Hilbert Transform and Gain/Phase Error Bounds for Rational Functions

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Abstract—It is well known that a function analytic in the right half plane can be constructed from its real part alone, or (modulo an additive constant) from its imaginary part alone via the Hilbert transform. It is also known that a stable minimum phase transfer function can be reconstructed from its gain alone, or (modulo a multiplicative constant) from its phase alone, via the Bode gain/phase relations. This paper considers the question of the continuity of these constructions, for example, whether small phase errors imply small errors in the calculated transfer function. This is considered in the context of rational functions, and the bound obtained depends on the McMillan degree of the function.

I. Introduction

It is part of the toolkit of techniques available in network and system theory that a function analytic in the right half plane can be obtained from its real part alone or (modulo an additive constant) from its imaginary part alone via the Hilbert transform. Similarly a stable, minimum phase transfer function can be constructed from its phase alone (to within a constant multiple), or from its magnitude alone [1]–[3] via what are commonly known as the Bode gain/phase relations. These constructions are an integral part of, for example, the synthesis of a complex signal from a real signal, certain network synthesis procedures, the design of phase equalizers and spectral factorization.

Given the limitations of data acquisition, however, it is important to assess the extent to which these relations are continuous. For example, do small errors in a (measured) phase characteristic lead to small errors in the computed gain, or could they lead to large errors in the gain?

As far as L_2, or square integral, errors are concerned, the situation is easily analyzed for the Hilbert transform. Consider the construction of a real stable transfer function $t(s)$ from its real part $r(j\omega)=1/2[t(j\omega)+t(-j\omega)]$. The construction can be achieved by inverse Fourier transformation, truncation of the resulting symmetric time function, and then Fourier transformation of the truncated time function—this is essentially equivalent to the frequency-domain convolution involved in a Hilbert transform. Suppose that $r(j\omega)\in L_2$ and denote by $\check{r}(t)$ the inverse Fourier transform of $r(j\omega)$. Notice that by Parseval's theorem, $\|\check{r}(t)\|_2=\|r(j\omega)\|_2(2\pi)^{-1/2}$. Now form $\check{t}(t)=2\check{r}(t)1(t)$, where $1(\cdot)$ is the unit step function. Obviously $\|\check{t}(t)\|_2=\|\check{r}(t)\|_2$. The transfer function $t(j\omega)$ is obtained as the Fourier transform of $\check{t}(t)$ and again by Parseval's theorem we have $\|t(j\omega)\|_2=(2\pi)1/2\|\check{t}(t)\|_2=\|r(j\omega)\|_2$. Since the construction of $t(j\omega)$ from $r(j\omega)$ is linear, it follows that an L_2 error of ϵ in $r(j\omega)$ will induce an L_2 error of ϵ in $t(j\omega)$.

The L_2 continuity of the gain/phase relations is less conspicuous, but has been considered in the context of the continuity of spectral factorization in [4]–[6]. Thus the Hilbert transform and the gain/phase relations are continuous in L_2 norm, so that for example a small L_2 error in the (measured) gain of a stable, minimum phase, transfer function will mean a small L_2 error in the computed transfer function. The L_2 norm, however, is not necessarily relevant in this context; since the L_2 norm measures the "average" error, a small L_2 error can still mean a very large error at some frequencies. If the error is to be contained at all frequencies, not just on average, then it is the L_∞ norm which is required, not the L_2 norm.

The techniques for the construction of a stable rational function from its real or imaginary part described, for example, in [2] seem to suggest that small L_∞ errors in the real part will lead to small errors in the computed function. For nonrational functions however, it can easily be shown that the Hilbert transform is not continuous in L_∞ norm (see example in Section III). The gain/phase relations are also known to be discontinuous in L_∞ norm for nonrational functions from the example in [6] and the comments of [3, pp. 431–432]. In [6] an additional condition is imposed on the derivative of the function class considered which restores the L_∞ continuity of the gain/phase relation. Interestingly, this derivative condition is automatically satisfied by rational functions. Thus assuming rationality, one could argue the L_∞ continuity of the Hilbert transform and the gain/phase relations. The purpose of this paper is to quantify the continuity of the Hilbert transform and the gain/phase relations, i.e., to construct error bounds as applicable to rational functions. As it turns out, the L_∞ error bounds obtained depend on the McMillan degree of the rational function, and tend to infinity with increasing McMillan degree. Consistency with results such as can be found in [6] is thus obtained.

Mathematically, the gain/phase relations can be obtained from the Hilbert transform simply by taking logarithms, since $\log(re^{i\theta})=\log(r)+i\theta$. The stable and minimum phase condition required to use the gain phase

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relations therefore derives from the need to satisfy the right half plane analyticity requirements of the Hilbert transform. Note however that imposing the condition of rationality destroys the connection between the Hilbert transform construction and the gain/phase relationship, since a function and its logarithm cannot both be rational. Thus error bound calculations for rational functions must be done separately for the two problems. Note also that the use of the Hilbert transform to set up gain/phase relations is restricted to the scalar case. Matrix spectral factorization of a power spectrum requires some other tool, and the continuity problem, whether for L_2 or L_∞ norm, must generally be differently approached.

The techniques used in this paper are not rooted in the classical history of the gain/phase relations, but are based on the recent introduction to system theory of Hankel singular values and Nehari's Theorem (see [7] and [8]) which has been stimulated by the problems of H_∞ optimal control and model reduction.

