Channels Leading to Rapid Error Recovery for Decision Feedback Equalizers

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Abstract—**When a decision feedback equalizer is used on a channel satisfying a simple passivity constraint (equivalently expressible in terms of gain-phase constraints) the error recovery time is finite, and thus recovery is rapid, regardless of the initial error state and the particular data sequence. This class of nontrivial channels includes cases of practical interest and identifies some channels for which a decision feedback equalizer is a practical option.**

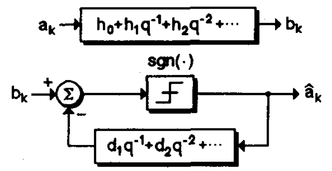

Fig. 1. Channel and DFE models.

I. Introduction

A DECISION feedback equalizer (DFE) is a simple hardware device to cancel intersymbol interference (ISI) generated by a distorting channel. However, one major problem with its operation is an effect known as error propagation [1] which we now describe (see Fig. 1). The DFE operates by feeding back past data estimates called decisions, which generally will not correspond to the actual input data sequence if recent past decision errors have been made. Because past decisions are used to cancel the ISI of the real data, any decision errors may lead to a deterioration of the effectiveness of the cancelling operation of the DFE at future times. Hence, errors in the present data estimate will increase the likelihood of future estimation errors, and so on.

The presence of error propagation means that DFE operation in practice may be unsatisfactory, in the sense that the time for the DFE to recover from any error condition may be unacceptably long [2], [3]. In fact, it has been shown that over the class of all finite inpulse response (FIR) channels of length N the mean error recovery time may be of order 2^N data periods (even for some which are minimum phase or near minimum phase), which is evidently totally impractical. It then becomes a problem to identify stronger hypotheses on the channel model for which the error recovery time is sufficiently short, as judged by practical standards. For these channels we can say then that a DFE is a practical option. Determining such a class of (nontrivial) channels is the objective of this paper.

In general, we can classify two classes of channel which are acceptable from a practical point of view—the distinction is not great. The first class needs some statistical model of the input sequence for its definition. To define the class, one then simply requests that the expected or mean time for error recovery for all initial error states is sufficiently short. However, this leaves open the possibility that there exist pathological input sequences [2], [4] for which errors are made after any arbitrary length of time even in the absence of noise. In a well-defined sense, however, these sequences are probability zero events [2], [4]. The second class of channels are those for which the error recovery time is finite for all possible input sequences and initial conditions. In this case the statistical model of the input is largely irrelevant. Further, this means that pathological input sequences are nonexistent, i.e., one can never be so unlucky as to have an input sequence for which the DFE never performs satisfactorily—a most attractive property. In our work we find a broad and robust class of channels for which the error recovery time is finite. As such we are defining a class of channels suffering from significant ISI for which a DFE may be effectively used. This class captures a range of practical channels as we will see from an example.

In the literature there has been very little written about the error recovery properties of DFE's. In fact only in [5], [6] has it been indicated theoretically that there are some nontrivial channels for which the DFE operates satisfactorily. The two prominent early references analyzing DFE's [1], [4] both give little comfort to the practicing engineer who finds their structural simplicity appealing. In [1], [4] the given bounds on recovery time and error probability actually correspond to the worst realizable channel models as was demonstrated in [2], [3]. Because the DFE has such deplorable performance when operating on channels with these bounds, the results are not very useful in practice (but of theoretical importance and interest). We note here also the work in [7], [8] which strives to reduce these bounds given explicit, i.e., specific, knowledge of the channels. In contrast here we give a broad general nontrivial condition on the channel parameters—specifically the coefficients satisfy a passivity constraint or equivalently a simple frequency response condition—to ensure good DFE error recovery performance.

The following sections are organized as follows. In Section II we define the DFE system of interest and we define our finite error recovery time problem. In Section III we give the necessary background on passivity theory. In Section IV we give our basic main result which establishes that whenever the channel satisfies a simple frequency domain constraint, the error recovery time of an ideal DFE is always finite. We also include four applications of this theorem including analysis of a real channel. Convergence rates and explicit bounds given an exponential overbound on the channel impulse response are the subject of Section V. Results of greater practical interest

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where we relax most of the major idealized assumptions of the previous sections are given in Section VI. We also give the result for M-ary data and relate the error recovery time bound back to the binary case. A formula for the error probability given a high signal to noise ratio channel is presented. The conclusions and summary are given in Section VII.

II. Problem Formulation and Definitions

A communication channel and general nonadaptive decision feedback equalizer (DFE) are shown in Fig. 1. The communication channel is modeled as a linear, time-invariant filter with impulse response,

$$h \triangleq \{h_0, h_1, h_2, \cdots\} \tag{2.1}$$

of possibly infinite dimension. This channel is driven by an input binary sequence $\{a_k\}$ where k is the discrete time index. No statistical model of $\{a_k\}$ is assumed nor needed. The M-ary $\{a_k\}$ case will also be treated in a later section. We note that in a more general context h could be thought of as the cascade (convolution) of the linear channel and a linear equalizer preceding the DFE.

The distorted output of the linear channel is b_k and is assumed noiseless. By studying the noiseless case we are creating a pointer to the important practical situation of a high SNR channel. (In a later section, we will introduce an additive noise signal into the analysis but only treat the asymptotic case as the noise variance tends to zero). At the receiving end we have a DFE consisting of a tapped delay line with impulse response

$$d \triangleq \{0, d_1, d_2, \cdots\} \tag{2.2}$$

fed by a binary output decision sequence $\{\hat{a}_k\}$ as described by Fig. 1.

The algebraic formulation of the system depicted in Fig. 1 is given by

$$\hat{a}_k = \operatorname{sgn}\left(h_0 a_k + \sum_{i=1}^{\infty} h_i a_{k-i} - \sum_{i=1}^{\infty} d_i \hat{a}_{k-i}\right); \quad h_0 \geq 0 \tag{2.3}$$

where ideally we would like $d_i = h_i$, $\forall i > 0$. Note also that we assume without loss of generality that $h_0 \geq 0$ (if $h_0 = 0$, see Section V). Hence, the study of error propagation under these ideal conditions leads to the equation

$$\hat{a}_k = \operatorname{sgn}(h_0 a_k + r_k) \tag{2.4a}$$

where

$$r_k \triangleq \sum_{i=1}^{\infty} h_i e_{k-i} \tag{2.4b}$$

and

$$e_k \triangleq a_k - \hat{a}_k. \tag{2.4c}$$

Most of the ideal assumptions represented in (2.4) will be relaxed in Section VI. Here it is convenient to treat the ideal case first so that we may focus on the technique employed and not get lost in a labyrinth of unimportant detail.

We now define what we mean by error recovery.

Definition: The DFE has recovered from error at time k if

$$\hat{a}_k = a_k, \text{ or equivalently, } e_k = 0 \ \forall k \geq K, \ \forall\{a_k\}.$$

Now if we rewrite (2.4) as $\hat{a}_k = \operatorname{sgn}((h_0 + r_k a_k) a_k)$ then it is clear that $h_0 > |r_k|$ ensures $h_0 + r_k a_k > 0$ and thus a sufficient condition for DFE recovery at time K is

$$h_0 > |r_k| \ \forall k \geq K, \quad \forall\{a_k\}. \tag{2.5}$$

However, this condition (2.5) is also necessary because the definition for error recovery stipulates no errors can be made when we consider all possible input sequences. So that a particular input sequence which is generated by $a_k = -\operatorname{sgn}(r_k)$ $\forall k \geq K$ must give no errors, and the desired conclusion follows.

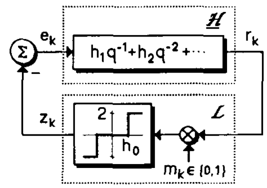

Fig. 2. Error propagation block diagram.

From (2.5) it is clear that the residual ISI term r_k is crucial in understanding the error propagation and error recovery mechanisms. We complete this section with a simple but fundamental lemma which is a mild generalization of the above analysis and so we omit a proof. In it we see that the only way an error can be made in the noiseless DFE is for the residual ISI r_k to have magnitude greater than h_0 and to be of opposite polarity to the binary data a_k.

Lemma 1: Let r_k in (2.4b) denote the ISI and h_0 the cursor. Then

(i) $|r_k| < h_0$ or $a_k = \operatorname{sgn}(r_k) \Rightarrow \hat{a}_k = a_k \Leftrightarrow e_k = 0$.

