{\partial\sigma} & \dfrac{\partial\,\mathrm{Im}\,f}{\partial\omega}\end{bmatrix}^{-1}\begin{bmatrix}\dfrac{\partial\,\mathrm{Re}\,f}{\partial\gamma_1} & \dfrac{\partial\,\mathrm{Re}\,f}{\partial\gamma_2} & \dfrac{\partial\,\mathrm{Re}\,f}{\partial\gamma_3}\\[2ex] \dfrac{\partial\,\mathrm{Im}\,f}{\partial\gamma_1} & \dfrac{\partial\,\mathrm{Im}\,f}{\partial\gamma_2} & \dfrac{\partial\,\mathrm{Im}\,f}{\partial\gamma_3}\end{bmatrix}\begin{bmatrix}\Delta\gamma_1\\ \Delta\gamma_2\\ \Delta\gamma_3\end{bmatrix}\end{aligned} \tag{4.1}$$

Now if we are on the boundary of the root set, there cannot exist perturbations $\Delta\gamma_i$ which can give arbitrary $\Delta\sigma$, $\Delta\omega$. So candidates for values of σ, ω and γ_i yielding a point on the boundary of the root set are given by

$$\operatorname{rank}\begin{bmatrix}\dfrac{\partial\,\mathrm{Re}\,f}{\partial\gamma_1} & \dfrac{\partial\,\mathrm{Re}\,f}{\partial\gamma_2} & \dfrac{\partial\,\mathrm{Re}\,f}{\partial\gamma_3}\\[2ex] \dfrac{\partial\,\mathrm{Im}\,f}{\partial\gamma_1} & \dfrac{\partial\,\mathrm{Im}\,f}{\partial\gamma_2} & \dfrac{\partial\,\mathrm{Im}\,f}{\partial\gamma_3}\end{bmatrix} \leqslant 1 \tag{4.2}$$

Equivalently

$$\frac{\partial(\mathrm{Re}\,f, \mathrm{Im}\,f)}{\partial(\gamma_1, \gamma_2)} = 0 \tag{4.3 a}$$

and

$$\frac{\partial(\mathrm{Re}\,f, \mathrm{Im}\,f)}{\partial(\gamma_1, \gamma_3)} = 0 \tag{4.3 b}$$

as well as

$$\mathrm{Re}\,f = 0, \quad \mathrm{Im}\,f = 0 \tag{4.3 c}$$

It is enough to look for purely imaginary points on the boundary of the root set, i.e. to set $\sigma = 0$. Then (4.3) represent four simultaneous equations in the four unknowns γ_1, γ_2, γ_3, ω. In general, there is a finite number of solutions. If and only if one of these solutions is real, with $\gamma_i \in [\underline{\gamma}_i, \bar{\gamma}_i]$, can there be a purely imaginary point on the boundary set.

The computation of solutions of simultaneous polynomial equations is a problem which has been studied. Older methods have depended on successive elimination of variables using resultants until a single equation in a single variable is obtained. This is solved, and then through successive back substitution, values of the other variables are obtained (see, for example, Bose 1982, and Hodge and Pedoe 1968).

Note that if ω is the variable eliminated from $\mathrm{Re}\,f = 0$, $\mathrm{Im}\,f = 0$ and all other equations are neglected, there results a single equation which corresponds to setting a Hurwitz determinant equal to zero. The terms in this equation depend on the γ_i. The question is then whether this determinant can be made zero for some choice of γ_i in the parameter region of interest or not. This is of course a natural question, and is roughly the approach expounded by Bose (1982).

Example 5

Consider

$$f(s) = s^3 + 1 + (s^2 + s)\gamma_2 + s\gamma_1\gamma_2 + \gamma_1\gamma_2\gamma_3$$

with $\gamma_i \in [\underline{\gamma}_i, \bar{\gamma}_i]$, $i = 1, 2, 3$.