The organization of the paper is as follows: Section II contains the necessary definitions and prerequisites. Section III considers the derivation of a bound on a function analytic in the right half complex plane given a bound on either its real part or its imaginary part. Section IV derives a bound on the relative error of a scalar, minimum phase, stable, function given a bound on its phase error and Section V extends this to the multivariable case. Section VI derives a bound on the relative error of a stable, minimum phase scalar, or matrix, transfer function given a bound on its gain error.

II. Preliminaries

The norm $|\cdot|$ of a $p \times p$ complex matrix is defined as $|M| = (\lambda_{\max} M^* M)^{1/2}$. The L_∞ norm $\|\cdot\|_\infty$ of a $p \times p$ matrix function of a complex variable s is defined by

$$\|M(j\omega)\|_\infty = \sup_\omega |M(j\omega)|$$

and $M(s) \in L_\infty$ if $\|M(j\omega)\|_\infty$ is finite. A matrix function $M(s) \in H_\infty$ if $M(s) \in L_\infty$ and is analytic in $\{s\colon \mathrm{Re}(s) \geqslant 0\}$. A matrix function $M(s)$ is real if $M(s)^* = M(\bar{s})^t$ and $M(s) \in RH_\infty$ if it is real, rational and in H_∞. $M(s) \in R_n H_\infty$ if $M(s) \in RH_\infty$ and has McMillan degree less than or equal to n. $M(s) \in L_\infty$ is all pass if $M(s)M(-\bar{s})^* = I$ and $M(-\bar{s})^* M(s) = I$; $M(s) \in H_\infty^-$ if $M(-s) \in H_\infty$. $M(s)$ is minimum phase if it is nonsingular in $\{s : \mathrm{Re}(s) \geqslant 0\}$.

Lemma 2.1: Let $T(s) \in R_n H_\infty$ with $\sigma_1(T)$ its maximum Hankel singular value.[1] Then

$$\sigma_1(T) \leqslant \|T(j\omega) \pm T(-j\omega)\|_\infty \quad (2.1)$$

$$\|T(j\omega) - T(\infty)\|_\infty \leqslant 2n\sigma_1(T) \quad (2.2)$$

$$\|T(j\omega)\|_\infty \leqslant 2n\sigma_1(T) + |T(\infty)|. \quad (2.3)$$

[1] For definition of Hankel singular values see [7]. Nehari's theorem is theorem 6.1 of [7].

Proof: By Nehari's Theorem (see theorem 6.1 of [7]),

$$\begin{aligned}
\sigma_1(T) &= \inf_{F(s) \in H_\infty^-} \|T(j\omega) + F(j\omega)\|_\infty \\
&\leqslant \|T(j\omega) \pm T(-j\omega)\|_\infty \\
&\qquad \text{since } \pm T(-s) \in H_\infty^- \\
\|T(j\omega) - T(\infty)\|_\infty &\leqslant 2(\sigma_1(T) + \cdots + \sigma_n(T)) \\
&\qquad \text{by Corollary 9.3 of [7]} \\
&\leqslant 2n\sigma_1(T) \\
\|T(j\omega)\|_\infty &= \|T(j\omega) - T(\infty) + T(\infty)\|_\infty \\
&\leqslant \|T(j\omega) - T(\infty)\|_\infty + |T(\infty)| \\
&\leqslant 2n\sigma_1(T) + |T(\infty)|.
\end{aligned}$$

▽▽▽

Lemma 2.2: Let A, B be matrices with B invertible and $|B^{-1}||A - B| < 1$. Then A^{-1} exists and

$$|A^{-1}| \leqslant \frac{|B^{-1}|}{1 - |B^{-1}||A - B|}. \quad (2.4)$$

Proof: Standard result. See e.g. [9, p. 70]. ▽▽▽

III. Real Part or Imaginary Part to Transfer Matrix Error Bound

Given an L_∞ bound on the real part of a rational transfer matrix on the imaginary axis we consider in this section the problem of obtaining an L_∞ bound on the transfer matrix itself. This calculation is envisaged as primarily applicable in the context of errors. That is, given the real parts of two transfer matrices on the imaginary axis and the L_∞ error between them, we find a bound on the L_∞ error between the transfer matrices themselves.

Definition 3.1: Let $T(s)$ be a real matrix function. Then $R(s)$ and $I(s)$ given by

$$R(s) = \left(T(s) + T(\bar{s})\right)/2 \quad (3.1a)$$

$$I(s) = \left(T(s) - T(\bar{s})\right)/(2j) \quad (3.1b)$$

are, respectively, the *real part* and the *imaginary part* of $T(s)$.

Example 3.1 ([2, pp. 298–299]): To illustrate the kind of discontinuity that can occur with the Hilbert transform when applied to nonrational functions, consider the real function defined by

$$r(j\omega) = \epsilon \text{ for } \omega \in [-\omega_0, \omega_0] \text{ and zero otherwise.} \quad (3.2)$$

The imaginary part of a stable function with real part $r(j\omega)$ is then given via the Hilbert transform as

$$i(j\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{r(j\tau)}{\omega - \tau}\, d\tau \quad (3.3)$$

$$= \frac{\epsilon}{\pi} \ln\left|\frac{\omega - \omega_0}{\omega + \omega_0}\right|. \quad (3.4)$$

Evidently, $i(j\omega)$ is unbounded at $\pm\omega_0$. Thus with $t(j\omega) = r(j\omega) + ji(j\omega)$, $\|t(j\omega)\|_\infty = \infty$, whilst $\|r(j\omega)\|_\infty = \epsilon$, for any $\epsilon > 0$ and any $\omega_0 > 0$. This means that although $r(j\omega)$ is arbitrarily close to the zero function, which has Hilbert transform zero, the Hilbert transform of $r(j\omega)$ is unbounded.