(ii) $|r_k| > h_0$ and $a_k = -\operatorname{sgn}(r_k) \Rightarrow \hat{a}_k = -a_k$

$$\Leftrightarrow e_k = -2\operatorname{sgn}(r_k).$$

Lemma 1 is significant because it characterizes the error propagation mechanism. The feedback system represented in Fig. 2 is a pictorial representation of Lemma 1. The upper block in Fig. 2 is just a block representation of equation (2.4b). It is modeled by a strictly causal convolutional operator $\mathcal{H}$ which maps the error sequence $e \triangleq \{e_k, k \geq 0\}$ to the residual ISI sequence $r \triangleq \{r_k, k \geq 0\}$ in accordance with (2.4b), i.e., $r = \mathcal{H}e \triangleq h \circledast e$ where $h \triangleq \{0, h_1, h_2, \cdots\}$. The lower block $\mathcal{L}$ in Fig. 2 consists of two parts. The first is a stochastic multiplier defined by

$$m_k \triangleq \begin{cases} 1 & \text{if } a_k = -\operatorname{sgn}(r_k); \\ 0 & \text{if } a_k = +\operatorname{sgn}(r_k), \end{cases} \tag{2.6}$$

whose function is clear from Lemma 1(i), i.e., if $m_k = 0 \Rightarrow a_k = \operatorname{sgn}(r_k) \Rightarrow e_k = 0$. Otherwise, m_k does nothing, i.e., takes the value unity (2.6). The second part of the lower block is a time-invariant nonlinearity which maps $\{m_k r_k\}$ into the sequence $z \equiv -e$. Note whenever the input $m_k r_k$ is less in magnitude than h_0 the output $z_k = -e_k$ is zero [as in Lemma 1(i)]. Otherwise, the output conforms to Lemma 1(ii). Note that in this block the stochastic multiplier and the nonlinearity may be commuted. The point of introducing the summation block to perform the inversion of z to e is that the blocks form a feedback system suggesting the use of stability arguments.

A significant observation we make concerning the lower block $\mathcal{L}$ in Fig. 2 is that it is a memoryless nonlinearity whose graph is confined to the first and third quadrants. As such whenever the output is nonzero we see the output preserves the sign of the input and therefore is a passive operator in the circuit theoretic sense [9]. In this paper, we transform the system in Fig. 2 such that the transformation of the upper block $\mathcal{H}$ becomes a strictly passive operator whilst the transformation of the lower block $\mathcal{L}$ remains passive. Then we utilize some standard results from input-output stability to show the DFE has a (quantifiable) finite recovery time. In the next section, we present the minimal set of definitions and notation needed to develop the general input-output stability result.

III. Passivity Analysis

The idea of reformulating the error recovery problem as a stability problem originated with Cantoni and Butler [4]. We take up this concept and it is natural to investigate the use of stability ideas in proving that under certain conditions a DFE has a finite recovery time (for all initial conditions and for all input sequences). The ideas we need have their origins within circuit theory. Our main result uses the Passivity Theory [9] to give an easily checked frequency domain condition that guarantees a finite recovery time.

We begin with some definitions which are standard in input-output stability theory [9]. We focus on a Hilbert space structure composed of real valued sequences indexed by $k \in \mathbb{Z}_+$ (nonnegative integers). Then if we have two sequences given by $x \triangleq \{x_0, x_1, \cdots\}$ and $y \triangleq \{y_0, y_1, \cdots\}$ then their inner product will be defined as

$$\langle x, y\rangle \triangleq \sum_{i=0}^{\infty} x_i y_i \tag{3.1}$$

where it is clear that $\langle x, y\rangle = \langle y, x\rangle$. This inner product (3.1) induces a natural Euclidean norm defined by

$$\|x\| \triangleq \langle x, x\rangle^{1/2} = \left(\sum_{i=0}^{\infty} x_i^2\right)^{1/2}. \tag{3.2}$$

We define the discrete function space l_2 which consists of all sequences satisfying

$$x \in l_2 \Leftrightarrow \|x\| < \infty. \tag{3.3}$$

Similarly, we have the space l_1 which consists of all sequences satisfying $x \in l_1 \Leftrightarrow \|x\|_1 \triangleq \Sigma_{i=0}^{\infty} |x_i| < \infty$. The space l_2 is generally too restrictive an arena for deriving results, so we introduce the standard concept of an extended space l_2^e [9], defined by

$$x \in l_2^e \Leftrightarrow \|P_T x\| < \infty, \ \forall T \in \mathbb{Z}_+ \tag{3.4}$$

where P_T is a truncation operator parametrized by $T \in \mathbb{Z}_+$ defined by

$$(P_T x)(k) \triangleq \begin{cases} x_k, & \text{if } k \le T; \\ 0, & \text{if } k > T. \end{cases}$$

Note (3.4) just says that $x \in l_2^e$ if and only if $|x_k| < \infty \ \forall k$. So, for example, if $x \triangleq \{x_k = 2^k, \forall k \in \mathbb{Z}_+\}$ then $x \in l_2^e$ but $x \notin l_2$.

From definitions (3.3) and (3.4) it is apparent that $l_2 \subset l_2^e$. In our work, all signals considered will lie in the extended space l_2^e (because we stipulate only that $h \in l_1$). However, it is of great interest to show that particular signals also lie in the subset l_2. For example, with the error signal, it is our aim to show $e \in l_2$. Then because $e_k \in \{-2, 0 +2\}$ we have the following fundamental observation:

$$e \in l_2 \Leftrightarrow e_k = 0, \ \exists K < \infty \text{ such that } \forall k \ge K, \tag{3.5}$$

i.e., the DFE has recovered from error at finite time K.

Now define $\|x\|_T \triangleq \|P_T x\|$, and $x_T \triangleq P_T x$. In relating l_2 and l_2^e we note the following important properties of the inner product and its induced norms which we will use later without explicit reference:

1) $\forall x \in l_2^e$ the mapping $T \mapsto \|x\|_T$ is monotonically increasing.

2) $\forall x \in l_2 \lim_{T\to\infty} \|x\|_T = \|x\|$.

3) $\forall x, y \in l_2^e$, $\forall T \in \mathbb{Z}_+$ we have $\langle x_T, y_T\rangle = \langle n_T, y\rangle = \langle x, y_T\rangle \triangleq \langle x, y\rangle_T$.

This leads to the crucial definitions of passivity.

Definition: An operator $\mathcal{H}: l_2^e \mapsto l_2^e$ is *passive* if $\exists$ constant

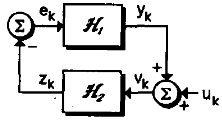

Fig. 3. Passivity theorem block diagram.

β such that

$$\langle \mathcal{H}x, x\rangle_T \ge \beta, \quad \forall x \in l_2^e \ \forall T \in \mathbb{Z}_+. \tag{3.6}$$

If $\mathcal{H}$ were linear then β could be taken as zero (this is a derived result, see [9]).

Definition: An operator $\mathcal{H}: l_2^e \mapsto l_2^e$ is *strictly passive* if $\exists \delta > 0$ and $\exists \beta$ such that

$$\langle \mathcal{H}x, x\rangle_T \ge \delta \|x\|_T^2 + \beta, \quad \forall x \in l_2^e \ \forall T \in \mathbb{Z}_+. \tag{3.7}$$

Again if $\mathcal{H}$ were linear then β could be taken as zero. We label δ as the *degree of passivity*.

As an example of passivity (but not strict passivity), which will be important later, let us check the claim at the end of Section II concerning the lower block $\mathcal{L}$ of Fig. 2. Suppose $x \in l_2^e$ is the input to an operator $\mathfrak{H}$ with output $y \triangleq \mathfrak{H}x$, which satisfies $y_k x_k \ge 0$, $\forall k \in \mathbb{Z}_+$ (a sign preserving operator). Then trivially

$$\langle \mathfrak{H}x, x\rangle_T = \sum_{k=0}^{T} y_k x_k \ge 0, \quad \forall x \in l_2^e \ \forall T \in \mathbb{Z}_+, \tag{3.8}$$

showing $\mathfrak{H}$ is passive according to definition (3.6) with $\beta = 0$. That is, if $\mathfrak{H}$ is a nonlinearity constrained to the first and third quadrants then it is passive (even if it is time-varying or has memory).