For investigation of the root boundary, it is necessary to check all sides of the γ-cube, i.e. $\gamma_i \in \{\underline{\gamma}_i, \bar{\gamma}_i\}$, $i = 1, 2, 3$, and all points of the interior of the parameter region with

$$\mathrm{Rank}\ \mathbb{J} \leqq 1, \quad \mathrm{Re}\,f = 0, \quad \mathrm{Im}\,f = 0$$

where $\mathbb{J}$ is the Jacobi matrix:

$$\mathbb{J} = \begin{bmatrix} \sigma\gamma_2 + \gamma_2\gamma_3 & \sigma^2 - \omega^2 + \sigma + \sigma\gamma_1 + \gamma_1\gamma_3 & \gamma_1\gamma_2 \\ \gamma_2 & 2\sigma + 1 + \gamma_1 & 0 \end{bmatrix}$$

the relevant equations are

$$\mathrm{Re}\,f = \sigma^3 - 3\sigma\omega^2 + 1 + \gamma_2(\sigma^2 - \omega^2 + \sigma) + \gamma_1\gamma_2\sigma + \gamma_1\gamma_2\sigma + \gamma_1\gamma_2\gamma_3$$

$$\mathrm{Im}\,f = (3\sigma^2 - \omega^2 + \gamma_2(2\sigma + 1) + \gamma_1\gamma_2)\omega$$

$$\gamma_1\gamma_2^2 = 0$$

$$\gamma_1\gamma_2(2\sigma + 1 + \gamma_1) = 0$$

$$(2\sigma + 1 + \gamma_1)(\sigma + \gamma_3) - \sigma^2 + \omega^2 - \sigma - \sigma\gamma_1 - \gamma_1\gamma_3 = 0$$

These equations must be fulfilled simultaneously.

For $\gamma_2 = 0$ the polynomial family degenerates to

$$f(s) = s^3 + 1$$

a simple polynomial. Rank $\mathbb{J}$ is zero. No special investigation is necessary.

For $\gamma_1 = 0$ we obtain

$$f(s) = s^3 + 1 + (s^2 + s)\gamma_2$$

and from rank $\mathbb{J} = 1$ the condition

$$(\gamma_2 - 2)\gamma_3 = 1$$

results, which is a hyperbola in the $\gamma_2\gamma_3$ plane. Along this hyperbola, there are pairs $\gamma_1 = 0$, γ_2, γ_3 which give possible boundary polynomials. However, from the degeneration of the polynomial family $f(s)$ it is obvious that rank $\mathbb{J} = 1$ follows from the degeneration of the parameter space to a straight line. Therefore, there exist internal points of the parameter space, which fulfil the necessary conditions for the root boundary. However, these points do not yield this boundary.

Next, the six sides of the parameter box must be checked. In general, this can be done by testing all boundaries of such a side and all candidates for the root boundary from the side inners. These are exactly the same steps as we have done before but on a lower dimensional space. In this way, we proceed for every side until the two-dimensional faces are reached.

In our special case, the box sides are directly the two-dimensional faces. The procedures of the last section can be used.

For $\gamma_1 =$ constant we have $\mathrm{Re} f = 0$ and $\mathrm{Im} f = 0$, and

$$(2\sigma + 1 + \gamma_1)\gamma_1\gamma_2 = 0$$

With $\gamma_1 \neq 0 \neq \gamma_2$, and $\mathrm{Im} f = 0$, we obtain

$$2\sigma + 1 + \gamma_1 = 0$$

$$\omega^2 = 3\sigma^2 + \gamma_2(2\sigma + 1) + \gamma_1\gamma_2$$

and then

$$\sigma = -\tfrac{1}{2}(1 + \gamma_1)$$

$$\omega^2 = 3\sigma^2$$

The critical points of the γ_1-sides yield only a point in the s-space and because of the continuity conditions not a significant part of the root boundary.