The following simple calculation establishes an L_∞ continuity result for a stable, rational, matrix function.

Theorem 3.1: Let $T(s) \in R_n H_\infty$ and let $R(s)$ denote the real part of $T(s)$. Suppose $R(s)$ satisfies

$$\|R(j\omega)\|_\infty \leqslant \epsilon. \tag{3.5}$$

Then

$$\|T(j\omega)\|_\infty \leqslant 4n\epsilon + |T(\infty)| \tag{3.6a}$$

$$\leqslant (4n+1)\epsilon. \tag{3.6b}$$

Proof:

$$\begin{aligned}\sigma_1(T) &\leqslant \|T(j\omega)+T(-j\omega)\|_\infty \quad \text{by (2.1)}\\ &= 2\|R(j\omega)\|_\infty \quad \text{by (3.1a)}\\ &\leqslant 2\epsilon.\end{aligned}$$

Substituting into (2.3) gives (3.6a). Observe that

$$|T(\infty)| = |(T(\infty)+T(-\infty))/2| = |R(\infty)| \leqslant \epsilon$$

from which (3.6b) follows. ▽▽▽

Theorem 3.2: Let $T(s) \in R_n H_\infty$ and let $I(s)$ denote the imaginary part of $T(s)$. Suppose $I(s)$ satisfies

$$\|I(j\omega)\|_\infty \leqslant \epsilon. \tag{3.7}$$

Then

$$\|T(j\omega)\|_\infty \leqslant 4n\epsilon + |T(\infty)|. \tag{3.8}$$

Proof:

$$\begin{aligned}\sigma_1(T) &= \|T(j\omega)-T(-j\omega)\|_\infty \quad \text{by (2.1)}\\ &= 2\|I(j\omega)\|_\infty \quad \text{by (3.1b)}\\ &\leqslant 2\epsilon. \end{aligned} \tag{3.9}$$

Substituting into (2.3) gives (3.8). ▽▽▽

Note that we cannot now simply replace $|T(\infty)|$ by ϵ in (3.8) as we did in (3.6), since $I(\infty) \equiv 0$ for any real $T(s)$.

Notice that the bounds obtained are affine in n, so that for high-order functions the Hilbert transform relating the real and imaginary parts of a function in H_∞ is increasingly sensitive to errors. Rational approximations to (3.2) could be found with increasing accuracy for higher and higher degree. Theorem 3.1 is thus consistent with the conclusion established in Example 3.1.

It is of interest to understand how conservative are the bounds of Theorems 3.1 and 3.2. Whether or not the bounds are tight is not known; the following example shows however that the affine-with-n nature of the bounds is the best that can be expected.

Example 3.2: Consider

$$T(s) = \sum_1^n \frac{\alpha_i}{s+\alpha_i}, \qquad \alpha_i > 0$$

where the values of the α_i are specified below. Now,

$$\left|\mathrm{Im}\left[\frac{\alpha_1}{j\omega+\alpha_1}\right]\right| = \frac{\omega\alpha_1}{\omega^2+\alpha_1^2}$$

and the maximum is achieved at $\omega = \alpha_1$, the maximum value being 1/2. For $\omega < \alpha_1$,

$$\left|\mathrm{Im}\left[\frac{\alpha_1}{j\omega+\alpha_1}\right]\right|$$

increases monotonically, starting at zero when $\omega = 0$, and for $\omega > \alpha_1$, it decreases monotonically, to zero at $\omega = \infty$. It follows that, given arbitrary $\delta > 0$, we can select successively $\alpha_2, \alpha_3, \cdots, \alpha_n$ with $\alpha_1 < \alpha_2 < \alpha_3 \cdots$ such that

$$\left|\mathrm{Im}\left[\frac{\alpha_i}{j\omega+\alpha_i}\right]\right| > \delta \text{ implies } \left|\mathrm{Im}\left[\frac{\alpha_j}{j\omega+\alpha_j}\right]\right| < \delta$$

for all $j \neq i$. As a consequence, we can ensure that

$$\|\mathrm{Im}\, T(j\omega)\|_\infty \leqslant \frac{1}{2} + (n-1)\delta$$

where δ is arbitrarily small. We notice also that

$$\|\mathrm{Re}\, T(j\omega)\|_\infty = n$$

[the maximum value of $|\mathrm{Re}\, T(j\omega)|$ is achieved at $\omega = 0$]. Hence we can find δ so that

$$\|T(j\omega)\|_\infty \geqslant 2n\|\mathrm{Im}\, T(j\omega)\|_\infty.$$

IV. Scalar Phase Error Bound to Transfer Function Error Bound

In this section we develop an L_∞ bound on the relative error of a stable, minimum phase, rational function given a bound on its phase error. As for the Hilbert transform, this bound depends on the McMillan degree of the rational function. In [6] an example of essentially the same type as given in Section III shows that the bound must become unbounded as the degree goes to infinity, and this is indeed the case for the bound obtained.

Definition 4.1 [10]: Let $t(s) \in RL_\infty$ be scalar. The *phase function* $t(s)$ is defined as $t(-s)^{-1}t(s)$.

The terminology "phase function" derives from the observation that, writing

$$t(j\omega) = |t(j\omega)|\exp(j\theta(\omega)) \tag{4.1}$$

where $\theta(\omega)$ denotes the *phase* of $t(j\omega)$, we have

$$t(-s)^{-1}t(s) = \exp(2j\theta(\omega)). \tag{4.2}$$

The phase function is in L_∞ and is all pass. Note however that if $t(s)$ has a simple zero on the $j\omega$-axis, then $\theta(\omega)$ jumps by π and the phase of $t(-j\omega)^{-1}t(j\omega)$ jumps by 2π at this zero. Since such a jump cannot be discerned from the phase function itself we see that the phase function carries no information about such zeros. Put another way, a pole–zero cancellation occurs in forming $t(-s)^{-1}t(s)$. A similar remark holds for multiple zeros.

