

## On the Robustness of Low-Order Schur Polynomials

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*Abstract*—Robust stability conditions for low-order Schur polynomials are obtained. In particular, conditions for degree  $n = 2, 3, 4,$  and  $5$  are explicitly obtained. These conditions relate to stability of the corner points for  $n = 2, 3$  and for corner and possible supplementary points for  $n = 4$  and  $5$ . Two counterexamples given in the literature are fully discussed in relation to the obtained conditions. Future research work on possible extension of the results to higher order Schur polynomials are discussed.

### I. INTRODUCTION

Since the first published results of Kharitonov [1], [2] for robust stability of Hurwitz polynomials for real and complex coefficients, extensive research efforts have been directed toward extending and simplifying these important results. Because of the relation with ideas of this paper, we mention the recent work of Anderson, Jury, and Mansour [3], where Kharitonov's theorem has been simplified for  $n = 3, 4,$  and  $5$ . Furthermore, it is shown in this work that for  $n \geq 6$  no further simplification of the theorem is possible. A desired extension of the theorem is the derivation of its discrete counterpart. This problem was first posed by Bose [4] as an open research topic. Several recent publications have been devoted to tackling this problem. In particular, the works of Bose and Zeheb [5], and Bose, Jury, and Zeheb [6] are directed towards this problem. However, the results obtained which are based on the bilinear transformation and application of Kharitonov's theorem give only *sufficient* conditions for robust stability of Schur polynomials. This is in contrast to the necessary and sufficient conditions for robust stability of Hurwitz polynomials. Recent published works of Hollot and Bartlett [7] have shed some light on the discrete counterpart of Kharitonov's theorem. The results obtained relate only to independent variation of about half the coefficients of the robust stability of Schur polynomials. Their results are utilized in this

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paper to help in obtaining conditions for robust stability of Schur polynomials for  $n=2, 3, 4$ , and  $5$ . A somewhat related, but different, approach to this problem was discussed by Soh *et al.* [8] for both the continuous and discrete cases.

The structure of the paper is as follows. In Section II we present three lemmas which are general enough to be used for possible solution of the problem for any degree polynomial. In Section III, we obtain explicit conditions for stable Schur polynomials for  $n=2$  and  $3$ . In Section IV, we obtain similar conditions for  $n=4$  and  $5$ . Also in this section, we discuss the two counterexamples given in the literature for  $n=4$  and point out the difficulty in obtaining a robust stability result. Also, we present a modification for these examples, based on our results, to overcome this difficulty. In section V, we discuss the difficulty in extending our approach for  $n \geq 6$ . Special cases for which our results can be extended to higher degree polynomials are mentioned. In the concluding Section VI, we present some problems for future research.

II. PRELIMINARIES AND GENERAL LEMMAS

Before we present the three general lemmas, we provide some preliminary definitions and remarks:

The objective of this paper is to obtain conditions for robustness of low-order "Schur polynomials", i.e., conditions for discrete time stability of an  $n$ th degree real polynomial are given by

$$F(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_{n-1} z + a_n, \quad n \geq 2 \tag{1}$$

with respect to parameter values that lie in a parameter hypercube

$$\Omega = \{(a_1 a_2 \dots a_n) | a_i \in [\underline{a}_i, \bar{a}_i], \quad i=1,2,\dots,n\} \tag{2}$$

where  $\underline{a}_i, \bar{a}_i$  are the minimum and maximum values of the parameters  $a_i$ , respectively. Equation (1) represents a monic polynomial; for some discussion of nonmonic polynomials, we refer to Sections III and V. As notation for the cornerpoints of the hypercube, we use

$$\xi = (b_1 \ b_2 \ \dots \ b_n), \quad b_i \in \{\underline{a}_i, \bar{a}_i\} \tag{3}$$

or

$$\left. \begin{aligned} \xi_0 &= (a_1 \ a_2 \ a_3 \ \dots \ a_n) \\ \xi_1 &= (\bar{a}_1 \ a_2 \ a_3 \ \dots \ a_n) \\ \xi_2 &= (a_1 \ \bar{a}_2 \ a_3 \ \dots \ a_n) \\ \xi_3 &= (\bar{a}_1 \ \bar{a}_2 \ a_3 \ \dots \ a_n) \\ \xi_4 &= (a_1 \ a_2 \ \bar{a}_3 \ a_4 \ \dots \ a_n) \\ &\vdots \\ \xi_{2^n-1} &= (\bar{a}_1 \ \bar{a}_2 \ \bar{a}_3 \ \dots \ \bar{a}_n). \end{aligned} \right\} \tag{4}$$

Now, we recall the discrete time stability constraints for  $F(z)$ , that is the necessary and sufficient conditions for all its roots to be inside the unit circle, given for example by Jury [9], [10]. The stability constraints for  $F(z)$  are as follows.

- 1) The critical stability constraints

$$F(1) > 0 \tag{5a}$$

$$(-1)^n F(-1) > 0 \tag{5b}$$

$$\Delta_{n-1}^- > 0 \tag{6}$$

where  $\Delta_{n-1}^-$  is an inner determinant [9].

- 2) The following inequalities involving inner determinants are besides (5) and (6) the complete necessary and sufficient stability constraints:

$$\begin{aligned} \Delta_i^+ &> 0, \\ \Delta_i^- &> 0, \end{aligned} \quad i=1,2,\dots,n-1. \tag{7}$$

The critical stability constraints will play an important role in our derivations. They guarantee that if we start from any stable point and move continuously in the parameter space, we need to check only these constraints to ensure retention of stability. For general discussion of the critical constraints we refer to Jury [10].

To prove robust stability, it is advisable to check first that the hypercube lies within the limits given by the coefficient conditions of Mansour [11]. The proof of robust stability in this paper will involve the following two steps:

- 1) From the formulation of the stability at some discrete points (corner points and if necessary other points too) of the hypercube  $\Omega$ , we prove the stability at every point of  $\Omega$ .