Our second example which we state as a lemma will be important later and relates to the definition of strict passivity (3.7) applied to *linear* operators. The proof is a simple adaptation from the continuous time proof given in [9], and is therefore omitted.

Lemma 2: Suppose $\mathcal{G}: l_2^e \mapsto l_2^e$ is defined by $\mathcal{G}u = g \circledast u$ where $g \triangleq \{g_0, g_1, \cdots\} \in l_1$. Let $\delta > 0$. Then

$$\langle \mathcal{G}u, u\rangle_T \ge \delta \|u\|_T^2, \ \forall u \in l_2^e, \ \forall T \in \mathbb{Z}_+ \Leftrightarrow \mathrm{Re}\,(\tilde{g}(e^{j\theta})) \ge \delta, \quad \forall \theta \in [0, 2\pi] \tag{3.9}$$

where $\tilde{g}(z) \triangleq \Sigma_{i=0}^{\infty} g_i z^{-i}$ is the Z-transform of the impulse response g.

Lemma 2 says that a linear convolutional operator is strictly passive if and only if its Nyquist plot belongs to $\{z \in \mathbb{C}: \mathrm{Re}(z) \ge \delta\}$ where $\mathbb{C}$ is the complex plane.

We now come to the main passivity theorem. Fig. 3 defines the signals and operators of interest. In it e and v are the input sequences to the operators $\mathcal{H}_1$ and $\mathcal{H}_2$ and $y = \mathcal{H}_1 e$ and $z = \mathcal{H}_2 v$ are the respective output sequences. There is a single external signal u. (Note in comparing Fig. 3 to Fig. 2 the idea of introducing an external signal u is to model an initial error state in the DFE at time $k = 0$ which triggers the error propagation. When we come to apply the passivity theorem in Section IV this will become clearer). All signals shown are assumed to lie in l_2^e. The following theorem and proof are an adaptation of a more general result in [9, p. 182].

(Passivity) Theorem 3: Suppose 1) Operator $\mathcal{H}_1$ is linear and strictly passive, i.e.,

$$\langle \mathcal{H}_1 e, e\rangle_T \ge \delta_1 \|e\|_T^2, \quad \forall e \in l_2^e \ \forall T \in \mathbb{Z}_+ \tag{3.10}$$

where $\delta_1 > 0$, and 2) operator $\mathcal{H}_2$ is a nonlinearity confined to the first and third quadrant, implying

$$\langle \mathcal{H}_2 v, v\rangle_T \ge 0, \quad \forall v \in l_2^e \ \forall T \in \mathbb{Z}_+ \tag{3.11}$$

by (3.8), and is thus passive. Then $u \in l_2 \Rightarrow e \in l_2$.

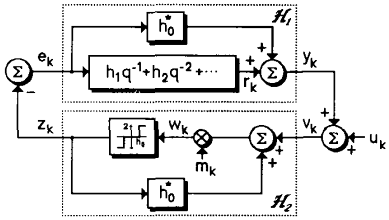

Fig. 4. Loop transformation.

Proof: We show $e \in l_2$ by determining upper and lower bounds on the quantity $\langle \mathcal{H}_1 e, e\rangle_T + \langle \mathcal{H}_2 v, v\rangle_T$. First we determine a lower bound. Using (3.10) and (3.11) we clearly have

$$\langle \mathcal{H}_1 e, e\rangle_T + \langle \mathcal{H}_2 v, v\rangle_T \geq \delta_1 \|e\|_T^2 \ \forall T \in \mathbb{Z}_+ \quad (3.12)$$

where, recall, $\delta_1 > 0$ is the constant associated with the degree of passivity of the $\mathcal{H}_1$ operator. An upper bound on (3.12) follows from the next simple calculation, using Fig. 3,

$$\begin{aligned}&\langle \mathcal{H}_1 e, e\rangle_T + \langle \mathcal{H}_2 v, v\rangle_T\\ &= \langle \mathcal{H}_1 e, e\rangle_T + \langle -e, v\rangle_T = \langle -e, v - \mathcal{H}_1 e\rangle_T\\ &= \langle -e, u\rangle_T \leq \|e\|_T \|u\|_T, \ \forall T \in \mathbb{Z}_+ \quad (3.13)\end{aligned}$$

where the last line is an application of the Cauchy–Schwartz inequality. Then combining (3.13) with (3.14) we obtain $\|e\|_T \leq \delta_1^{-1}\|u\|_T \ \forall T \in \mathbb{Z}_+$ whenever $\|e\|_T > 0$. Letting $T \to \infty$ we find

$$\|e\| \leq \delta_1^{-1}\|u\|, \quad (3.14)$$

i.e., $u \in l_2 \Rightarrow e \in l_2$ as desired. □

IV. Sufficient Conditions for a Finite Recovery Time

In this section, we transform the system in Fig. 2 so that we may apply the general passivity theorem of the last section. This involves two steps. The first step is to apply a loop transformation because $\mathcal{H}$ (Fig. 2) is not passive. The second step is to model the effects of initial conditions at time $k = 0$, i.e., an initial (arbitrary) error state, by an external signal u as in the passivity theorem.

We apply a loop transformation [9] to the system in Fig. 2 to obtain the new system shown in Fig. 4. Note that the effect of the newly introduced feedforward and feedback paths with gains h_0^* is to cancel exactly. The upper block labeled $\mathcal{H}_1$ has impulse response given by

$$\{h_0^*, h_1, h_2, \cdots\} \quad (4.1)$$

where h_0^* is a finite gain associated with the feedforward path. For the passivity theorem to apply we need (4.1) strictly passive, i.e., h_0^* sufficiently positive, and we have available Lemma 2 as a test in the frequency domain.

In the lower block labeled $\mathcal{H}_2$, which includes the positive feedback of gain h_0^*, we need to be concerned that we have not destroyed the passivity of the original lower block (Fig. 2). The following lemma with proof now applies. The symbol definitions are given in Fig. 4.

Lemma 4: If $0 \leq h_0^* \leq h_0/2$ then $\mathcal{H}_2$ (Fig. 4) is passive.

Proof: The $\mathcal{H}_2$ block has input v_k and output $z_k \in \{-2, 0, +2\}$. We attempt to show $v_k z_k \geq 0$, $\forall k$ which ensures passivity. From Fig. 4 the input w_k to the sector nonlinearity within the $\mathcal{H}_2$ block is given by $w_k = m_k(v_k + h_0^* z_k)$ from which we have after multiplying through by z_k,

$$m_k v_k z_k = (w_k - m_k h_0^* z_k) z_k, \qquad \forall k \in \mathbb{Z}_+. \quad (4.2)$$

We have three cases according to the three possible values of $z_k \in \{-2, 0 +2\}$ (see Fig. 4): 1) $z_k = +2 \Rightarrow m_k = 1$ and $w_k \geq h_0$, which implies from (4.2) that $v_k z_k = 2(w_k - 2h_0^*) \geq 2(h_0 - 2h_0^*) \geq 0$, given $0 \leq h_0^* \leq h_0/2$, i.e., $v_k z_k \geq 0$; 2) $z_k = -2 \Rightarrow m_k = 1$ and $w_k \leq -h_0$ leading to $v_k z_k \geq 0$ by symmetry; and 3) $z_k = 0$ which gives $v_k z_k = 0$ because $v \in l_2^e$, i.e., $|v_k| < \infty$ $\forall k$. Thus, $v_k z_k \geq 0$ $\forall k$ in every case, implying $\mathcal{H}_2$ is passive by (3.8). □

Another condition which needs to be fulfilled in Theorem 3 is $u \in l_2$. This condition will necessitate some hypothesis on the channel h to be fulfilled. The signal u for our application will model the effects of initial conditions in the $\mathcal{H}_1$ block since all our sequences are defined only for $k \geq 0$, whereas the real system may have been operating from the distant past, i.e., $k = -\infty$. Note that this signal u, as shown in Fig. 4, is unaffected by the introduction of h_0^*. From Fig. 2, we use superposition on the upper $\mathcal{H}$ linear operator of impulse response $\{0, h_1, h_2, \cdots\}$ to represent the effects of arbitrary initial conditions, i.e., an arbitrary initial error state via the signal

$$u_k \triangleq \sum_{i=k+1}^{\infty} h_i \eta_{k-i}, \qquad k \in \mathbb{Z}_+ \quad (4.3)$$

where values $\eta - 1, \eta - 2, \eta - 3, \cdots$, taking values in $\{-2, 0, +2\}$ define the initial (error) state at time $k = 0$. (Every initial condition can be represented in this form.) To ensure $u \in l_2$ we impose some sufficient conditions on the channel h.