Lemma 4.1: Let $t(s) \in R_n H_\infty$, scalar satisfy

$$\left\|1 - t(-j\omega)^{-1}t(j\omega)\right\|_\infty \leqslant \epsilon \tag{4.3}$$

for some $\epsilon < (2n)^{-1}$. Then

$$\|t(j\omega) - t(\infty)\|_\infty \leqslant 2n\epsilon(1-2n\epsilon)^{-1}|t(\infty)|. \tag{4.4}$$

Proof:

$$\begin{aligned}\sigma_1(t) &\leqslant \|t(-j\omega)-t(j\omega)\|_\infty \quad \text{by (2.1)}\\ &\leqslant \left\|t(-j\omega)\left(1-t(-j\omega)^{-1}t(j\omega)\right)\right\|_\infty\\ &\leqslant \epsilon\|t(j\omega)\|_\infty \quad \text{by (4.3)}\\ &\leqslant \epsilon\left(2n\sigma_1(t)+|t(\infty)|\right) \quad \text{by (2.3)}.\end{aligned}$$

Thus

$$\sigma_1(t) \leqslant \epsilon(1-2n\epsilon)^{-1}|t(\infty)|. \tag{4.5}$$

Substituting (4.5) into (2.2) gives (4.4). ▽▽▽

The lemma says that if the phase of a rational transfer function is close to zero at all frequencies (i.e., the phase function is close to 1, so (4.3) is satisfied), then the transfer function is almost constant, i.e., $\|t(j\omega)-t(\infty)\|_\infty$ is small. This is not of much apparent use in this form, but Lemma 4.1 can be used to derive an error bound between two transfer functions given an error bound between their phase functions. Before we do this however, we consider some examples.

It is first of all important to note that the condition $\epsilon<(2n)^{-1}$ of Lemma 4.1 cannot possibly hold unless $t(\infty)$ is nonzero and $t(s)$ is minimum phase: Let $t(s)$ have n (stable) poles, z_+ stable zeros, z_0 imaginary axis zeros and z_- unstable zeros. The total phase change from $\omega=0$ to $\omega=\infty$ of the phase function of $t(j\omega)$ is then $(z_+-z_--n)\pi$. Then, unless $z_-=0$ and $z_0=0$ (i.e., $t(s)$ is minimum phase) and $z_+=n$ (i.e., $t(\infty)$ nonzero), the phase is $\pm\pi$ at some frequency ω_0 (with $\omega_0=\infty$ allowed as a possibility). The phase function is, therefore, -1 at ω_0 by (4.2), so $\|1-t(-j\omega)^{-1}t(j\omega)\|_\infty \geqslant 2$ if $t(\infty)=0$ or $t(s)$ nonminimum phase.

In light of (4.6) it is natural to ask whether the conditions of the lemma can be true for any $t(s)$, or if there are $t(s)$ for any value of ϵ.

Example 4.1: Consider $t_\delta(s)$ defined by

$$t_\delta(s)=(s+1+\delta)(s+1)^{-1}, \qquad \delta>0. \tag{4.6}$$

From (4.2) it is not difficult to show that

$$\left|1-t(-j\omega)^{-1}t(j\omega)\right|=2|\sin(\theta(\omega))|. \tag{4.7}$$

A little algebra gives

$$\begin{aligned}\sin\theta_\delta(\omega)&=-\delta\omega\left[(\omega^2+1)(\omega^2+1+\delta^2+2\delta)\right]^{-1/2}\\ &\leqslant -\delta\omega(\omega^2+1)^{-1}\end{aligned} \tag{4.8}$$

with the inequality tending to equality as $\delta\to 0$. It follows that

$$\|\sin\theta_\delta(\omega)\|_\infty \leqslant \delta/2 \text{ (approx. = for small } \delta).$$

Thus by (4.7) we can take $\epsilon=\delta$ in Lemma 4.1. So for any ϵ, it is possible to find a transfer function (which is not constant) such that (4.3) is satisfied. Applying now Lemma 4.1 we get

$$\|t_\delta(j\omega)-1\|_\infty \leqslant 2\delta(1-2\delta)^{-1}. \tag{4.9}$$

However,

$$t_\delta(j\omega)-1=\delta(j\omega+1)^{-1}$$

so that

$$\|t_\delta(j\omega)-1\|_\infty=\delta. \tag{4.10}$$

Thus the error bound is, for this example, conservative by about a factor of 2.

The main application of Lemma 4.1 is to show that two stable, minimum phase, rational functions with similar phase must be similar.

