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CONJECTURE IN SYSTEM THEORY*

This letter proves that a time-invariant infinite-dimensional linear network cannot have identical behaviour to a time-varying finite-dimensional network, the proof depending heavily on advanced concepts in system theory.

The purpose of this note is to answer rigorously the following question (which, it is clear, has equivalent statements in nonelectrical-engineering contexts):

Can a time-invariant infinite-dimensional (linear) network (for example, a transmission line) have identical port behaviour to a time-varying finite-dimensional network (but not to a time-invariant finite-dimensional network)?

By 'finite-dimensional' we mean 'possessing a finite number of energy-storage elements' or, equivalently, 'possessing a state-space description, using a finite number of first-order differential equations'.

The answer is in the negative, as one might expect; but the proof depends surprisingly heavily on advanced concepts in system theory. The question may readily be reformulated in state-space terms. We assume that there exist matrix functions $F(\cdot)$, $G(\cdot)$ and $H(\cdot)$, taken to have elements of bounded variation for obvious physical reasons; so that the system

$$\dot{x} = F(t)x + G(t)u \quad \ldots \quad (1a)$$

$$y = H(t)x \quad \ldots \quad (1b)$$

is zero-state time-invariant; i.e. its impulse response $h(t, \tau)$ can be expressed in the form $h(t-\tau)$. Clearly, this must be the case if the time-varying system is to act like a time-invariant system, regardless of the dimensionality of the latter. [Naturally, this imposes certain constraints on $F(\cdot)$, $G(\cdot)$ and $H(\cdot)$.[1]]

We assume further that $h(\cdot)$ does *not* have a rational Laplace transform, and we attempt to deduce a contradiction. [If $h(\cdot)$ did possess a rational Laplace transform, this would contradict the assumption that $h(\cdot)$ cannot be the impulse response of a time-invariant finite-dimensional network.] Note that eqns. 1 actually imply more than mere finite dimensionality; intrinsically, these equations imply that the Laplace transform of $h(\cdot)$ has no pole at infinity. We assume that this is the case, for, if not, our theory will apply to $\mathscr{L}^{-1}\left\{\frac{\mathscr{L}h(\cdot)}{s^n}\right\}$, where n is a suitably large integer, and $\mathscr{L}, \mathscr{L}^{-1}$ denote the Laplace and inverse Laplace operations, respectively.

For convenience, let us take $h(\cdot)$ to be a scalar and make the following definitions of vector functions $\phi(\cdot)$ and $\psi(\cdot)$:

$$\phi(t) = \Phi'(t, 0)H'(t) \quad \ldots \quad (2a)$$

$$\psi(t) = \Phi(0, t)G(t) \quad \ldots \quad (2b)$$

where $\Phi(t, \tau)$ is the transition matrix of eqn. $1a$. Then, as is well known,[1]

$$h(t-\tau) = \phi'(t)\psi(\tau)1(t-\tau) \quad \ldots \quad (3)$$

where $1(t)$ is the unit step function. We may legitimately assume that the elements of $\phi(\cdot)$ and $\psi(\cdot)$ are linearly independent over $(-\infty, \infty)$, for, otherwise, we could set

$$h(t-\tau) = \phi_r'(t)\psi_r(\tau)1(t-\tau) \quad \ldots \quad (4)$$

where ϕ_r has fewer elements than ϕ, ψ_r has fewer elements than ψ, and the elements of ϕ_r and of ψ_r are, separately, linearly independent over $(-\infty, \infty)$.

From Reference 2, p. 11, we observe that $\phi(\cdot)$ and $\psi(\cdot)$ are differentiable almost everywhere, because $F(\cdot)$, $G(\cdot)$ and $H(\cdot)$ are of bounded variation. Hence we may write

$$\frac{\partial}{\partial t}\{h(t-\tau)\} = -\frac{\partial}{\partial \tau}\{h(t-\tau)\} \quad \ldots \quad (5)$$

or

$$\dot{\phi}'(t)\psi(\tau) = -\phi'(t)\dot{\psi}(\tau) \quad \ldots \quad (6)$$

Let ϕ have n entries, and select times $t_1, t_2, \ldots t_n$, existing by the linear independence of the entries of $\phi(\cdot)$ so that

$$\begin{bmatrix} \phi'(t_1) \\ \cdot \\ \cdot \\ \phi'(t_n) \end{bmatrix}$$

is nonsingular. Then

$$\dot{\psi}(\tau) = -A\psi(\tau) \quad \ldots \quad (7)$$

where

$$A = \begin{bmatrix} \phi'(t_1) \\ \cdot \\ \cdot \\ \phi'(t_n) \end{bmatrix}^{-1} \begin{bmatrix} \dot{\phi}'(t_1) \\ \cdot \\ \cdot \\ \dot{\phi}'(t_n) \end{bmatrix} \quad \ldots \quad (8)$$

It follows that

$$\psi(t) = e^{-At}\psi_0 \quad \ldots \quad (9)$$

for some constant vector ψ_0. Similar arguments establish that

$$\phi(t) = e^{A't}\phi_0 \quad \ldots \quad (10)$$

and thus

$$h(t-\tau) = \phi_0' e^{A(t-\tau)}\psi_0 \quad \ldots \quad (11)$$

Consequently, $h(\cdot)$ has a rational transform; i.e.

$$H(s) = \phi_0'(sI - A)^{-1}\psi_0 \quad \ldots \quad (12)$$

which contradicts the earlier assumption that $h(\cdot)$ has a nonrational Laplace transform.

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