- 2) We reduce the stability tests at all the pertinent discrete points to the minimum number and complexity, whenever possible.

Now, we prove three lemmas, which are general enough that they can be utilized for any  $n$ ; however, for low  $n$ , they simplify the proofs of robust stability considerably.

Lemma 1

Necessary conditions for robust stability of  $F(z)$  in (1) with respect to the parameter cube  $\Omega$  are

$$1 + \underline{a}_1 + \underline{a}_2 + \underline{a}_3 + \dots + \underline{a}_n > 0 \tag{8a}$$

$$1 - \bar{a}_1 + \underline{a}_2 - \bar{a}_3 + \dots + (-1)^n \bar{a}_n > 0. \tag{8b}$$

For  $a_n$  in (8b), take  $\bar{a}_n$  for  $n$  odd and  $\underline{a}_n$  for  $n$  even. If (8a) and (8b) are satisfied, then (5) will be satisfied for all other points in the parameter cube.

*Proof:* The necessary conditions for  $F(z)$  in (1) to be stable include the linear stability constraints (5). For robust stability, these constraints must be fulfilled in all points of the parameter cube (8). Now, for all points of (2), we have

$$1 + a_1 + a_2 + a_3 + \dots + a_n \geq 1 + \underline{a}_1 + \underline{a}_2 + \dots + \underline{a}_n. \tag{9}$$

Therefore, if (8a) holds, (5a) holds for all points of (2). Similarly, we prove that (8b) implies (5b) following the same argument.  $\square$

Lemma 2

Given two stable parameter points  $c \in \Omega$ ,  $d \in \Omega$  of the polynomial (1), then,  $F(z)$  is stable for all points on the straight line between  $c$  and  $d$ , i.e. for  $\lambda \in (0,1)$

$$a_i = c_i + \lambda(d_i - c_i), \quad i=1,2,\dots,n; \quad n \geq 2 \tag{10}$$

if for  $\Delta_{n-1}^-$  in (6) there exists  $\tilde{\lambda} \in [0,1]$  such that

$$\left. \begin{aligned} \frac{d}{d\lambda} \Delta_{n-1}^-(\lambda) &\geq 0, & 0 \leq \lambda \leq \tilde{\lambda} \\ &\leq 0, & \tilde{\lambda} < \lambda \leq 1. \end{aligned} \right\} \tag{11}$$

*Proof:* For this proof, only critical stability constraints need to be checked. The linear constraints (5) are fulfilled for all points between  $c$  and  $d$  because from

$$1 + c_1 + c_2 + c_3 + \dots + c_n > 0 \tag{12}$$

$$1 + d_1 + d_2 + d_3 + \dots + d_n > 0 \tag{13}$$

if we multiply the condition (12) with  $1 - \lambda$ , the condition (13) by  $\lambda$  and add them, there results for  $\lambda \in [0,1]$ ,

$$1 + (1 - \lambda)c_1 + \lambda d_1 + (1 - \lambda)c_2 + \lambda d_2 + \dots > 0. \tag{14}$$

Inequality (14) is exactly (5a) for a point  $a_i$  in (10). Similarly, the proof for (5b) can be readily obtained.

The critical stability condition  $\Delta_{n-1}^-$  in (6) is a smooth function of the parameters  $a_i$ . Therefore its minimum with respect to  $\lambda$  is either at the boundary or at a *local minimum point*. But, from (11), local minimum points cannot exist and the *global minimum* is on the *boundary*. Hence, if the stability condition (6) is satisfied for the points  $c, d$  then it is satisfied for all points of the straight line between them.  $\square$

We may extend Lemma 2 to the case of functions with a local minimum. This extension is presented in Lemma 3.

#### Lemma 3

Given two stable parameter points  $c \in \Omega$ ,  $d \in \Omega$  of the polynomial (1), then  $F(z)$  is stable for all points on the straight line between  $c$  and  $d$  with  $a_i$  in (10) if, for  $\Delta_{n-1}^-$  in (6), there exist  $\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3$  such that:

$$\left. \begin{aligned} \frac{d}{d\lambda} \Delta_{n-1}^-(\lambda) &\geq 0, & 0 &\leq \lambda \leq \tilde{\lambda}_1 \\ &\leq 0, & \tilde{\lambda}_1 &\leq \lambda < \tilde{\lambda}_2 \\ &\geq 0, & \tilde{\lambda}_2 &\leq \lambda < \tilde{\lambda}_3 \\ &\leq 0, & \tilde{\lambda}_3 &\leq \lambda \leq 1 \end{aligned} \right\} \quad (15)$$

and  $\Delta_{n-1}^-$  is positive for  $\lambda = \tilde{\lambda}_2$ .

The proof follows using the same arguments as for Lemma 2.

### III. ROBUST STABILITY CONDITIONS FOR $n = 2, 3$

Case  $n = 2$ , let

$$F(z) = z^2 + a_1 z + a_2, \quad a_i \in [a_i, \bar{a}_i], \quad i=1,2. \quad (16)$$

The necessary and sufficient stability conditions for  $F(z)$  are [9]

$$F(1) = 1 + a_1 + a_2 > 0 \quad (17)$$

$$F(-1) = 1 - a_1 + a_2 > 0$$

$$1 - a_2 > 0. \quad (18)$$

From (17) and (18) we have the stability region shown in Fig. 1. Hence, the following theorems:

#### Theorem 1

The polynomial  $F(z)$ , for  $n = 2$ , is robust stable with respect to the cube (2), *iff* it is stable for all the four corner points.

#### Theorem 2

Assume Lemma 1 holds. The polynomial  $F(z)$ , for  $n = 2$ , is robust stable with respect to the cube (2), *iff* (19) is satisfied

$$1 - \bar{a}_2 > 0. \quad (19)$$

The proof is trivial and, therefore, omitted.