Theorem 4.1: Let $v_1(s)$ and $v_2(s)$ be such that $t(s)\equiv v_1(s)^{-1}v_2(s)\in R_nH_\infty$, $t(\infty)=1$ and satisfy, for some $\epsilon<(2n)^{-1}$

$$\left\|v_1(-j\omega)^{-1}v_1(j\omega)-v_2(-j\omega)^{-1}v_2(j\omega)\right\|_\infty \leqslant \epsilon. \tag{4.11}$$

Then

$$\left\|v_1(j\omega)^{-1}\left(v_1(j\omega)-v_2(j\omega)\right)\right\|_\infty \leqslant 2n\epsilon(1-2n\epsilon)^{-1} \tag{4.12a}$$

$$\|v_1(j\omega)-v_2(j\omega)\|_\infty \leqslant 2n\epsilon(1-2n\epsilon)^{-1}\|v_1(j\omega)\|_\infty. \tag{4.12b}$$

Proof:

$$\begin{aligned}&\left\|v_1(-j\omega)^{-1}v_1(j\omega)-v_2(-j\omega)^{-1}v_2(j\omega)\right\|_\infty\\ &\quad=\left\|v_1(-j\omega)^{-1}v_1(j\omega)\left(1-t(-j\omega)^{-1}t(j\omega)\right)\right\|_\infty\\ &\quad=\left\|1-t(-j\omega)^{-1}t(j\omega)\right\|_\infty\end{aligned} \tag{4.13}$$

since $v_1(-s)^{-1}v_1(s)$ is all pass. By (4.13) and (4.11) we see that (4.3) is satisfied for $t(s)$. Thus

$$\begin{aligned}&\left\|v_1(j\omega)^{-1}\left(v_1(j\omega)-v_2(j\omega)\right)\right\|_\infty\\ &\quad=\|(1-t(j\omega))\|_\infty\\ &\quad\leqslant 2n\epsilon(1-2n\epsilon)^{-1} \quad \text{by Lemma 4.1.}\end{aligned}$$

▽▽▽

Equation (4.12b) of course follows by Cauchy's inequality.

There are several points to note about the error bound derived in this section. The results are applicable only to rational functions. The condition $\epsilon<(2n)^{-1}$ can of course be rewritten as $n<(2\epsilon)^{-1}$. This, therefore, for given ϵ, puts a definite degree bound on the functions $t(s)$ to which the results are applicable.

Theorem 4.1 gives a *relative error bound* on $v_1(s)-v_2(s)$. Thus if $v_1(j\omega_0)$ is (close to) zero for some ω_0, one cannot get good phase matching between $v_1(s)$ and $v_2(s)$ unless $v_2(j\omega_0)$ is also (close to) zero. That is, if $v_2(j\omega)$ satisfies (4.11) (and the theorem conditions), then (4.12a) implies $v_2(j\omega_0)$ must be (close to) zero when $v_1(j\omega_0)$ is (close to) zero.

It has been observed in [11] that the phase matching and relative error approaches to power spectrum approximation are interrelated, a particular implementation of the phase matching approach being equivalent to an approach to optimal relative error approximation. Theorem 4.1 implies that this relationship is not accidental since, for any stable minimum phase rational function, a phase error bound implies a relative error bound.

The condition $t(s) \in RH_\infty$ is satisfied when $v_i(s) \in RH_\infty$ and $v_1(s)$ is minimum phase, or more generally if the zeros of $v_1(s)$ in $\{s: \operatorname{Re}(s) \geqslant 0\}$ are the same as those of $v_2(s)$ in $\{s: \operatorname{Re}(s) \geqslant 0\}$.

Note also that the particular formulation of Theorem 4.1 is at least partially motivated by the power spectrum approximation method of phase matching, where a phase error bound of the form (4.12) has been obtained [12], [13].

The following example illustrates that the linear dependence on n is the best possible result that one can expect, although it does not show that the bound of the theorem is necessarily tight.

Example 4.2: Consider the transfer function

$$v_1(s) = \prod_{i=1}^{n} \frac{s+\alpha_i(1+\delta)}{s+\alpha_i}$$

with α_i specified below, and δ suitably small, as described below. Now notice that the phase of

$$w_1(j\omega) = \frac{j\omega + \alpha_1(1+\delta)}{j\omega + \alpha_1}$$

is zero at dc and $\omega = \infty$, and has a maximum lag at $\omega = \alpha_1\sqrt{(1+\delta)}$ which is

$$\tan^{-1}\frac{\delta}{2\sqrt{1+\delta}} = \sin^{-1}\frac{\delta}{2\left(1+\dfrac{\delta}{2}\right)}.$$

The phase lag increases monotonically from 0 to this value over $[0, \alpha_1\sqrt{(1+\delta)}\,]$, and then decreases monotonically to zero over $[\alpha_1\sqrt{(1+\delta)}, \infty]$. It follows that

$$\left\| w_1(-j\omega)^{-1} w_1(j\omega) - 1 \right\|_\infty = \frac{\delta}{1+\dfrac{\delta}{2}}.$$

Now choose the α_i very far apart, so that wherever the phase lag of $\omega_i(j\omega)$ exceeds ϵ_0 where ϵ_0 is chosen with $n\epsilon_0 < \delta$, the phase lag of $w_k(j\omega)$ for $k \neq i$ is less than ϵ_0. It follows that:

$$\left\| v_1(-j\omega)^{-1} v_1(j\omega)^{-1} \right\|_\infty \leqslant \frac{\delta}{1+\dfrac{\delta}{2}} + (n-1)\epsilon_0 < 2\delta.$$

By identifying $v_2(j\omega)$ in Theorem 4.1 with 1, it follows that (4.11) in effect holds, with $\epsilon = 2\delta$. Further

$$\begin{aligned} \| v_1(j\omega) - 1 \|_\infty &= \left\| \prod_{i=1}^{n} \left(1 + \frac{\alpha_i \delta}{j\omega + \alpha_i}\right) - 1 \right\|_\infty \\ &= \prod_{i=1}^{n} (1+\delta)^n - 1 \\ &\geqslant n\delta \\ &= \frac{n\epsilon}{2}. \end{aligned}$$