In the same way one may proceed for γ_2- and γ_3-sides.

Recently, methods for solving simultaneous polynomial equations based on homotopy theory have been suggested (see Watson *et al.* 1987).

A number of further points should be noted. First, in this section, no special use has been made of the multilinearity, i.e. the same ideas will apply even if the dependence of the coefficients on the parameters is a general polynomial dependence.

Secondly, it is easy to recover various special cases. Suppose following Panier *et al.* (1987) that the coefficients of even powers of $f(\cdot)$ depend on $\gamma_1, \ldots, \gamma_r$ and the coefficients of odd powers of $f(\cdot)$ depend on $\gamma_{r+1}, \ldots, \gamma_m$. The condition (4.2) then becomes

$$\mathrm{rank} \begin{bmatrix} \dfrac{\partial \,\mathrm{Re} f}{\partial \gamma_1} & \cdots & \dfrac{\partial \,\mathrm{Re} f}{\partial \gamma_r} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \dfrac{\partial \,\mathrm{Im} f}{\partial \gamma_{r+1}} & \cdots & \dfrac{\partial \,\mathrm{Im} f}{\partial \gamma_m} \end{bmatrix} \leqslant 1$$

Suppose for example that

$$\frac{\partial \operatorname{Re} f}{\partial \gamma_i} = 0, \quad i = 1, \dots, r$$

and

$$\frac{\partial \operatorname{Im} f}{\partial \gamma_j} = 0, \quad j = r+1, \dots, m-1$$

Now $\operatorname{Re} f$ is multilinear in $\gamma_1, \dots, \gamma_r$, and accordingly can take no extreme value inside the region $\gamma_i \in [\underline{\gamma}_i, \bar{\gamma}_i]$ unless that value is also assumed on the boundary. Thus if for some ω,

$$\frac{\partial \operatorname{Re} f}{\partial \gamma_i} = 0$$

this continues to hold when the γ_i take extreme values. Similarly, if

$$\frac{\partial \operatorname{Im} f}{\partial \gamma_j} = 0, \quad j = r+1, \dots, m-1$$

for some ω, these equations continue to hold when the γ_j take extreme values. Hence the rank condition, if fulfilled anywhere, is necessarily fulfilled when all but one γ_i, say γ_m, take extreme values, i.e. it is fulfilled on the edge defined by variable γ_m. Consequently, it is only necessary to check edges for stability.

Another special case is provided by the shaping conditions of Djaferis and Hollot (1988). To fix ideas, suppose that f depends on four parameters, with

$$f(s; \gamma_1, \gamma_2, \gamma_3, \gamma_4) = \varphi_1(s) f_1(\gamma_1, \gamma_2) + \varphi_2(s) f_2(\gamma_1, \gamma_2) + \varphi_3(s) f_3(\gamma_1, \gamma_4) + \varphi_4(s) f_4(\gamma_3, \gamma_4)$$

The $\varphi_i(s)$ are of course independent of the γ_i. Moreover, $\varphi_i(j\omega) = E_{\varphi_i}(j\omega) + j\omega O_{\varphi_i}(j\omega)$ where

$$E_{\varphi_i}(j\omega) = \frac{1}{2}[\varphi_i(j\omega) + \varphi_i(-j\omega)], \; O_{\varphi_i}(j\omega) = \frac{1}{2\omega}[\varphi_i(j\omega) - \varphi_i(-j\omega)]$$

and the side conditions (shaping conditions)

$$E_{\varphi_1} O_{\varphi_2} - E_{\varphi_2} O_{\varphi_1} = 0, \quad E_{\varphi_3} O_{\varphi_4} - E_{\varphi_4} O_{\varphi_3} = 0$$