Lemma 1 and condition (19) have for  $n = 2$  an easy interpretation in the parameter space as shown in Fig. 2.

Case  $n = 3$ , let

$$F(z) = z^3 + a_1 z^2 + a_2 z + a_3, \quad a_i \in [a_i, \bar{a}_i], \quad i=1,2,3. \quad (20)$$

The necessary and sufficient conditions for stability are [9]

$$\left. \begin{aligned} F(1) &= 1 + a_1 + a_2 + a_3 > 0 \\ -F(-1) &= 1 - a_1 + a_2 - a_3 > 0 \end{aligned} \right\} \quad (21)$$

$$f = \Delta_2^- = 1 - a_3^2 - a_2 + a_1 a_3 > 0 \quad (22)$$

$$\left. \begin{aligned} 1 + a_3 &> 0 \\ 1 - a_3 &> 0. \end{aligned} \right\} \quad (23)$$

From Hollot and Bartlett [7], when  $a_1$  is fixed  $F(z)$  is robust

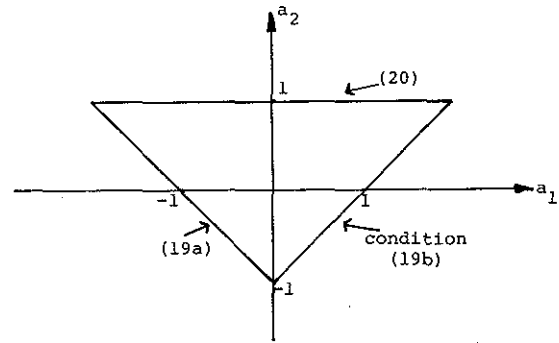


Fig. 1. The stability triangle for  $F(z)$ ,  $n = 2$ .

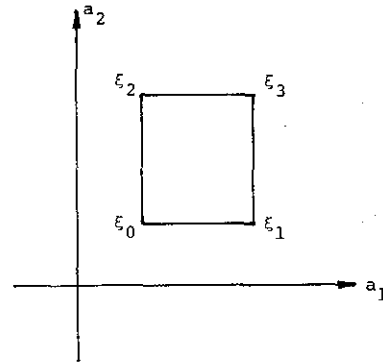


Fig. 2. Robust conditions in parameter plane.

stable (stable for all variations of  $a_2, a_3$ ) *iff* it is stable at all the points  $(a_1, b_2, b_3)$ .

To complete the test allowing all variations of  $a_1, a_2$  and  $a_3$ , we must prove stability with respect to  $a_1$  only. Since all critical stability constraints (21) and (22) are linear in  $a_1$ , we see that if  $F(z)$  is stable for the two parameter points  $(a_1, c_2, c_3)$  and  $(\bar{a}_1, c_2, c_3)$ , it is stable for all variations of  $a_1$  between  $a_1$  and  $\bar{a}_1$  (Lemma 2). Combining this observation with [7], we conclude the stability for all points in the parameter space. Thus we have the following theorem:

#### Theorem 3

The polynomial  $F(z)$  for  $n = 3$  is stable for all parameters in the cube  $a_i \in [a_i, \bar{a}_i]$ ,  $i=1,2,3$  *iff* it is stable in all 8 corner points  $\xi_j$ .

*Proof:* The necessity condition is obvious. To prove sufficiency, we start with the four "a<sub>1</sub>" edges between the corner points  $\xi_i \rightarrow \xi_{i+1}$ ,  $i=0,2,4,6$ . As mentioned earlier, due to the linearity (Lemma 2) of the stability conditions with respect to  $a_1$ , it follows that the stability at the *end points* guarantees the stability of all points of these edges (straight lines). Together with [7], we have stability for the whole cube  $a_i \in [a_i, \bar{a}_i]$ ,  $i=1,2,3$ . For another proof see also [12].  $\square$

#### Simplification of the Robust Stability Test

We try now to minimize the effort, for stability tests on all 8 corner points (Theorem 3). For one corner point we have to check all stability conditions. Starting from this stable corner point, we have to satisfy only the critical constraints (21) and (22). Because of Lemma 1, we concentrate on (22). The left side of inequality (22) is decreasing with respect to  $a_2$ . So, if it is satisfied at  $\bar{a}_2$  (maximum value), it is satisfied for all  $a_2 < \bar{a}_2$ . Therefore, we need to check this condition only at the corner points where  $\bar{a}_2$  appears i.e.,  $\xi_j$ ,  $j=2,3,6,7$ .

Though (22) is linear in  $a_1$ , the issue is more involved since the behavior of the left side of (22) as a function of  $a_1$  depends on the sign of  $a_3$ . For positive  $a_3$  the minimum of  $a_1$  i.e.  $\underline{a}_1$  minimizes the left side of (22) with respect to  $a_1$ . For negative  $a_3$  the value  $\bar{a}_1$  is the critical point. For  $\underline{a}_3 < 0 < \bar{a}_3$ , the minimum of the quadratic critical constraint (22) occurs at  $\underline{a}_1\bar{a}_3$  or  $\bar{a}_1\underline{a}_3$ . Hence we choose for these cases different corner points depending on  $\underline{a}_3, \bar{a}_3$ . The above test can be summarized by the following theorem.