V. Multivariable Phase Error to Transfer Matrix Error Bound

A natural quandary which arises in attempting to generalize the results of Section IV to matrix functions is the problem of the definition of multivariable phase. We could, by analogy with Definition 4.1, define the multivariable phase function as $T(-s)^{-1}T(s)$, which allows Lemma 4.1 to be written, with T replacing t, for the multivariable case. However $T(-s)^{-1}T(s)$ is not in general all pass, a property we would like to have in a phase matrix. Accordingly we adopt an approach suggested by the phase matching algorithm for power spectrum approximation [14]. If $V(s) \in RH_\infty$ and is minimum phase, define $W(s) \in RH_\infty$ and minimum phase by

$$V(s)V(-s)' = W(-s)'W(s). \tag{5.1}$$

Now $W(s)$ is unique up to premultiplication by an orthogonal matrix [15], so we can uniquely specify $W(s)$ from $V(s)$ by (5.1), the stable/minimum phase condition and the normalization

$$E(s) = V(-s)^{-1}W(s)' \qquad \text{with } E(\infty) = I. \tag{5.2}$$

The all-pass matrix $E(s)$ will be called, by analogy with the scalar case (where $W(s) = V(s)$), the *phase matrix of* $V(s)$.

An important property of this definition is that when $W(s)$ is nonsingular at infinity, stable and minimum phase, $W(s)$ can be reconstructed from $E(s)$ to within a multiplicative constant [5], [12]. In case $W(s)$ has zeros on the $j\omega$-axis (including infinity) it can still be found, but there is further freedom.

The above construction is actually valid when $V(s)$ has imaginary axis zeros, as well as zeros in $\{s: \operatorname{Re}(s) < 0\}$. For Theorem 5.1 below, however, it will turn out that in effect attention is restricted to $V(s)$ without imaginary axis zeros.

For nonminimum phase, but stable, $V(s)$, the situation is somewhat more complex. Let $V(s) \in RH_\infty$ have inner–outer factorization

$$V(s) = V_m(s)E_p(s) \tag{5.3}$$

where $V_m(s)$, $E_p(s) \in RH_\infty$, $V_m(s)$ is minimum phase and $E_p(s)$ is all pass. The matrix functions $V_m(s)$ and $E_p(s)$ are unique up to multiplication by a constant orthogonal matrix so we define $E_p(s)$ and $V_m(s)$ uniquely by the normalization $E_p(s) = I$. Now let $W_m(s)$ satisfy

$$V_m(s)V_m(-s)' = W_m(-s)'W_m(s) \tag{5.4a}$$

with $W_m(s) \in RH_\infty$ and minimum phase. Now define $W(s)$ by

$$W(s) = E_p(s)W_m(s) \tag{5.4b}$$

and define the phase matrix by (5.2) as before. This yields

$$E(s) = E_p(-s)^{-1}\left[V_m(-s)^{-1}W_m(s)'\right]E_p(s)'$$

which suggests that the phase of $V(s)$, viz. $E(s)$, "exceeds" the phase of $V_m(s)$, viz. $V_m(-s)^{-1}W_m(s)'$, by the phase of $E_p(s)$, viz. $E_p(-s)^{-1}E_p(s)'$.

Note that in order to interpret $E(s)$ given by (5.2) as the phase matrix of $V(s)$, it is necessary to impose the construction above. The results however depend only on $V(s)$, $W(s) \in RH_\infty$ and satisfying (5.1).

The scalar results of the previous section can now be extended to matrix functions.

Lemma 5.1: Let $T(s) \in R_n H_\infty$ and $G(s) \in H_\infty$ satisfy

$$\left\| I - T(-j\omega)^{-1} G(j\omega)' \right\|_\infty \leqslant \delta$$

for some $0 < \delta < (2n)^{-1}$. Then

$$\| T(j\omega) - T(\infty) \|_\infty \leqslant 2n\delta(1-2n\delta)^{-1} |T(\infty)|.$$

Proof: as for Lemma 4.1. ▽▽▽

Lemma 5.2: Let $V(s)$, $W(s) \in L_\infty$ satisfy (5.1). Then

$$\| V(-j\omega) \|_\infty = \| W(j\omega)' \|_\infty$$
$$\left\| V(-j\omega)^{-1} \right\|_\infty = \left\| W(j\omega)^{-\prime} \right\|_\infty.$$

Proof: By (5.1), $V(-s) = W(s)' W(-s) V(s)^{-\prime} = W(s)' E(-s)'$, from which the result follows, since $E(-s)'$ is all pass. ▽▽▽

Theorem 5.1: Let $V_1(s)$, $W_1(s)$ and $V_2(s)$, $W_2(s)$ satisfy (5.1) and suppose $T(s) \equiv V_2(s) V_1(s)^{-1} \in R_n H_\infty$ with $T(\infty) = I$ and $G(s) \equiv W_1(s)^{-1} W_2(s) \in H_\infty$. Suppose

$$\left\| V_1(-j\omega) W_1(j\omega)' - V_2(-j\omega)^{-1} W_2(j\omega)' \right\|_\infty \leqslant \epsilon \quad (5.5)$$

with $\delta < (2n)^{-1}$, where

$$\delta = \epsilon \| V(j\omega) \|_\infty \left\| V(j\omega)^{-1} \right\|_\infty. \quad (5.6)$$

Then

$$\left\| [V_1(j\omega) - V_2(j\omega)] V_1(j\omega)^{-1} \right\|_\infty \leqslant 2n\delta(1-2n\delta)^{-1}. \quad (5.7)$$