hold identically in ω. The 2×4 jacobian matrix condition becomes

$$\operatorname{rank} \begin{bmatrix} E_{\varphi_1} \frac{\partial f_1}{\partial \gamma_1} + E_{\varphi_2} \frac{\partial f_2}{\partial \gamma_1} & E_{\varphi_1} \frac{\partial f_1}{\partial \gamma_2} + E_{\varphi_2} \frac{\partial f_2}{\partial \gamma_2} & E_{\varphi_3} \frac{\partial f_3}{\partial \gamma_3} + E_{\varphi_4} \frac{\partial f_4}{\partial \gamma_3} & E_{\varphi_3} \frac{\partial f_3}{\partial \gamma_4} + E_{\varphi_4} \frac{\partial f_4}{\partial \gamma_4} \\ O_{\varphi_1} \frac{\partial f_1}{\partial \gamma_1} + O_{\varphi_2} \frac{\partial f_2}{\partial \gamma_1} & O_{\varphi_1} \frac{\partial f_1}{\partial \gamma_2} + O_{\varphi_2} \frac{\partial f_2}{\partial \gamma_2} & O_{\varphi_3} \frac{\partial f_3}{\partial \gamma_3} + O_{\varphi_4} \frac{\partial f_4}{\partial \gamma_3} & O_{\varphi_3} \frac{\partial f_3}{\partial \gamma_4} + O_{\varphi_4} \frac{\partial f_4}{\partial \gamma_4} \end{bmatrix} \leqslant 1$$

Now the shaping conditions ensure that the minors formed from columns 1 and 2 and from columns 3 and 4 are zero automatically. Suppose the minor formed from columns 1 and 3 is zero. By the multiplinearity, column 1 is independent of γ_1 and column 3 is independent of γ_3. The special form of f ensures that column 1 is independent of γ_3 and column 3 is independent of γ_1. Hence, if the minor formed from columns 1 and 3 is zero, it must remain so if γ_1 and γ_3 are varied to extreme values. The shaping condition ensures that the minors formed from columns 1 and 2 and columns 3 and 4 remain zero with this variation of γ_1 and γ_3. Similarly, one can argue that γ_2 and γ_4 could be varied to their extreme values. Hence if the jacobian matrix has reduced rank somewhere, it has this property for all γ_i. A consequence of this is that the image of $f(j\omega; \gamma_1, ..., \gamma_m)$ as ω waries, $\gamma_i \in [\underline{\gamma}_i, \bar{\gamma}_i]$ is a set bounded by the images of the edges. In general, this is a polytope. However, with the jacobian matrix of rank 1, the image will be a line segment, and when of rank 0, it will be a point.

A third special case can be obtained by limiting the way in which the non-linear parameter dependence arises. Specifically, assume that any one γ_i can occur bilinearly with at most one other parameter γ_j, and that in the polynomial $f(s; \gamma_1, ..., \gamma_m)$ the s-polynomial multiplying $\gamma_i\gamma_j$ is either even or odd. An easy calculation shows that this ensures that all 2×2 minors of the generalized jacobian matrix are linear in the parameters. The solution of the associated simultaneous equations is made much easier in these circumstances.

5. Differing and converging necessary and sufficient conditions

We have referred earlier to the work of de Gaston and Safonov (1988), who exploited the observations of Zadeh and Desoer (1963) that the image of $f(j\omega; \gamma_1, ..., \gamma_m)$ for fixed ω and $\gamma_i \in [\underline{\gamma}_i, \bar{\gamma}_i]$ lies in the convex hull of $f(j\omega; \gamma_1, ..., \gamma_m)$ for $\gamma_i \in \{\underline{\gamma}_i, \bar{\gamma}_i\}$ to develop a test for stability based on Nyquist ideas.

It is possible to exploit the observation of Zadeh and Desoer (1963) in another way. Denote the corners of the allowed parameter space region by $A_1, ..., A_k$ where $k = 2^n$. Denote the corresponding points in the n-dimensional coefficient space by $B_1, ..., B_k$. Now a perusal of the argument of Zadeh and Desoer (which involves scalar functions dependent multilinearly on m parameters) shows that it extends very straightforwardly to vector functions. As a result, the image in coefficient space of any point in the allowed parameter region, i.e. in the convex hull of $A_1, ..., A_k$ necessarily lies in the convex hull of $B_1, ..., B_k$.