**Theorem 4**

Assume Lemma 1 holds. The polynomial  $F(z)$  for  $n=3$  is robust stable with respect to the cube (2), iff the following conditions are satisfied:

- 1) Inequality (22) holds at the corner points

$$\left. \begin{aligned} 0 < \underline{a}_3 & \quad \xi_2 = (\underline{a}_1, \bar{a}_2, \underline{a}_3) \wedge \xi_6 = (\underline{a}_1, \bar{a}_2, \bar{a}_3) \\ \underline{a}_3 < 0 < \bar{a}_3, & \quad \xi_3 = (\bar{a}_1, \bar{a}_2, \underline{a}_3) \wedge \xi_6 = (\underline{a}_1, \bar{a}_2, \bar{a}_3) \\ \bar{a}_3 < 0, & \quad \xi_3 = (\bar{a}_1, \bar{a}_2, \underline{a}_3) \wedge \xi_7 = (\bar{a}_1, \bar{a}_2, \bar{a}_3). \end{aligned} \right\} \quad (24)$$

- 2) Auxiliary constraints

$$1 + \underline{a}_3 > 0, \quad 1 - \bar{a}_3 > 0. \quad (25)$$

*Proof:* Lemma 1 and inequality (25) guarantee that (21) and (23) hold for any point in the parameter cube. The argument preceding the theorem guarantees that (22) holds for any point in the parameter cube when it holds for the points defined by (24).  $\square$

**Remarks**

- 1) If  $\underline{a}_3$  or  $\bar{a}_3$  are zero, then  $F(z)$ ,  $n=3$ , will have a root at  $z=0$ , and the stability test is reduced to that of  $n=2$ . The inequality (22) reduces to  $1 - a_2 > 0$ , at the appropriate point  $a_2 = \bar{a}_2$ .
- 2) It is possible to prove robust stability for  $n=3$ , similarly to  $n=2$ , without reliance on [7]. This is due to the fact that (22) is linear with respect to  $a_1$  and  $a_2$  and negative quadratic with respect to  $a_3$ . However, for  $n > 3$ , the use of the results of [7] proved to be necessary, as seen in the next section.
- 3) It is possible to extend the above results to the case of a nonmonic polynomial  $F(z)$ . The critical stability constraint (22) becomes now for positive  $a_0$

$$f = a_0^2 - a_3^2 - a_0 a_2 + a_1 a_3 > 0. \quad (26)$$

Because of the positive quadratic dependence of (26) on  $a_0$ , we must check stability not only at the corner points, but also for all points of local minima  $(\beta_0, b_1, b_2, b_3)$  which satisfy

$$0 < \underline{a}_0 < \beta_0 < \bar{a}_0 \quad (27)$$

where  $\beta_0 = b_2/2$ ,  $b_i \in \{\underline{a}_i, \bar{a}_i\}$ ,  $i=1,2,3$ . Depending on inequality (27) we have to check 8, 4, or no supplementary points for stability to guarantee the robustness.

**IV. ROBUST STABILITY CONDITIONS FOR  $n=4,5$**

Case  $n=4$ , let

$$\begin{aligned} F(z) &= z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4, \\ a_i &\in [\underline{a}_i, \bar{a}_i], \quad i=1, \dots, 4. \end{aligned} \quad (28)$$

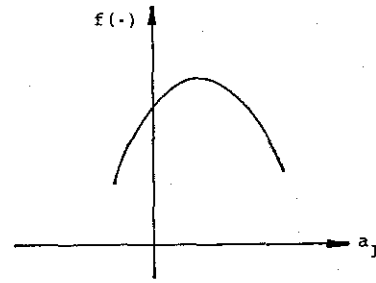


Fig. 3. Shape of  $f(\cdot)$  for  $a_4 > 0$ .

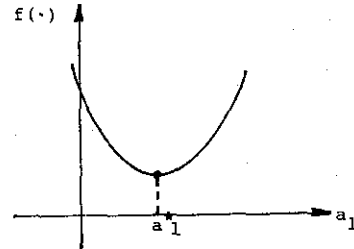


Fig. 4. Shape of  $f(\cdot)$  for  $a_4 < 0$ .

The stability conditions are [9]

$$\left. \begin{aligned} F(1) &= 1 + a_1 + a_2 + a_3 + a_4 > 0 \\ F(-1) &= 1 - a_1 + a_2 - a_3 + a_4 > 0 \end{aligned} \right\} \quad (29)$$

$$\begin{aligned} f = \Delta_3 &= a_4^3 + 2a_2 a_4 + a_1 a_3 - a_4 - a_2 - a_4 a_1^2 - a_4^2 \\ &\quad - a_4^2 a_2 - a_3^2 + 1 + a_1 a_3 a_4 > 0 \end{aligned} \quad (30)$$

and

$$\begin{aligned} 3 - a_2 + 3a_4 &> 0 \\ a_0 - a_4 &> 0. \end{aligned} \quad (31)$$

Equations (29) and (30) constitute the critical stability constraints [9], [10].

**Robust Stability Conditions**

From Hollot and Bartlett [7] it is shown that  $F(z)$  is stable for all variations of  $a_i$ ,  $i=2,3,4$ , if and only if it is stable in the respective corner points. To cover the robust case for  $\underline{a}_i$ ,  $i=1, \dots, 4$ , the main difficulty is to prove stability with respect to  $a_1$  variations. At first, we demonstrate that the stability in all corner points, and for certain supplementary points, will guarantee stability for the whole parameter cube (2). This is shown as follows:

The condition in  $f(\cdot) > 0$  in (30) is quadratic in  $a_1$ . Its shape as a function of  $a_1$  depends on the sign of  $a_4$ . If  $a_4 > 0$  the shape is, as shown in Fig. 3. Therefore, if  $f(\cdot) > 0$  for  $\underline{a}_1$  and  $\bar{a}_1$ , then  $f(\cdot) > 0$  for all  $a_1 \in [\underline{a}_1, \bar{a}_1]$  with all other parameters  $a_i$ ,  $i=2,3,4$  constant. This result is obtainable from Lemma 2 directly.

For  $a_4 < 0$ , the shape of  $f(\cdot)$  is as shown in Fig. 4. Thus we have to use Lemma 3 to check the stability condition  $f(\cdot) > 0$  for all points between  $\underline{a}_1$  and  $\bar{a}_1$ . The point  $a_1^*$  and the corresponding minimum value of  $f(\cdot)$  provide such information. To ascertain the required conditions for  $a_4 < 0$ , we have to deal with the following special cases:

- 1)  $\bar{a}_1 < a_1^*$  or  $a_1^* < \underline{a}_1$ .