Lemma 5: Suppose $h \in l_2^e$ satisfies $|h_m| = O(m^{-\eta})$ as $m \to \infty$ where η is constant. Then

$$1)\ \eta > 1 \Rightarrow h \in l_1, \quad (4.4a)$$

$$2)\ \eta > \frac{3}{2} \Rightarrow u \in l_2. \quad (4.4b)$$

Proof: 1) Is straightforward. 2) It is easy to show from (4.3) that $|u_k| \leq 2\sum_{i=k+1}^{\infty}|h_i| = O(k^{-\eta+1})$ as $k \to \infty$, by using integral approximations to the summations. Then $p_k \triangleq u_k^2 = O(k^{-2\eta+2})$ as $k \to \infty$. However, $u \in l_2$ if and only if $p \in l_1$. Using (4.4a) on p this implies $2\eta - 2 > 1$, i.e., $\eta > 3/2$, is sufficient. □

We state our first main DFE result.

Theorem 6: Suppose a channel $h \triangleq \{h_0, h_1, \cdots\}$, used for binary transmissions of symbols $\{a_k\}$, satisfies $|h_m| = O(m^{-3/2-\epsilon})$ as $m \to \infty$ where $\epsilon > 0$. Suppose $\exists\delta > 0$ such that

$$\mathrm{Re}\left(\tilde{h}(e^{j\theta}) - \frac{h_0}{2}\right) \equiv \frac{h_0}{2} + \sum_{m=1}^{\infty} h_m \cos(m\theta) \geq \delta, \qquad \forall\theta \in [0, 2\pi] \quad (4.5)$$

where $\tilde{h}(z)$ denotes the Z-transform of h. Then given an ideal DFE output sequence $\{\hat{a}_k\}$ generated through (2.4), then for some $K < \infty$, we have $\hat{a}_k = a_k$, $\forall k \geq K$.

Proof: By Lemma 5(1) the constraint on the channel implies $h \in l_1$, thus $\tilde{h}(e^{j\theta})$ exists, and we can use Lemma 2. Set $h_0^* = h_0/2$ in Fig. 4. By Lemma 2 we have $\mathrm{Re}\,(\tilde{h}(e^{j\theta}) - h_0/2) \geq \delta$, $\forall\theta \in [0, 2\pi]$ if and only if operator $\mathcal{H}_1$ in Fig. 4 is linear and strictly passive. Operator $\mathcal{H}_2$ in Fig. 4, on the other hand, is passive by Lemma 4. By Lemma 5(2) the constraint on the channel implies $u \in l_2$, therefore, Theorem 3 applies and we deduce $e \in l_2$, which proves the result. □

A somewhat clearer and conceptually simpler result takes the form.

Corollary 7: DFE's with weights correctly adjusted to the coefficients of an exponentially stable channel h whose Nyquist plot $\tilde{h}(e^{j\theta})$ satisfies

$$\mathrm{Re}\,(\tilde{h}(e^{j\theta})) > \frac{h_0}{2}, \quad \forall\theta \in [0, 2\pi]$$

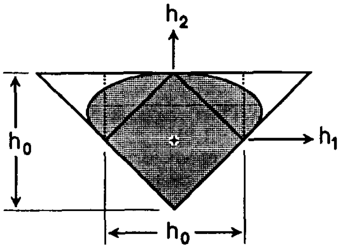

Fig. 5. 3 tap FIR channel regions.

have finite error recovery times regardless of the initial conditions, and regardless of the input sequence.

An explicit bound on the error recovery time is the subject of Section V. Note, a finite recovery time means there are not any pathological input sequences [2], [4]. Now we look at some applications of Theorem 6.

Example (1): Suppose (4.4b) is satisfied, $\exists\delta' > 0$ and

$$\frac{h_0}{2} \geq \sum_{i=1}^{\infty} |h_i| + \delta'. \tag{4.6}$$

Then it follows that (4.5) is trivially satisfied. In fact condition (4.6) is equivalent to $h_0 > |r_k|$ $\forall k \in \mathbb{Z}_+$. In this case, the DFE has always recovered by (2.5), i.e., it never makes errors (in the absence of noise).

Example (2): Let the channel be FIR with impulse response $\{h_0, h_1, h_2\}$ (such that $h_0 > 0$). Then condition (4.5) is simply

$$\frac{h_0}{2} + h_1 \cos\theta + h_2 \cos 2\theta \geq \delta, \qquad \forall\theta \in [0, 2\pi]. \tag{4.7}$$

This defines a region as $\delta \to 0$ in (h_1, h_2)-space shown shaded in Fig. 5 (see [10]). Note this ice-cream cone region consists of two straight boundaries which tangent into an ellipse at points $(\pm 2/3 h_0, 1/6 h_0)$. This region is shown sandwiched between two other regions: (1) an inner diamond which is (4.6), and (2) an outer triangle which is the region which defines the necessary and sufficient conditions for a finite recovery time (demonstrated in [2]). This highlights that the converse of Theorem 6 is false (see Section VI.B).

Example (3): Let $h \triangleq \{h_m = \beta^m \cos(m\omega), \forall m \in \mathbb{Z}_+\}$ where $0 \leq \beta < 1$ and $\omega \in [0, 2\pi]$. Then $\tilde{g}(z) \triangleq \tilde{h}(z) - 1/2$ is given by

$$\tilde{g}(z) = \frac{\frac{1}{2}(z^2 - \beta^2)}{z^2 - 2\beta\cos\omega z + \beta^2}.$$

We need $\tilde{g}(e^{j\theta})$ to have positive real part and this can be shown to be equivalent to checking

$$\text{Re}\,((e^{j2\theta} - \beta^2)(e^{-j2\theta} - 2\beta\cos\omega e^{-j\theta} + \beta^2)) > 0, \qquad \forall\theta \in [0, \pi]. \tag{4.8}$$

But the left-hand side of (4.8) can be decomposed thus

$$\begin{aligned} 1 - 2\beta\cos\omega\cos\theta(1-\beta^2) - \beta^4 &= (1-\beta^2)((1-\beta)^2 + 2\beta(1-\cos\theta\cos\omega)) \\ &\geq (1-\beta^2)(1-\beta)^2 > 0. \end{aligned}$$

Thus all decaying exponential channels with impressed sinusoidal oscillation (of the appropriate phase) have a finite recovery time. This same result but restricted to the case where $\omega = 0$ was proven in [6]. Note that one can extend this result to $h \triangleq \{h_m = \alpha^m \cos(m\omega + \phi), \forall m \in \mathbb{Z}_+\}$ to conclude that one can tradeoff α against ϕ and maintain $\tilde{g}(z)$ strictly passive provided ϕ remains close to zero.

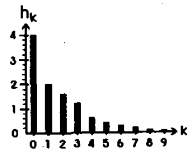

Fig. 6. 3 km twisted pair cable response.

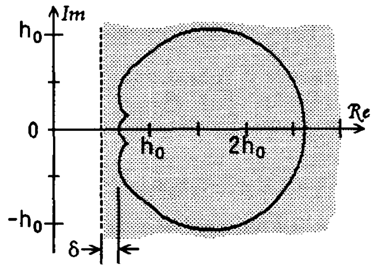

Fig. 7. Nyquist plot for twisted pair cable.

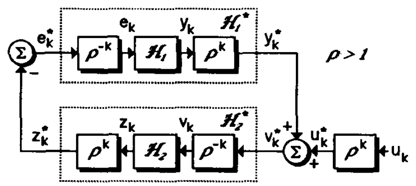

Fig. 8. Convergence rates via multipliers.

Example (4): Fig. 6 shows the (sampled) measured impulse response of a 3 km twisted pair copper cable which is the line between a subscriber and a local exchange [11]. Fig. 7 shows the Nyquist plot of the same channel. Since the closed Nyquist curve lies completely to the right of the line Re $(\tilde{h}(e^{j\theta})) = h_0/2$ (the shaded region in Fig. 6), then Theorem 6 establishes any error recovery time is finite. Also shown in Fig. 7 is a geometrical interpretation of δ in Theorem 6. This type of channel is ideal for the use of a DFE.