Figure 6 depicts a rectangular region in parameter space and certain straight lines

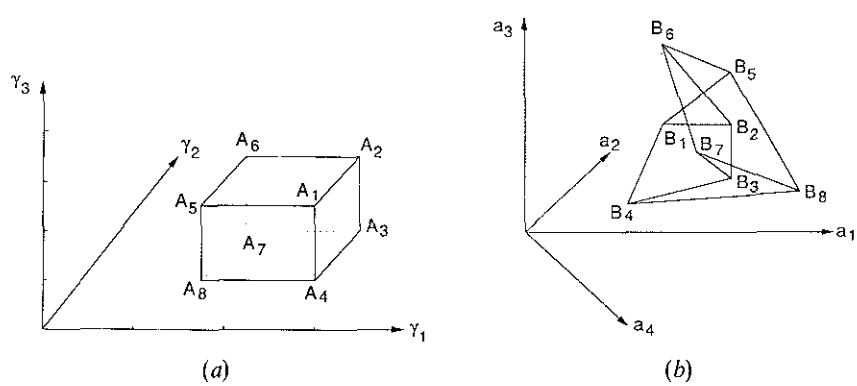

Figure 6. Parameter and coefficient space convex hull covering: (a) parameter space three-dimensional; (b) coefficient space four-dimensional.

in its image in coefficient space. These straight lines are the images of the edges in parameter space. It is possible to construct the convex hull of $B_1, \ldots, B_8$ by joining with straight lines all possible so-far unjoined pairs of points and then 'filling in' the enclosed region. Thus straight lines such as $B_4 B_7$, $B_1 B_6$, etc. must be joined. Note that $B_4 B_7$ is *not* the image of the straight line $A_4 A_7$ (on a certain face) in parameter space. The image of $A_4 A_7$ will in general be curved, and be within the convex hull determined by B_1 through B_8.

A necessary condition for robust stability is clearly that the edges in parameter space (or their images in coefficient space) are all stable. A sufficient condition is that *the straight lines joining all possible pairs of corner points in coefficient space* (i.e. those which are images of parameter space edges and those which are not) must be stable; for the edge theorem of Bartlett *et al.* (1988) ensures that all points in coefficient space in the convex hull of $B_1, \ldots, B_k$ will be stable, and so in particular those that are images of points in the defined region of parameter space.

Now if the necessary conditions for stability are fulfilled and the sufficiency ones are not, one can proceed in a similar fashion to de Gaston and Safonov (1988). That is, one partitions the original rectangular box in parameter space in two, and develops separate necessity and sufficiency conditions. More precisely, if in the example $B_2 B_7$ proves to contain an unstable polynomial one could make a slice in parameter space parallel to $A_1 A_2 A_3 A_4$ or parallel to $A_1 A_2 A_6 A_5$ thus ensuring that A_2, A_7 go into different rectangular boxes. Then the line $B_2 B_7$ will no longer enter into a sufficiency condition.

To the original necessity conditions are added four more, while a number of the original sufficiency conditions fall away to be replaced by a greater number of collectively less demanding conditions.

6. Concluding remarks

This paper has extended consideration of the robust polynomial stability problem by allowing mild forms of non-linear dependence of the polynomial coefficients on variable parameters. It is seen very rapidly that even this mild form of dependence introduces substantial complications, so that for example the edge theorem applicable with affine dependence is probably no longer a valid tool. The key to examining interior points in parameter space is to consider a generalized jacobian matrix and study the points where its rank is 1 or 0. Various special cases can be identified which allow a conclusion like that of the edge theorem to be applied. It would be interesting to expand this range of special cases.

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