As shown in Figs. 5 and 6, the positivity of  $f(\cdot)$  at the end points  $\underline{a}_1$  and  $\bar{a}_1$  ensures positivity in all points in between.

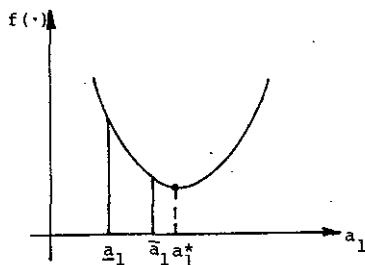


Fig. 5. Shape of \$f(\cdot)\$ for \$\bar{a}\_1 < a\_1^\*\$.

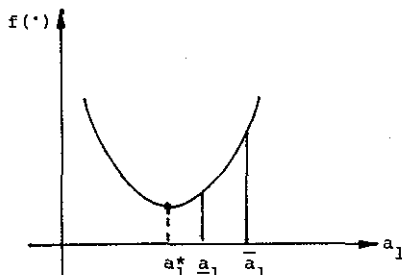


Fig. 6. Shape of \$f(\cdot)\$ for \$a\_1^\* < \bar{a}\_1\$.

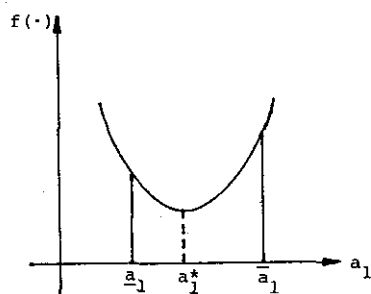


Fig. 7. Shape of \$f(\cdot)\$ for \$\bar{a}\_1 < a\_1^\* < \bar{a}\_1\$.

2) \$a\_1 < a\_1^\* < \bar{a}\_1\$.

As shown in Fig. 7, we have to check the positivity of \$f(\cdot)\$ at the point \$a\_1^\*\$. This point is not at a corner. Its appearance represents a departure from the earlier discussed robust stability condition for \$n = 2, 3\$.

For the given values of \$a\_2, a\_3\$, and \$a\_4\$, it is easy to calculate the minimizing point \$a\_1^\*\$. Setting \$\partial f(\cdot)/\partial a\_1\$ equal zero gives

$$a_1^* = \frac{a_3}{2a_4} (1 + a_4).$$

We are now ready to formulate the robust stability conditions for the parameter cube in (2) as follows:

**Theorem 5**

Assume the stability of \$F(z)\$ for \$n = 4\$ in all 16 corner points \$\xi\_i\$ of the parameter cube (2). Then, \$F(z)\$ is stable in all points of the cube, *iff* all the supplementary points \$(b\_1^\*, b\_2, b\_3, b\_4)\$

$$b_1^* = \frac{b_3}{2b_4} (1 + b_4), \quad b_i \in \{a_i, \bar{a}_i\}, \quad i = 2, 3, 4 \quad (32)$$

which satisfy \$b\_4 < 0, \bar{a}\_1 < b\_1^\* < \bar{a}\_1\$ are stable.

*Proof:* To prove this theorem, we follow a similar proof as for \$n = 3\$. Consider first the 8 "\$a\_1\$"-edges of the cube between the corners, \$\xi\_i\$ and \$\xi\_{i+1}, i = 0, 2, 4, \dots, 14\$. These edges can also be designated as \$(a\_1, b\_2, b\_3, b\_4)\$. From the above discussion, it is evident that if \$a\_1^\* \in [a\_1, \bar{a}\_1]\$ then \$(a\_1^\*, b\_2, b\_3, b\_4)\$ must be stable.

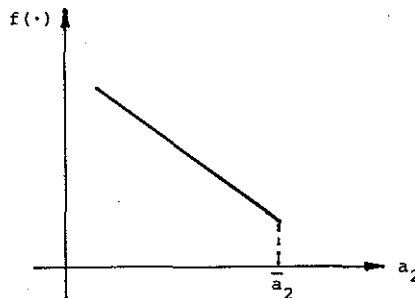


Fig. 8. Shape of \$f(\cdot)\$ as a function of \$a\_2\$.

Equation (32) gives us all such points on the 8 edges. Thus, we prove stability of these edges, *iff* besides the stability of the corner points, also the points of (32) are stable. Now, we proceed with the variations of \$a\_2\$. Appealing to [7], we can at once extend the proof of the stability region to all \$a\_1, a\_2\$ sides of the parameter cube. Continuing in the same fashion and using [7], for the variations of \$a\_3\$ and \$a\_4\$ we prove stability of the whole cube (2). \$\square\$

*Simplification of the Robust Stability Test*

Similar to the case \$n = 3\$ we start with a stable corner point. Thus only the critical constraints need be checked. Beside the conditions of Lemma 1, we need to check only (30). This equation is linear in \$a\_2\$, quadratic in \$a\_1\$ and \$a\_3\$, and cubic in \$a\_4\$. For \$a\_2\$, the function \$f(\cdot)\$ of (30) is linear decreasing. Hence, if \$f(\cdot)\$ is positive for \$\bar{a}\_2\$, it is positive for all \$a\_2 < \bar{a}\_2\$ (for all other parameters constant) (Fig. 8). For \$a\_3\$, the shape is parabolic, as shown in Fig. 3. Thus if \$f(\cdot)\$ is positive at the end points, it is positive for all points between \$\underline{a}\_3, \bar{a}\_3\$. Because of the cubic dependence of \$f(\cdot)\$ on \$a\_4\$ it is not easy to proceed in the same manner. Therefore we shall check all stability conditions for two corner points—one of these for \$\underline{a}\_4\$ and the other for \$\bar{a}\_4\$. For the variation of \$a\_1\$, the preceding discussions give us all the needed stability tests. Hence, based on Theorem 5 we have:

*Theorem 6*

Assume Lemma 1 holds. The polynomial \$F(z)\$, \$n = 4\$, is a robust Schur polynomial for all \$a\_i \in [a\_i, \bar{a}\_i], i = 1, 2, \dots, n\$, *iff* the following conditions are satisfied:

- 1) Condition (31) is satisfied for the two corner points,
 
$$\xi_2 = (a_1 \bar{a}_2 a_3 a_4) \text{ and } \xi_{15} = (\bar{a}_1 \bar{a}_2 \bar{a}_3 \bar{a}_4). \quad (33)$$
- 2) Condition (30) is satisfied for all corner points \$(b\_1 \bar{a}\_2 b\_3 b\_4)\$ and supplementary points, \$(b\_1^\* \bar{a}\_2 b\_3 b\_4)\$ that satisfy (32).