V. Convergence Rates and Explicit Bounds

Theorem 6 gives no indication of the maximum time one needs to wait before the DFE returns to an error-free mode. Intuitively, the more dissipative the upper block $\mathcal{H}_1$ is (Fig. 3), i.e., the greater is δ_1, the more rapidly the error signal should go to zero. We investigate this intuitive insight further.

Consider Fig. 8 which shows the use of multipliers [9] to transform Fig. 3. The signals e_k, y_k, u_k, v_k, and z_k are identical to those in Fig. 3, being unaffected by the introduction of the multipliers. We will be applying Theorem 6 to the new starred system where $\mathcal{H}_1^*$ maps $e_k^* = \rho^k e_k$ to $y_k^* = \rho^k y_k$, and $\mathcal{H}_2^*$ maps $v_k^* = \rho^k v_k$ to $z_k^* = \rho^k z_k$. The new external signal is now $u_k^* = \rho^k u_k$. We take the multiplier $\rho > 1$.

Now since we take $\rho > 1$ we trivially have $\text{sgn}(v_k) = \text{sgn}(v_k^*)$ and $\text{sgn}(z_k) = \text{sgn}(z_k^*)$. Thus $\mathcal{H}_2^*$ is passive since $\mathcal{H}_2$ is so. To check that $\mathcal{H}_1^*$ is strictly passive is simplified by linearity. It is easy to see that the Z-transforms of $\mathcal{H}_1^*$: $e_k^* \mapsto y_k^*$ and $\mathcal{H}_1$: $e_k \mapsto y_k$ are related through

$$\tilde{h}_1^*(z) = \tilde{h}_1(z\rho^{-1}). \tag{5.1}$$

This implies that more stringent conditions need to be enforced on the channel h than those given by Lemma 5 if the starred signals are to belong to l_2 and then other conditions need to be checked for passivity. Appealing to Lemma 5, the only sensible condition for stability takes the following form.

Assumption:

$$\text{For some } 0<\gamma<1,\ |h_i|<B\gamma^i,\ \forall i \in \mathbb{Z}_+. \tag{5.2}$$

This assumption ensures $\tilde{h}_1^*(z)$ has an impulse response in l_1. Then $\mathcal{H}_1^*$ is strictly passive if and only if for some $\delta_1^* > 0$

$$\text{Re}\left(\tilde{h}_1^*(e^{j\theta})-\frac{h_0}{2}\right)\geq\delta_1^*,\quad \forall\theta \in [0, 2\pi]. \tag{5.3}$$

The main difficulty before we can invoke Theorem 3 is to show $u^* \in l_2$. Using (5.2) and (4.3), we may prove the following, noting $u_k^* = \rho^k u_k$,

$$\begin{aligned}\|u^*\|^2 &\triangleq \sum_{k=0}^{\infty}\rho^{2k}\left|\sum_{i=k+1}^{\infty}h_i\eta_{k-i}\right|^2\\ &\leq 4B^2\sum_{k=0}^{\infty}\rho^{2k}\left|\sum_{i=k+1}^{\infty}\gamma^i\right|^2\\ &=\frac{4B^2\gamma^2}{(1-\gamma)^2}\sum_{k=0}^{\infty}(\rho\gamma)^{2k}=\frac{4B^2\gamma^2}{(1-\gamma)^2(1-(\rho\gamma)^2)}\end{aligned} \tag{5.4}$$

provided $|\rho\gamma| < 1$, i.e., $\|u^*\| < \infty$. Thus with an exponential overbound of the channel and $|\rho\gamma| < 1$, Theorem 3 applies to the starred system in Fig. 8 and we conclude from (3.14) that

$$\begin{aligned}\|e^*\|^2 &\triangleq \sum_{k=0}^{\infty}|e_k^*|^2\leq\delta_1^{*-2}\|u^*\|^2\\ &\leq\delta_1^{*-2}\frac{4B^2\gamma^2}{(1-\gamma)^2(1-(\rho\gamma)^2)}\end{aligned} \tag{5.5}$$

i.e., $e^* \in l_2$ provided $|\rho\gamma| < 1$). This provides an exponential rate of decay on $|e_k| = \rho^{-k}|e_k^*| \leq \rho^{-k}\|e^*\|$. However e_k is restricted to the set $\{-2, 0, +2\}$ and therefore must be zero after some time $K(\rho) \in \mathbb{Z}_+$ which is the least integer satisfying

$$2>\frac{2B\gamma}{\delta_1^*}\frac{\rho^{-K(\rho)}}{(1-\gamma)\sqrt{(1-(\rho\gamma)^2)}}, \tag{5.6}$$

i.e., the least integer $K(\rho) \in \mathbb{Z}_+$ such that,

$$K(\rho)\geq\log_\rho(B\gamma)-\log_\rho(\delta_1^*(1-\gamma^2))-\frac{1}{2}\log_\rho(1-(\rho\gamma)^2). \tag{5.7}$$

This $K(\rho)$ is an explicit error recovery time bound that we desired. We will not elaborate further but rather give an example which makes the above analysis clearer and shows how to determine a suitable multiplier ρ.

We consider the special case of Example (3) given in Section IV by setting $\omega = 0$, i.e., $h_i = \gamma^i$, $\forall i \in \mathbb{Z}_+$ for some $0 < \gamma < 1$ (this case resembles Fig. 6). This channel trivially satisfies (5.2) with $B = 1$. For this channel it can be shown using elementary analysis that

$$\text{Re}\left(\tilde{h}^*(e^{j\theta})-\frac{1}{2}\right)=\frac{\frac{1}{2}(1-(\rho\gamma)^2)}{1-2\rho\gamma\cos\theta+(\rho\gamma)^2} \tag{5.8}$$

where ρ is chosen such that $\gamma < \rho\gamma < 1$. [Note also $h_i^* = (\rho\gamma)^i$, $\forall i \in \mathbb{Z}_+$, by (5.1)]. From (5.8) the δ_1^* associated with strict passivity of $\mathcal{H}_1^*$ is given by $\delta_1^* = 1/2(1 - \rho\gamma)/(1 + \rho\gamma)$, being the minimum of (5.8) achieved when $\theta = \pi$. We can now use (5.7) to compute the bound on the error recovery time for various $\rho > 1$. To obtain the tightest bound we can optimize over $1 < \rho < 1/\gamma$, noting $K(\rho) \to \infty$ whenever $\rho \to 1/\gamma$ or $\rho \to 1$. We give three numerical examples: 1) $\gamma = 0.50$ then using (5.7), we can determine an optimum $\rho \approx 1.642$ yielding $\delta_1^* = 0.0492$ leading to $K_{\text{opt}} \approx K(1.642) = 8$, 2) $\gamma = 0.81$ with optimum $\rho \approx 1.194$ yielding $\delta_1^* = 0.0083$ leading to $K_{\text{opt}} \approx K(1.194) = 43$, and 3) $\gamma = 0.95$ with $\rho \approx 1.047$ yielding $\delta_1^* = 0.0014$ leading to $K_{\text{opt}} \approx K(1.047) = 258$.

TABLE I
ERROR RECOVERY TIME BOUNDS

Analysis Technique	$\gamma = 0.50$	$\gamma = 0.81$	$\gamma = 0.95$
Passivity Theory (5.7)	8	43	258
Exponential Results [6]	2	11	71
Markov Processes [1-4]	6*	4094*	5×10^{21}*

* These bounds are on the mean not the maximum recovery time.

These bounds are conservative by the nature of the analysis. In [6] (but only for exponential channels), it is shown that the tight bounds on the maximum error recovery times are 2, 11, and 71, respectively. It is interesting to compare both sets of bounds (see the first two rows of Table I) with mean error recovery time bounds which can be deduced from the DFE literature based on Markov processes [1]–[4]. Of course being statistical bounds we need a statistical model of the input sequence $\{a_k\}$—an independent, equiprobable binary distribution being standard. This does not invalidate the comparison because the error recovery time bound $K(\rho)$ in (5.7) always *overbounds* the true mean error recovery time.