Proof:

$$\begin{aligned} &\left\| 1 - V_1(-j\omega) V_2(-j\omega)^{-1} W_2(j\omega)' W_1(j\omega)^{-\prime} \right\|_\infty \\ &\quad = \left\| V_1(-j\omega) \left[V_1(-j\omega)^{-1} W_1(j\omega)' \right. \right. \\ &\qquad \left. \left. - V_2(-j\omega)^{-1} W_2(j\omega)' \right] W_1(j\omega)^{-\prime} \right\|_\infty \\ &\quad \leqslant \epsilon \| V(j\omega) \|_\infty \left\| V(j\omega)^{-1} \right\|_\infty \end{aligned} \quad (5.8)$$

by Lemma 5.2 and (5.5). Let $T(s) = V_2(s) V_1(s)^{-1}$ and $G(s) = W_1(s)^{-1} W_2(s)$. Then by Lemma 5.1, with $\delta = \epsilon \|V(j\omega)\|_\infty \|V(j\omega)^{-1}\|_\infty$, we have

$$\begin{aligned} &\left\| [V_1(j\omega) - V_2(j\omega)] V_1(j\omega)^{-1} \right\|_\infty \\ &\quad = \| I - T(j\omega) \|_\infty \\ &\quad \leqslant 2n\delta(1-2n\delta)^{-1}. \end{aligned}$$

▽▽▽

As mentioned in the introduction to this section, the particular form of the theorem is motivated by the phase matching approach to stochastic approximation, where a phase error bound of the type in (5.5) has been obtained [12], [13].

Note that the bound is "degraded" somewhat by the condition number factor $\|V(j\omega)\|_\infty \|V(j\omega)^{-1}\|_\infty$, which certainly means that $V(s)^{-1} \in L_\infty$, or that $V(s)$ have no zeros on the imaginary axis. This restriction did not appear in the scalar case, since in (5.8) we can commute $W_1(j\omega)'$ across next to $V_1(-j\omega)^{-1}$, and then omit it since $V_1(-s)^{-1} W_1(s)'$ is all pass.

It is to be observed that the theorem is nonsymmetric with respect to the V's and the W's. This is to enable less restrictive assumptions than would otherwise be the case—we could impose the same restrictions on the W's as on the V's and get a corresponding result for the W's. The result for the W's can in fact be obtained straight from Theorem 5.1, since for $E_1(s)$ all pass,

$$\begin{aligned} \| E_1(j\omega) - E_2(j\omega) \|_\infty &= \left\| E_1(-j\omega)' - E_2(-j\omega)' \right\|_\infty \\ &= \left\| E_1(j\omega)^{-1} - E_2(j\omega)^{-1} \right\|_\infty. \end{aligned}$$

This swaps the roles of the V's and the W's in Theorem 5.1.

It is possible to contemplate an alternative construction for the phase matrix of $V(s)$ than that given above: one could use the polar decomposition

$$V(j\omega) = G(j\omega) U(j\omega) \quad (5.9)$$

where G is positive definite hermitian and U is all pass. In case $V(j\omega) = v(j\omega)$ is scalar, this leads to $v^{-1}(-j\omega) v(j\omega) = u^2(j\omega)$, and so there is a close relation between the two definitions. In the matrix case, we are unaware whether $W(j\omega) = U(j\omega) G(j\omega)$ or something very similar. Because $U(j\omega)$ is not rational, and because of the prevalent use of $E(j\omega)$ in literature on spectrum approximation, we have elected to work with $E(j\omega)$. The question of whether $U(j\omega)$ or $E(j\omega)$ is more natural is obviously debatable, with the answer depending on one's view of what is natural.

VI. Multivariable Gain to Transfer Matrix Error Bounds

This section considers the problem of obtaining a bound on the error between two stable, rational transfer matrices given a bound on their gain error. (There seems no point in a separate analysis for scalar transfer functions.) The gain of a transfer matrix $T(s)$ is defined by the *gain matrix function* $T(s)T(s)^*$.

Theorem 6.1: Let $T(s) \in R_n H_\infty$, minimum phase and $T(\infty)$ invertible ($T(s) \in R_n H_\infty$ and $T(s)^{-1} \in RH_\infty$). Let $B = T(\infty) T(\infty)^*$ and suppose

$$\| T(j\omega) T(j\omega)^* - T(\infty) T(\infty)^* \|_\infty \leqslant \epsilon \quad (6.1)$$

with $\epsilon < |B^{-1}|^{-1}$. Let

$$\delta = \left\{ |B^{-1}| \left[1 - \epsilon |B^{-1}| \right]^{-1} \right\}^{1/2}. \quad (6.2)$$

Then

$$\| T(j\omega) - T(\infty) \|_\infty \leqslant 2n\delta\epsilon. \quad (6.3)$$

Let $G(s) \in R_n H_\infty$, minimum phase with $G(\infty) = T(\infty)'$

satisfy

$$G(-s)'G(s) = T(s)T(-s)'. \tag{6.4}$$

Then the phase matrix $T(-s)^{-1}G(s)'$ satisfies

$$\left\| I - T(-j\omega)^{-1}G(j\omega)' \right\|_\infty \leqslant 4n\epsilon\delta^2. \tag{6.5}$$

Proof: Firstly, observe

$$\begin{aligned}\left\| T(j\omega)^{-1} \right\|_\infty &= \left\| [T(j\omega)T(j\omega)^*]^{-1} \right\|_\infty^{1/2} \\ &\leqslant \left\{ |B^{-1}| [1 - \epsilon |B^{-1}|]^{-1} \right\}^{1/2} \end{aligned} \tag{6.6}$$

by Lemma 2.2. Thus

$$\begin{aligned}&\| T(j\omega) - BT(j\omega)^{-*} \|_\infty \\ &\quad= \| (T(j\omega)T(j\omega)^* - B)T(j\omega)^{-*} \|_\infty \\ &\quad\leqslant \epsilon \| T(j\omega)^{-*} \|_\infty \quad \text{by (6.1)} \\ &\quad\leqslant \epsilon\delta \quad \text{by (6.6)}. \end{aligned} \tag{6.7}$$