We may note that if \$a\_4 > 0\$, then the stability conditions are checked only at the corner points, similar to the \$n = 2, 3\$ cases. The two corner points (33) can be chosen arbitrarily with respect to \$a\_1\$ and \$a\_3\$. In the proof of robust stability we used Hollot and Bartlett's results [7]. This is in contrast to cases \$n = 2\$ and 3, for which robust stability can also be proven directly as remarked for \$n = 3\$.

*Discussion of the Two Counterexamples [5], [7]*

- In the paper of Bose-Zeheb [5], the following counterexample (attributed to Gershak, and communicated by Vorghese) is given to show that for the discrete stability the corner points are not sufficient for robustness:

$$F(z) = z^4 + a_1 z^3 + 1.5z^2 - \frac{1}{3}, \quad a_1 \in \left[ -\frac{17}{8}, \frac{17}{8} \right] \quad (34)$$

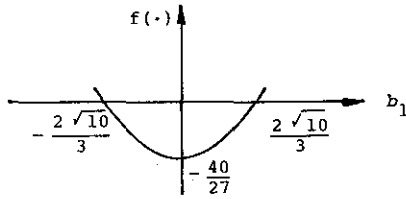


Fig. 9. Shape of  $f(\cdot)$  as a function of  $a_1$ .

To discuss this example in relation to the previous results, we note first that  $a_4 = -1/3 < 0$ . Hence, we need to calculate  $b_1^*$  from (32) which is zero for the above example. From (30),  $f = -40/27 < 0$  for  $a_1 = 0$ . Hence,  $F(z)$  is not robust stable. To obtain an insight into this example, we compute  $f(\cdot)$  from (30) using the parameters of (34). Thus we obtain  $f(\cdot)$  as a function of  $a_1$

$$f(a_1) = \frac{1}{3} \left( a_1^2 - \frac{40}{9} \right)$$

as shown in Fig. 9. The plot crosses the  $a_1$  axis at the points  $\bar{a}_1 = \pm 2\sqrt{10}/3 = \pm 2.108$  and, therefore, for  $|a_1| > 2\sqrt{10}/3$   $f(\cdot)$  is positive. Thus for (34) stability constraint  $f(\cdot)$  is satisfied at the corner points.

Now, we modify the above counterexample to obtain

$$F(z) = z^4 + a_1 z^3 + \frac{1}{3} z^2 - \frac{1}{3}, \quad a_1 \in \left[ -\frac{9}{10}, \frac{9}{10} \right].$$

Because  $a_4 = -1/3 < 0$ , we compute  $b_1^*$  which is again zero. However,  $f(a_1 = 0) = 16/27 > 0$ , which, according to Theorem 6, implies that  $F(z)$  is stable everywhere.

- The following counterexample is presented by Hollot and Bartlett [7]:

$$F(z) = z^4 + a_1 z^3 + 1.35z^2 + 0.243z - 0.2916, \quad a_1 \in [-2.3, 1.7]. \quad (35)$$

The polynomial (35) is stable at 1.7 and -2.3, but unstable at  $a_1 = -1.3$ . Similarly to the first counterexample, we notice that  $a_4 < 0$ . Hence, we need to calculate  $b_1^*$  from (32), which is  $b_1^* = -0.29517$ . Because  $-2.3 < b_1^* < 1.7$ , and  $f(b_1^*) = -1.15479$  it follows that  $F(z)$  is not stable at this point.

To modify the above example to be robust with respect to  $a_1 \in [\underline{a}_1, \bar{a}_1]$ , we obtain e.g. the range  $a_1 \in [1.7, 1.8]$ . It is of interest to note that with this choice of  $a_1$ , we obtain the condition  $b_1^* = -0.29517 < \underline{a}_1$  and  $f(\cdot)$  is positive for all interesting  $a_1$  as seen in Fig. 6.

Another modification of the example (35) yields,

$$F(z) = z^4 + a_1 z^3 + 1.29z^2 + 0.243z + 0.2916, \quad a_1 \in [0.3, 0.8]. \quad (36)$$

One can readily check that the above example is robust stable with respect to  $a_1$ . Because  $a_4 = 0.2916 > 0$ , we need to check stability only at corner points.

Case  $n = 5$ , let

$$F(z) = z^5 + a_1 z^4 + a_2 z^3 + a_3 z^2 + a_4 z + a_5, \quad a_i \in [\underline{a}_i, \bar{a}_i], \quad i = 1, \dots, 5. \quad (37)$$

To obtain the necessary and sufficient stability conditions, we

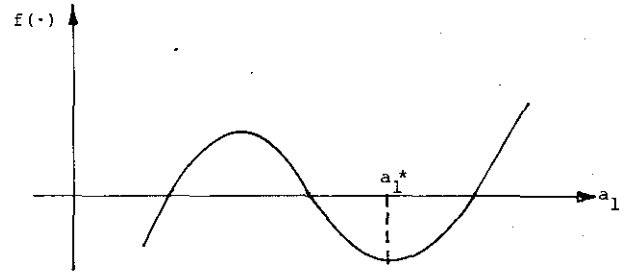


Fig. 10. Shape of  $f(\cdot)$ , if local minimum exists and  $a_5 > 0$ .