To compute the mean error recovery time bounds based on the work in [1] we define an effective channel length n for the exponential channel. This is given by the minimum n such that

$$2\sum_{i=n+1}^{\infty}\gamma^i=\frac{2\gamma^{n+1}}{1-\gamma}<1. \tag{5.9}$$

The meaning attached to the quantity n is simply that the DFE needs to make n consecutive correct decisions to recover from any error state with $\hat{a}_{k-1} \neq a_{k-1}$ (k being the present instant of time). Now for the worst case channels implicitly considered in [1]–[4], subject to (5.9), the probability of making an error is precisely 1/2 for every decision before recovery (i.e., before n consecutive correct decisions have been made). By the theory of success runs [4] the mean recovery time is given by $2(2^n - 1)$. Looking at our three examples we have: 1) $\gamma = 0.50$ implying $n = 2$ and thus a mean recovery time of 6, 2) $\gamma = 0.81$ implying $n = 11$ and thus a mean recovery time of 4094, and 3) $\gamma = 0.95$ implying $n = 71$ and thus a mean recovery time of 5×10^{21}. These three bounds are displayed in the third row of Table I.

Table I shows that using the theory of Markov processes one may get ridiculously conservative results, even though we have (minimally) exploited some structural assumptions (5.9). Also note that here the Markov techniques are incapable of telling us directly that the recovery time is finite. The Markov mean bounds in Table I can presumably be improved on by the techniques in [7], [8]. However, the amount of computation that would be necessary looks formidable.

VI. Some Generalizations

A. Error Recovery Under Imperfect Equalization

This subsection represents a three-fold generalization of the previous results. These modifications involve, in part, relaxation of some of the previous assumptions regarding the model of the system under study. The generalizations are as follows: 1) the DFE tapped delay line is assumed to be FIR of length N rather than IIR, whilst the channel may be IIR; 2) the assumption that $d_i = h_i$, $\forall i \geq 1$ is relaxed to a condition which stipulates the d_i are sufficiently close but not necessarily equal to some ideal values, and 3) the results are generalized to the situation where error-free behavior is characterized by $\hat{a}_k = \text{sgn}(h_\delta)a_{k-\delta}$, $\forall k \geq K$ for some fixed delay $\delta \in \{0, 1, \cdots, N\}$ rather than $\hat{a}_k = a_k$, $\forall k \geq K$. All these generalizations will be treated in parallel. A key feature of the analysis performed in this subsection is showing explicitly the close relationship between eye diagrams and rates of error recovery.

As some motivation to studying delay-type behavior, alluded to above, consider the situation where a DFE has its taps adapted blindly, i.e., without a training sequence. In this case, it was shown in [12] that the DFE taps may adapt not only to an (ideal) equilibrium where $d_i = h_i$, $i \in \{1, 2, \cdots, N\}$ but also to a delay equilibrium where $d_i = \text{sgn}(h_\delta)h_{i+\delta}$, $i \in \{1, 2, \cdots, N\}$ provided certain conditions are met. We will show that when in the vicinity of a *delay equilibrium*, after some finite time K, all decisions will be of the form $\hat{a}_k = \text{sgn}(h_\delta)a_{k-\delta}$ $\forall k \geq K$, hence the terminology.

To analyze nonideal behavior we take (2.3) and set $d_i = 0$ for $i > N$, i.e., the tapped delay line is FIR of length N rather than IIR. Define $\sigma_\delta \triangleq \text{sgn}(h_\delta)$. We can decompose (2.3) as follows:

$$\hat{a}_k = \text{sgn}\left(\sum_{i=0}^{\infty} h_i a_{k-i} - \sum_{i=1}^{N} d_i \hat{a}_{k-i}\right) \quad (6.1a)$$

$$= \text{sgn}\,(h_\delta a_{k-\delta} + r_k(\delta) + s_k(\delta) + t_k(\delta)) \quad (6.1b)$$

where

$$r_k(\delta) \triangleq \sum_{i=\delta+1}^{N+\delta} h_i(a_{k-i} - \sigma_\delta \hat{a}_{k+\delta-i}) \quad (6.1c)$$

$$s_k(\delta) \triangleq \sum_{i=0}^{\delta-1} h_i a_{k-i} + \sigma_\delta \sum_{i=\delta+1}^{N+\delta} (h_i - \sigma_\delta d_{i-\delta})\hat{a}_{k+\delta-i} \quad (6.1d)$$

and

$$t_k(\delta) \triangleq \sum_{i=N+\delta+1}^{\infty} h_i a_{k-i}. \quad (6.1e)$$

In (6.1), 1) $r_k(\delta)$ acts as the basic residual ISI term [note if we let $\delta = 0$ and $N \to \infty$ then (6.1c) becomes (2.4b)]; 2) $s_k(\delta)$ is a term which generally gets smaller as the taps $(d_1, d_2, \cdots, d_N)'$ approach the δ-delay equilibrium at $\sigma_\delta(h_{\delta+1}, h_{\delta+2}, \cdots h_{\delta+N})'$, and includes any precursor; and 3) $t_k(\delta)$ is that part of the tail of the channel which cannot be modeled by the DFE because the tapped delay line is FIR.

Beginning with $t_k(\delta)$ in (6.1e), it is clear that we need

$$|t_k(\delta)| \leq \sum_{i=N+\delta+1}^{\infty} |h_i| \triangleq \Phi, \ \forall k \in \mathbb{Z}_+ \quad (6.2)$$

with Φ sufficiently small else the DFE problem is not well posed, i.e., N, the number of DFE taps, needs to be chosen large enough in the first place so that the DFE can effectively cancel the ISI.

Now when in the vicinity of a delay equilibrium we claim $\hat{a}_k = \sigma_\delta a_{k-\delta}$ $\forall k \geq K$ provided certain conditions are met, which we now determine. Define new (delay) errors

$$e_k(\delta) \triangleq a_{k-\delta} - \sigma_\delta \hat{a}_k \quad (6.3)$$

then the basic residual ISI term $r_k(\delta)$ (6.1c) may be written

$$r_k(\delta) \triangleq \sum_{i=\delta+1}^{N+\delta} h_i e_{k+\delta-i}(\delta) \quad (6.4)$$

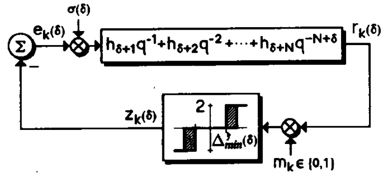

Fig. 9. Imperfect equalization error propagation.

and will be zero whenever we make N consecutive correct δ-delay decisions. Now suppose

$$\Delta_{\min}(\delta) \triangleq |h_\delta| - \sum_{i=0}^{\delta-1} |h_i| - \sum_{i=\delta+1}^{N+\delta} |h_i - \sigma_\delta d_{i-\delta}| - \sum_{i=N+\delta+1}^{\infty} |h_i| > 0. \quad (6.5)$$

Then some perusal, based on (6.1b) when (6.5) holds, will show that whenever N consecutive correct decisions are made, all future decisions will be correct (in the delay sense) because $r_k(\delta) = 0$ and h_δ is larger in magnitude than $s_k(\delta) + t_k(\delta)$ can ever be. This defines a new form of error recovery, i.e., (6.5) is a sufficient condition for all decisions to be (delay) correct whenever N consecutive δ-delay decisions have been made. Note that if all decisions are to be of the form $\hat{a}_k = \sigma_\delta a_{k-\delta}$ for all input sequences, given N consecutive correct decisions have been made, then condition (6.5) is also a necessary condition (see [12] which treats a similar problem).

Define $\Delta_k(\delta) \triangleq (h_\delta a_{k-\delta} + s_k(\delta) + t_k(\delta))\sigma_\delta a_{k-\delta}$, noting that by (6.5) we have $\Delta_k(\delta) \geq \Delta_{\min}(\delta) > 0$. From (6.1b), $\hat{a}_k = \text{sgn}(\sigma_\delta a_{k-\delta}\Delta_k(\delta) + r_k(\delta))$; then clearly the analogue of Lemma 1 is as follows.

Lemma 8: Suppose condition (6.5) holds. Then

1) $|r_k(\delta)| < \Delta_k(\delta)$ or $a_{k-\delta} = \sigma_\delta\,\text{sgn}\,(r_k(\delta)) \Rightarrow \hat{a}_k = \sigma_\delta a_{k-\delta}$.

2) $|r_k(\delta)| > \Delta_k(\delta)$ and $a_{k-\delta} = -\sigma_\delta\,\text{sgn}\,(r_k(\delta)) \Rightarrow \hat{a}_k = -\sigma_\delta a_{k-\delta}$.

Thus we have the picture in Fig. 9 which differs marginally from Fig. 2. Note the lower block is sector bounded within the 1st and 3rd quadrants whilst $\Delta_{\min}(\delta) > 0$. The critical value at which $r_k(\delta)$ causes a change from $z_k = 0$ to $z_k = +2$ is $\Delta_k(\delta)$ and is thus time-varying [but bounded below by $\Delta_{\min}(\delta)$]—we have depicted this behavior by a fuzziness of the switching value in the nonlinearity in Fig. 9. The generalization of Theorem 6 is then as follows.