Now $BT(j\omega)^{-*} = BT(-j\omega)^{-\prime} \in RH_\infty^-$, so by Nehari's Theorem

$$\sigma_1(T) \leqslant \| T(j\omega) - BT(j\omega)^{-*} \|_\infty. \tag{6.8}$$

The result (6.3) now follows from (6.7), (6.8), and (2.2).

$$\begin{aligned}&\left\| I - T(-j\omega)^{-1}G(j\omega)' \right\|_\infty \\ &\quad\leqslant \left\| T(j\omega)^{-1} \right\|_\infty \left\| T(-j\omega) - G(j\omega)' \right\|_\infty \\ &\quad\leqslant \delta \left[\| T(j\omega) - T(\infty) \|_\infty + \left\| G(j\omega)' - G(\infty)' \right\|_\infty \right] \end{aligned} \tag{6.9}$$

by (6.6) and the triangle inequality. Now

$$\begin{aligned}&\left\| G(j\omega)' \left[G(j\omega)' \right]^* - G(\infty)'G(\infty) \right\|_\infty \\ &\quad= \left\| G(j\omega)'G(-j\omega) - B \right\|_\infty \\ &\quad= \left\| G(-j\omega)'G(j\omega) - B \right\|_\infty \\ &\quad= \| T(j\omega)T(j\omega)^* - B \|_\infty \qquad \text{by (6.4)} \\ &\quad\leqslant \epsilon \qquad \text{by (6.1)}. \end{aligned}$$

Thus (6.1) and (6.3), with $T(j\omega)$ replaced with $G(j\omega)'$, give

$$\left\| G(j\omega)' - G(\infty)' \right\|_\infty \leqslant 2n\delta\epsilon. \tag{6.10}$$

Combining (6.3), (6.9), and (6.10) gives the result. ▽▽▽

Theorem 6.1 says that if a transfer matrix has almost constant gain, then the transfer function is almost constant. This can be used to derive a bound on the relative error between two transfer matrices given an error bound on their gains.

Theorem 6.2: Let $V_1(s)$, $V_2(s)$ be such that $T(s) = V_1(s)^{-1}V_2(s) \in R_nH_\infty$, minimum phase and $T(\infty) = I$. Suppose

$$\| V_1(s)V_1(s)^* - V_2(s)V_2(s)^* \|_\infty \leqslant \epsilon. \tag{6.11}$$

Let $\gamma = \epsilon \left\| V_1(j\omega)^{-1} \right\|_\infty^2$ and suppose $\gamma < 1$. Then

$$\left\| V_1(j\omega)^{-1} [V_1(j\omega) - V_2(j\omega)] \right\|_\infty \leqslant 2n\gamma(1-\gamma)^{-1/2} \tag{6.12a}$$

$$\| V_1(j\omega) - V_2(j\omega) \|_\infty \leqslant 2n\gamma(1-\gamma)^{-1/2} \tag{6.12b}$$

Proof:

$$\begin{aligned}\| T(j\omega)T(j\omega)^* - I \|_\infty &= \left\| V_1(j\omega)^{-1} [V_1(j\omega)V_1(j\omega)^* \right. \\ &\qquad \left. - V_2(j\omega)V_2(j\omega)] V_1(j\omega)^{-*} \right\|_\infty \\ &\leqslant \epsilon \left\| V_1(j\omega)^{-1} \right\|_\infty^2 = \gamma. \end{aligned} \tag{6.13}$$

Hence, by Theorem 6.1,

$$\begin{aligned}\left\| V_1(j\omega)^{-1} [V_1(j\omega) - V_2(j\omega)] \right\|_\infty &= \| T(j\omega) - I \|_\infty \\ &\leqslant 2n\gamma(1-\gamma)^{-1/2}. \end{aligned}$$ ▽▽▽

Notice that the conditions on $T(s)$ are satisfied for $V_i(s) \in R_{n/2}H_\infty$ and minimum phase and having the same zero structure at infinity.

Theorems 6.1 and 6.2 apply to scalar transfer functions *without* improvement of the various bounding constants.

Again, the question arises as to how conservative the bounds are. Preliminary calculations suggest that if $V_1 = 1$ and V_2 is defined as the minimum phase stable transfer function of degree n such that

$$|V_2(j\omega)|^2 = \frac{\omega^{2n} + (1-\delta)}{\omega^{2n} + (1+\delta)}$$

then a bound for $\|1 - V_2\|_\infty$ which is linear in n can be obtained. Actually, $|1 - V_2(j1)|$ appears linear in n.

VII. Conclusion

Given an L_∞ error bound on the real part, or the imaginary part, of a rational function, analytic in the right half plane, an L_∞ error bound on the function itself has been obtained which depends linearly on the McMillan degree of the function.

Given an L_∞ bound on the phase error or the gain error of a stable, minimum phase rational function, an L_∞ bound on the relative error in the function itself has been obtained and again the bound depends on the McMillan degree of the transfer function.

Certainly for some transfer functions, the bounds we have given will be very loose. However, we have shown with examples that the linear dependence of general bounds with McMillan degree is inescapable, and for such examples the bounds are far from loose.

The results can be used to extend the phase error bounds obtained for the phase matching approach to power spectrum approximation in [12], [13] to a bound on the relative error between the full order and reduced order process models.

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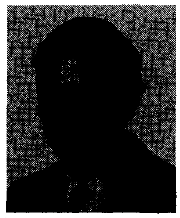

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