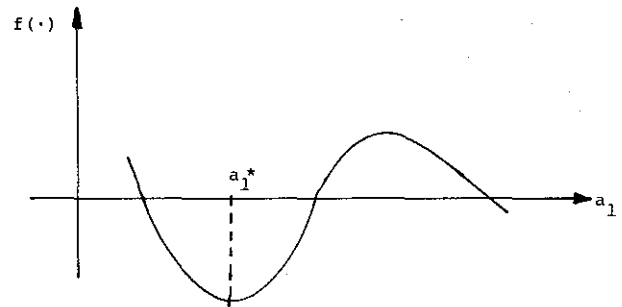


Fig. 11. Shape of  $f(\cdot)$ , if local minimum exists and  $a_5 < 0$ .

invoke the simplified Anderson–Jury criterion [10] as follows:

$$\begin{aligned} F(1) &= 1 + a_1 + a_2 + a_3 + a_4 + a_5 > 0 \\ -F(-1) &= 1 - a_1 + a_2 - a_3 + a_4 - a_5 > 0 \end{aligned} \quad (38)$$

$$\begin{aligned} f(\cdot) = \Delta_4 = & (a_5^2 - 1)^2 - (a_5 a_1 - a_4)^2 + (a_5 a_1 - a_4) \\ & \cdot (a_1^2 + a_4 a_2 - a_4^2 - a_3 a_1 - a_5^2 + 1) \\ & + (a_5 a_2 - a_3)(a_5 a_4 - a_1 - a_5 a_2 + a_3) \\ & - (a_5 a_3 - a_2)(a_5^2 - 1 - 2a_5 a_1 + 2a_4) \\ & > 0 \end{aligned} \quad (39)$$

and

$$\left. \begin{aligned} 1 - a_4 + a_1 a_5 - a_5^2 &> 0 \\ 10 - 2a_1 - 2a_2 + 2a_3 + 2a_4 - 10a_5 &> 0 \\ 5 + 3a_1 + a_2 + a_3 - 3a_4 - 5a_5 &> 0. \end{aligned} \right\} \quad (40)$$

### Robust Stability Conditions

From Hollot and Bartlett [7] we have that  $F(z)$  is stable for all parameter values  $a_i$  of the cube (2),  $i = 3, 4, 5$  ( $a_1, a_2$  are constant), iff it is stable in all the respective corner points. To prove robustness for the whole cube, we have to prove stability with respect to  $a_1$  and  $a_2$  only.

The function  $f(\cdot)$  in (39) is quadratic in  $a_2$ , with a finite maximum, and cubic in  $a_1$ . For the variation of  $a_2$ , we need to test stability only at the extreme points  $\underline{a}_2, \bar{a}_2$ , as seen in Fig. 3 and deduced from Lemma 2.

The variation  $f(\cdot)$  with respect to  $a_1$  is more complicated. We distinguish between two different shapes of  $f(\cdot)$ , depending on the sign of  $a_5$ . If a local minimum exists, the shape of  $f(\cdot)$  is shown in Figs. 10 and 11 for  $a_5 > 0$  and  $a_5 < 0$  respectively.

Therefore, besides stability testing at the extreme points of  $a_1$ , stability at  $a_1^*$ , (if it exists and if  $a_1^* \in [\underline{a}_1, \bar{a}_1]$ ), should also be checked, to guarantee stability for all points between the extreme cases. The local minimum points discussed in Lemma 3 can be obtained (see Appendix) as follows:

$$a_1^* = \frac{a_1 + \text{sgn}(a_5)\sqrt{D}}{3a_5}$$

where

$$D = \alpha_1^2 - 3a_5\alpha_2$$

$$\alpha_1 = a_4 + a_3a_5 + a_5^2$$

$$\alpha_2 = 2a_4a_5 - a_5^3 + a_5 - a_4^2a_5 + 2a_5^2a_3 + a_3a_4$$

$$+ a_3 + a_2a_4a_5 - 3a_2a_5.$$

One may note that if a real  $a_1^*$  does not exist, i.e., if  $D < 0$ , it indicates that  $f(\cdot)$  has no local extremum with respect to  $a_1$ . For such a case, we do not need to check stability at any supplementary point, similar to  $n = 2, 3$  and the special case of  $n = 4$ . Now, we are ready to prove stability for the whole parameter cube (2), as follows:

#### Theorem 7

Assume stability of  $F(z)$  for  $n = 5$  in all 32 corner points of  $a_i \in [\underline{a}_i, \bar{a}_i]$ ,  $i = 1, \dots, 5$ . Then  $F(z)$  is stable everywhere in the cube (2), iff it is stable in all points  $(b_1^* b_2 b_3 b_4 b_5)$  that satisfy

$$D^* > 0, \quad \underline{a}_1 < b_1^* < \bar{a}_1 \quad (41)$$

where

$$D^* = \beta_1^2 - 3b_5\beta_2; \quad b_1^* = \frac{\beta_1 + \text{sgn}(b_5)\sqrt{D^*}}{3b_5}$$

$$b_i = \{\underline{a}_i, \bar{a}_i\}, \quad i = 2, 3, 4, 5. \quad (42)$$

The  $\beta_i$ 's are obtained from  $\alpha_i$ 's by replacing the  $a_i$ 's by  $b_i$ 's.