(Imperfect Equalization) Theorem 9: Suppose the parameters of a binary linear channel $h \triangleq \{h_0, h_1, \cdots\}$ and of the DFE tapped delay line $d \triangleq \{0, d_1, d_2, \cdots\}$ satisfy (6.5) for some (at most one) delay $\delta \in \{0, 1, \cdots, N\}$ and $\sigma_\delta \triangleq \text{sgn}(h_\delta)$. Further, suppose $\exists \xi > 0$ such that

$$\frac{\Delta_{\min}(\delta)}{2} + \sigma_\delta \sum_{m=1}^{N} h_{m+\delta}\cos(m\theta) \geq \xi, \quad \forall \theta \in [0, 2\pi].$$

Given a nonideal DFE output sequence $\{\hat{a}_k\}$ generated through (6.1a), then for some $K < \infty$, we have $\hat{a}_k = \sigma_\delta a_{k-\delta}$, $\forall k \geq K$.

Remarks:

1) The asymptotic condition on h in Theorem 6, in reality, controls the behavior of the tail of the ideal DFE tap setting, not the tail of the channel. That is why such a condition is absent in Theorem 9. However we still have $h \in l_1$ because $\Phi < \infty$ (6.2) is implied by (6.5) and this implies $h \in l_1$.

2) Note condition (6.5) stipulates that the d_i need to be

sufficiently close to the $\sigma_\delta h_{i+\delta}$ (in an l_1-norm sense) if a certain operator is to be strictly passive. Note that the worse the mismatch, the less $\Delta_{\min}(\delta)$ will be. This forms a convenient geometrical picture to replace the messy algebra.

4) Note $\Delta_{\min}(\delta)$ may be interpreted precisely as the amount that a certain eye diagram is open after recovery. Thus, the wider the postrecovery eye can be, the more rapid one can expect recovery to be. Theorem 9 is saying that given the eye is initially closed (an arbitrary error state) it will always open after at most some finite time K (and again this is quantifiable).

B. Comparison with the Exact Theory

An exact theory treating error recovery capable (in principle) of providing necessary and sufficient conditions on the system parameters for finite error recovery times and related problems can be found in [2]. One conclusion of [2] is that if inclusion in a certain region of the channel parameter space is a necessary and sufficient condition for a (guaranteed) finite error recovery time then that region is, without exception, a union of a (countable) number of polytopes, i.e., the region is bounded by hyperplanes. In contrast, the region determined in Theorem 6 has in general some curved boundaries (see Fig. 5). Thus we can see immediately that Theorem 6 can only be a sufficiency result—a conclusion we arrived at earlier. However, it is quite easy to strengthen Theorem 6 such that the region appearing in (4.5) is replaced by a suitable union of polytopes which contains the region (4.5). For example, in Fig. 5, the passivity analysis ice-cream cone region can in fact be replaced by the outer triangle in the theorem statement. The reason is the following. The property which defines the polytopes in [2] is that all points interior to a given polytope have indistinguishable error recovery properties [a manifestation of the sgn(·) quantization in (2.3)]. Let us refer to all points inside a given polytope as *isomorphic,* then we have the following straightforward extension of Theorem 6.

[Extended] Theorem 9: All channels h which are isomorphic to at least one channel satisfying the passivity constraint in (4.5) have a guaranteed finite recovery time.

Remarks:

1) For example in Fig. 5 the outer triangle is composed of five polytopes (see also [2]) each of which intersect with the passivity region in (4.7). Therefore, e.g., $h = \{2, 1.5, 0.75\}$ violates (4.7) (e.g., at $\theta = 135°$) but is isomorphic to $h = \{2, 1.1, 0.50\}$ which does satisfies (4.7).
2) The degree of passivity $\delta_1 > 0$ that we can associate with any channel h can be maximized by searching over all channels which are isomorphic to h, thus giving a tighter overbound on the error recovery rate, e.g., $h = \{2, 0, 0.9\}$ has $\delta_1 = 0.1$ but is isomorphic to $h = \{2, 0, 0\}$ with $\delta_1 = 1$. This may explain why the passivity theory does not give the tight result of [6] in Table I.
3) For FIR channels with less than or equal to three parameters, Theorem 9 provides both necessary and sufficient conditions for a guaranteed finite recovery time. It is not known whether this property holds for higher dimensions.

C. M-ary Results

The theory developed for binary systems can be extended to larger alphabets where M symbols are used. We outline some of the important differences. For brevity we restrict attention to zero delay systems. Let $\{a_k\} \in \{1-M, 3-M, \cdots, M-1\}$ where M is positive and even. The standard decision function $\mathcal{Q}_M(\cdot)$ which replaces sgn(·) in the binary analysis is defined by

$$\mathcal{Q}_M(x) \triangleq \sum_{k=1-M/2}^{M/2-1} \operatorname{sgn}(x+2k). \tag{6.6}$$

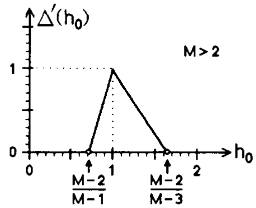

Fig. 10. M-ary $\Delta'(h_0)$ function.

The M-ary version of (2.4a) where we have ideal equalization, becomes

$$\hat{a}_k = \mathcal{Q}_M(h_0 a_k + r_k), \quad h_0 > 0 \tag{6.7}$$

where r_k is as in (2.4b) with the exception that $e_k \in \{0, \pm 2, \cdots, \pm 2(M-1)\}$.

Now suppose we had no residual ISI, i.e., $r_k = 0$, then (6.7) reduces to

$$\hat{a}_k = \mathcal{Q}_M(h_0 a_k) \tag{6.8}$$

from which it is clear that (with $M \geq 4$) we need $h_0 \approx 1$ for error-free behavior. (This differs from the binary case, $M = 2$ where it was only necessary that $h_0 > 0$.) Elaborating, we have the following.

Lemma 10: Given $e_{k-i} = 0$, $\forall i \in \mathbb{Z}_+$, and $M \geq 4$ even, then

$$\hat{a}_k = a_k, \quad \forall a_k \Leftrightarrow \frac{M-2}{M-1} < h_0 < \frac{M-2}{M-3}. \tag{6.9}$$

Proof: (Outline): If h_0 exceeds the upper bound in (6.9) then $a_k = M-3$ gets decoded as $\hat{a}_k = M-1$ in (6.8). Similarly, if h_0 is less than the lower bound in (6.9) then $a_k = M-1$ gets decoded as $\hat{a}_k = M-3$. These symbols define the critical cases. □

So, in summary, we require the right-hand condition in (6.9) to be in force if the M-ary error recovery problem is to be *well posed.*

Consider the error propagation mechanism for the well-posed M-ary problem. We now verify that the operator $\mathcal{L}$ which maps r to $z = -e$ (the residual ISI to the negative of the errors) is passive, indeed sector bounded. The proof is not difficult and given only in outline.

Lemma 11: Let $r_k \triangleq \sum_{i=1}^{\infty} h_i e_{k-i}$ and $z_k \triangleq -e_k = \hat{a}_k - a_k$. Then the operator $\mathcal{L}: r \mapsto z$ is sector bounded according to

$$0 \leq \frac{z_k}{r_k} \leq \frac{2}{\Delta'(h_0)} \tag{6.10}$$

where (see Fig. 11)

$$\Delta'(h_0) \triangleq \begin{cases} (M-1)h_0 - (M-2) & \text{if } h_0 \leq 1; \\ -(M-3)h_0 + (M-2) & \text{if } h_0 \geq 1, \end{cases} \tag{6.11}$$

provided the M-ary error recovery problem is well posed, i.e., h_0 satisfies (6.9).