*Proof:* Consider first the 16  $a_1$ -edges between  $\xi_i$  and  $\xi_{i+1}$ ,  $i = 0, 2, 4, \dots, 30$ . If the points  $b_1^*$  lie on some of these edges (only one  $b_1^*$  is possible in any one edge), we have to check the critical stability conditions  $f(\cdot)$  for all such points to guarantee stability for every point on these edges. Note, due to the critical stability conditions, we do not need to consider conditions (40) since we have corner point stability by hypothesis. Equations (41), (42) give stability for the supplementary points. Hence, we have stability for all the " $a_1$ " edges. Now, we proceed with the variation of  $a_2$ . In this case, because of the shape of  $f(\cdot)$  with respect to  $a_2$  (Fig. 3), we consider only the end points (Lemma 2). Therefore, for variations with respect to  $a_1$  and  $a_2$ , we have stability for all points in the 8 " $a_1 a_2$ "-sides of the parameter cube. Based on the results of Hollot and Bartlett [7], we complete the proof of robust stability for all the parameters of the cube.  $\square$

#### Simplification of the Robust Stability Test

To minimize the computational efforts required for robust stability test, we note that  $f(\cdot)$ , eqn. (39), is quartic in  $a_5$ , cubic in  $a_1, a_4$ , and quadratic in  $a_2, a_3$ . Its shape with respect to  $a_3$  is similar to Fig. 3, so we need to check only at the end points (Lemma 2). The shape with respect to  $a_1, a_2$  has been discussed earlier. The dependence of  $f(\cdot)$  on  $a_4, a_5$  is too complicated for this kind of reduction of stability tests. We shall therefore calculate all stability conditions on 4 corner points each of which for one combination of the extreme values of  $a_4, a_5$ . Thus we have:

#### Theorem 8

Assume Lemma 1 holds. The polynomial  $F(z)$ ,  $n = 5$ , is a robust Schur polynomial for all parameter points of the cube (2), iff the following conditions are satisfied:

1. The stability conditions in (40) are satisfied for the four corner points

$$(\underline{a}_1 \underline{a}_2 \underline{a}_3 b_4 b_5) \quad (43)$$

or for any other combination of the extreme values of  $a_1, a_2$ , and  $a_3$ .

2. The condition (39) for  $f(\cdot)$  is satisfied for all 32 corner points and for all the supplementary points,  $(b_1^* b_2 b_3 b_4 b_5)$  that satisfy (41) and (42).

*Proof:* Due to Theorem 7 we need to check only stability at discrete points. For the four corner points (43) all stability conditions are satisfied. These four corner points define four different groups of 8 corners each. Now, we can prove all stability conditions in each of those four groups separately. We start in each group with the stable point, and thus we need to check only the critical stability constraints. Due to Lemma 1, the auxiliary constraints  $F(1)$  and  $F(-1)$  are satisfied for all points of the cube. For  $a_2$  ( $a_3$ ) variations we need to check only  $f(\cdot) > 0$  at the end points (Lemma 2). To check the variation along  $a_1$ , we check  $f(\cdot) > 0$  for end points and for all supplementary points. Due to Theorem 7 this completes the proof.  $\square$

## V. DISCUSSION OF ROBUST STABILITY FOR $n \geq 6$

To exemplify the difficulty of extending this approach to higher order  $n$ , consider the case  $n = 6$ .

From [10], the critical stability constraint  $f(\cdot)$  is given by

$$f(\cdot) = \Delta_5^-$$

$$= \det \begin{bmatrix} 1 & a_1 & a_2 & a_3 & a_4 - a_6 \\ 0 & 1 & a_1 & a_2 - a_6 & a_3 - a_5 \\ 0 & 0 & 1 - a_6 & a_1 - a_5 & a_2 - a_4 \\ 0 & -a_6 & -a_5 & 1 - a_4 & a_1 - a_3 \\ -a_6 & -a_5 & -a_4 & -a_3 & 1 - a_2 \end{bmatrix}. \quad (44)$$

In expanding the determinant (44), we obtain  $f(\cdot)$  quartic in  $a_1$  and cubic in  $a_2$ . Thus, in distinction to the case  $n = 5$ , it may be necessary to determine the minimum of a function of two variables analytically to get the supplementary points. If  $a_2$  or  $a_1$  are constant (or missing), then for monic polynomials one can extend the results of Hollot and Bartlett [7] for  $n = 6$ . Similar situations can be obtained for  $n > 6$ . Because of the mentioned difficulty, we have not explored the case  $n = 6$  any further.

## VI. CONCLUSIONS

In this paper, robust Schur stability conditions were obtained for polynomials of orders two to five. For  $n = 2$  and 3 the conditions obtained are related to stability of the corner points, while for  $n = 4$  and 5, the conditions are related to stability of corner and possible supplementary points. The number of points increases substantially as the polynomial order increases. The obtained results are of importance in the robust design of control systems. The works of Hollot and Bartlett [7] have aided the authors in developing the results. The difficulty in extending the approach of the paper to higher order polynomials is discussed. Special cases for such extensions are mentioned.

Further applications of the results obtained may be of use in the stability study of two-dimensional systems. Some examples for the application of the stability conditions are given and, in particular, the two counterexamples presented in the literature, are discussed.

## APPENDIX

Derivation of  $a_1^*$ , for  $n = 5$ :

Form  $\partial f / \partial a_1$  to get

$$\frac{\partial f}{\partial a_1} = 3a_5a_1^2 - 2a_1(a_4 + a_3a_5 + a_5^2)$$

$$+ (2a_4a_5 - a_5^3 + a_5 - a_4^2a_5 + 2a_5^2a_3$$

$$+ a_3a_4 + a_3 + a_2a_4a_5 - 3a_4a_5)$$

$$= 3a_5a_1^2 - 2a_1\alpha_1 + \alpha_2 \quad (A1)$$

where  $\alpha_1$  and  $\alpha_2$  are as defined before Theorem 7. The roots of  $\partial f/\partial a_1 = 0$  are

$$a_1^* = \frac{\alpha_1 \pm \sqrt{\alpha_1^2 - 3a_5\alpha_2}}{3a_5} \quad (\text{A2})$$

The minimum of  $f$  is obtained with the plus sign for  $a_5 > 0$  and negative sign for  $a_5 < 0$ , i.e.,

$$a_1^* = \frac{\alpha_1 + \text{sgn}(a_5)\sqrt{D}}{3a_5}$$

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