Proof: (Outline): That z_k/r_k is nonnegative will be implicit in the following development. To compute the upper bound we search over all possible values of $z_k \in \{0, \pm 2, \cdots, \pm 2(M-1)\}$. Note we can restrict attention to the set $\{2, 4, \cdots, 2(M-1)\}$ by symmetry (and discarding the zero error case which cannot violate passivity). We begin with $z_k = 2$. By definition this implies $z_k = 2 \Leftrightarrow \hat{a}_k = \mathcal{Q}_M(h_0 a_k + r_k) = a_k + 2$ which in turn implies

$$a_k + 1 < h_0 a_k + r_k < a_k + 3, \quad \forall a_k \in \{1-M, \cdots, M-5\} \tag{6.12a}$$

and

$$a_k+1<h_0a_k+r_k, \quad \text{for } a_k=M-3. \qquad (6.12b)$$

With $z_k = 2$ fixed, the two critical inequalities which minimize r_k [in the light of (6.10)] are: 1) the LHS of (6.12a) with $a_k = 1 - M$, and 2) (6.12b) where $a_k = M - 3$. Imposing further that $r_k > 0$ (i.e., stipulating that the nonlinearity lies in the 1st quadrant) leads to the two line segments which define $\Delta'(h_0)$ (6.11), shown in Fig. 10. We can verify that other values of $z_k > 2$ do not yield higher values for the ratio z_k/r_k and the methodology is the same as the above. □

Fig. 10 shows the function $\Delta'(\cdot)$ versus h_0. At a conceptual level $\Delta'(\cdot)$ may be thought of as an effective cursor replacing h_0. Note $\Delta'(M - 2/M - 1) = 0$, $\Delta'(M - 2/M - 3) = 0$ and $\Delta'(1) = 1$ (the maximum). In analogy to Theorem 6 (and Fig. 3), we have the M-ary result where $M \geq 4$ is even.

(M-ary) Theorem 12: Suppose $a_k \in \{1 - M, 3 - M, \cdots, M - 1\}$ is the input to a linear channel $h \triangleq \{h_0, h_1, \cdots\}$ which satisfies (6.9) (to be well posed) and $|h_m| = O(m^{-3/2-\epsilon})$ as $m \to \infty$ where $\epsilon > 0$. Suppose $\exists\delta > 0$ such that

$$\frac{\Delta'(h_0)}{2}+\sum_{m=1}^{\infty} h_m \cos(m\theta)\geq\delta, \quad \forall\theta \in [0, 2\pi] \qquad (6.13)$$

where $\Delta'(h_0)$ is given by (6.11). Given an ideal DFE output sequence $\{\hat{a}_k\}$ generated through (6.7) then for some $K < \infty$, we have $\hat{a}_k = a_k$, $\forall k \geq K$.

Remarks:

1) Theorems relating the rates of convergence and robustness for the M-ary case can be generated by analogy with the binary case.
2) The error recovery rate is most rapid with $h_0 = 1$ which implies $\Delta'(h_0) = 1$ because this makes (6.13) the most strictly passive, which is in accord with intuition. If h_0 differs from 1 there will be a diminishing of passivity and hence a drop in the rate of error recovery (this is represented graphically in Fig. 10). This highlights the crucial role that gain compensation plays in the M-ary case (not a consideration for the binary case).
3) Normalized channels where h is scaled such that $h_0 = 1$ (e.g., if we had ideal gain compensation in the DFE), which results in a finite recovery time for binary symbols will also have a finite recovery time for the M-ary case because then conditions (6.13) and (4.5) are identical. The explicit error recovery times, however, will be different as we now indicate. Letting $K_M(\rho)$ denote the error recovery time bound for the M-ary case, in analogy to (5.7), then this is related to the binary error recovery time bound $K(\rho)$ via

$$K_M(\rho)=K(\rho)+\log_\rho(M-1).$$

To prove this note that in a calculation which mimics (5.4) the factor of 4 (the maximum binary error squared) is replaced by $4(M - 1)^2$ (the maximum M-ary error squared).

D. Noise and Asymptotic Error Probability Bounds

In [3] it is shown how the mean error recovery time is related to the error probability in the important case of a high SNR channel. To calculate an error probability bound we include additive channel noise with variance σ_n^2 into the analysis, and following [1] we define the fully open eye error probability as $\epsilon \triangleq \Pr(\hat{a}_k \neq a_k | r_k = 0)$ where r_k is the residual ISI (2.4b), and $\epsilon = O(\sigma^2)$ (Chebyshev's Inequality, [3]). We can then use the techniques in [3] to bound asymptotically the stationary error probability $P_E \triangleq \Pr(\hat{a}_k \neq a_k)$ for channels satisfying the conditions (5.2) and (5.3), via

$$P_E<\frac{\epsilon}{2}(K(\rho)+2) \text{ as } \sigma_n^2\to 0 \qquad (6.14)$$

where $K(\rho)$ is the passivity analysis error recovery time bound which appears in (5.7). From Table I, bound (6.14) may be anything up to a factor of 10^{20} tighter than the oft-cited result in [1] which says $P_E \leq \epsilon\, 2^n$ (effectively derived by replacing $K(\rho)$ by $2(2^n - 1)$ the worst case error recovery time implicit in [4]).

VII. Conclusions

A. Summary

We make a list of the main contributions of this paper.

1) Any channel satisfying the Nyquist conditions in Theorem 6 (or its variants Theorem 9 and Theorem 12) will have a finite recovery time regardless of the initial conditions and regardless of the particular input sequence. These channels possess no pathological input sequences.

2) The maximum time to recover from error can be bounded in terms of the degree of strict passivity of a operator derived from the channel parameters. This degree of passivity is intimately related to a posterror recovery eye diagram opening and an overbound on the rate of recovery.

3) With imperfect equalization this (post recovery) eye closes in proportion to an l_1-norm measure between the ideal DFE tapped delay line parameters and the actual values. The eye also closes when we use too few tap parameters in the DFE.

4) In the absence of ideal equalization, it is possible for the DFE to exhibit nice error recovery properties in a delay sense, i.e., the DFE output always settles down in a finite time to a fixed delay of the input with a possible (fixed) sign inversion. The conditions under which this behavior is possible are stringent and have been determined.

5) The techniques extend naturally to M-ary systems. Under ideal gain compensation (scaling of h such that $h_0 = 1$) any channel which behaves satisfactorily for binary signals will be satisfactory for M-ary signals (and vice-versa) because the conditions for passivity will then be identical.

6) A bound on the error probability for high signal-to-noise ratio channels has been given based on the passivity techniques.

B. Discussion

Up until now there has been scant theoretical justification that nontrivial, nonadaptive DFE's behave satisfactorily because of error propagation, perhaps only [5], [6] being relevant. This is in stark contrast with the purported popularity of DFE's in practice. Previous theoretical work [1], [4], [7], [8] concentrated on bounds which turn out to be hopelessly conservative in the majority of cases. These latter bounds will not be improved without relying heavily on explicit knowledge of the channel to be equalized—this was emphasized in [2], [3] and [6]. In this paper, we have determined some nontrivial broad classes of channels for which a DFE can be effectively used (the results in [5], [6] are very narrow and are subsumed by our present analysis). This class, motivated by the work in [6], includes channels which have near exponential impulse responses—thus capable of modelling twisted pair cable [11]. This provides some theoretical justification to the (controlled) use of DFE's in practice.

As well as defining a nontrivial case of channels for which the DFE behaves satisfactorily, the passivity analysis appears to provide an opportunity to clarify the role and function of a DFE. Recently, the intuition that sensibly the DFE can only be used on minimum phase channels was shown to be misguided [2]. In [2] it is highlighted that minimum phaseness or near minimum phaseness of

$$h=\{h_0, h_1, h_2, \cdots\} \qquad (7.1)$$

is not enough to imply satisfactory DFE error recovery. In comparison we have shown that the stronger notion of strict passivity of the object (or its generalizations)

$$\left\{\frac{h_0}{2}, h_1, h_2, \cdots\right\} \tag{7.2}$$

is a concept which leads to a sensible decision feedback equalization problem (for both binary and M-ary alphabets). Naturally, strict passivity of (7.2) implies strict passivity and thus minimum phaseness of (7.1) [but not vice versa]. If the channel fails the passivity condition it is our contention that a linear equalizer with a DFE must be used. Note that our analysis covers the case of a cascade of a linear equalizer with a DFE because we can interpret h as being not just the channel impulse response but alternatively as the convolution of the channel impulse response with the linear equalizer. We interpret the function of the linear equalizer as being to transform the channel into a passive object which aligns well with the intuition that the linear equalizer is needed to remove precursor ISI. We believe this is a new way of viewing the digital equalization problem and we are pursuing these investigations further.

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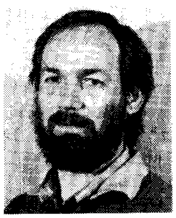

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