THE GENERATION OF ALL q-MARKOV COVERS*

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Abstract. Recently developed model reduction procedures lead (for the single input-output case) to q-th degree systems with first q Markov parameters and first q covariance parameters (Markov parameters of the causal part of the power spectrum) taking prescribed value. We show that such systems form a class, and parameterize the class. For continuous-time systems the class include an infinite set, whereas for discrete-time the class has only two members. The result is important in that it illustrates the freedom available in a model reduction procedure.

Keywords. Markov parameters, covariances, model reduction, realizations of linear systems.

1. INTRODUCTION

There have been many approaches advanced in the literature for solving the problem of model reduction, including aggregation methods [1], and balanced and Hankel norm approximation [2-6]. A potential drawback of these methods is that the reduced-order models are not guaranteed to match any of the rms values of the model output. Such matching may be important for systems with performance requirements stated in terms of rms values of outputs (e.g. antenna pointing, vibration control in flexible structures). For linear stationary systems, the output autocorrelation is $Ey(t+k)y^*(k) = \sum_{i=0}^{\infty} R_i(\frac{t^i}{i!})$ and the output power spectrum $\sum_{i=0}^{\infty}\{R_i s^{-(i+1)} + R_i'(-s)^{-(i+1)}\}$. If the impulse response is $\sum_{i=0}^{\infty} M_i(\frac{t^i}{i!})$, then the transfer function matrix is $\sum_{i=0}^{\infty} M_i s^{-(i+1)}$. Similar notation explains the discrete-time situations where s is replaced by the z transform and t, s take on integer values. Matching of *both* a subset of the M_i and the R_i is clearly a prima facie attractive basis for model reduction. The tasks of matching *only* the M_i or *only* the R_i have been considered separately in [11-12].

Recently, methods have been evolved for achieving such matching. More precisely, these methods have taken a full order model, and shown how to approximate it with a lower order model with the same $M_o, ..., M_{q-1}$ and $R_o, ..., R_{q-1}$. Such approximations have been named q Markov covers (COVER = COVariance Equivalent Realization) in [7]. Similar approximations appear in [9] for single input/output systems and in [10] for the multi-input output case. These derivations are much less direct than ours and there is no attempt in [9-10] to characterize the class of all q-Markov covers, as offered in this paper. In case the original system is single-input, single-output, the reduced order model is of degree q, (at least generally). The situation is more complicated for multivariable system approximation.

The algorithms of [7], [9], [10], find a q Markov cover. There is no attempt to characterize all, or even to consider the question, is a q Markov cover unique. (It is not.) Our purpose in this paper is to show how given one q-Markov cover one can construct all (with the same $M_o, ..., M_{q-1}$ and $R_o, ..., R_{q-1}$). The parameterization in this paper when combined with the algorithm of [7] then provides a solution to the problem of finding all q-Markov cover approximations of a high order system.

An important tool in our analysis is a specialized Hessenberg form state-variable realization, which we term the normalized Hessenberg form. Any transfer function (matrix) has a unique normalized Hessenberg state-variable realization. parameterization of q-Markov covers is easy, in terms of the entries of this form. After defining the form in Section 2, we describe in Section 3 all realizations which correspond to the same set of $M_o, ..., M_{q-1}$ (but not necessarily $R_o, ..., R_{q-1}$). In Section 4, we explain how some of the R_i are automatically defined by the M_j, and then in Section 5, we indicate how, when the necessary R_i are specified, the collection of all scalar system q-Markov covers can be parametered. In Section 6 we consider the multivariable problem, and in Section 7 we solve the corresponding discrete-time problem. Some concluding remarks are offered in Section 8.

2. NORMALIZED HESSENBERG FORM

Let W(s) be a degree q strictly proper stable transfer function. In this section, we shall describe the construction of a particular state variable realization of W(s) which we term the normalized Hessenberg form, and we shall establish a uniqueness result concerning this form.

Given an arbitrary minimal realization $\{A_1, B_1, C_1\}$ of W(s) one can find a coordinate basis transformation generating a second realization {A,B,C} of so-called Hessenberg form, [8], i.e.

$$C = [x \;\; 0 \;\; \ldots \;\; 0]$$

$$A = \begin{bmatrix} x & x & 0 & \cdots & 0 \\ x & x & x & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ x & x & x & \cdots & x \\ x & x & x & \cdots & x \end{bmatrix} \quad B = \begin{bmatrix} x \\ x \\ \cdot \\ \cdot \\ \cdot \\ x \\ x \end{bmatrix} \qquad (2.1)$$

Minimality ensures that $c_1 \neq 0$, $a_{i,i+1} \neq 0$ for $i = 1, \ldots, q-1$.

The normalized Hessenberg form is constructed as follows. Let X be the positive definite solution of

$$AX + XA' = -BB' \qquad (2.2)$$

[Recall that A is stable, and {A,B,C} is minimal]. Now let T be a lower triangular matrix such that

$$TT' = X \qquad (2.3)$$

Then set

$$\bar{A} = T^{-1}AT \quad \bar{B} = T^{-1}B \quad \bar{C} = CT \qquad (2.4)$$

The lower triangular nature of T ensures that $\bar{A}$, $\bar{B}$, $\bar{C}$ inherit the structural constraints of A, B, C including the nonzero nature of $\bar{c}_1$, $\bar{a}_{i,i+1}$ while also

$$\bar{A} + \bar{A}' = -\bar{B}\bar{B}' \qquad (2.5)$$

A triple $\{\bar{A}, \bar{B}, \bar{C}\}$ of this form has been described in [7] as being a cost-decoupled Hessenberg form.

To obtain the normalized Hessenberg form, one further coordinate basis transformation is made. The transforming matrix is diag $\{\pm 1, \pm 1, \ldots, \pm 1\}$ where the signs are chosen so that after transformation the first entry c_1 of the new C vector and the superdiagonal entries $a_{i,i+1}$, $i = 1, \ldots, q-1$ of the new A matrix are all positive. The transformation leaves undisturbed the zero pattern and (2.5) holds with the transformed matrices. Summing up, we have

Definition. *A minimal realization {A,B,C} of a stable q-th degree transfer function W(s) is a triple with the zero pattern of (2.1), with*

$$A + A' = -BB' \qquad (2.6)$$

and with

$$c_1 > 0 \quad a_{i,i+1} > 0 \quad i = 1, \ldots, q-1 .$$

and

Lemma 2.1. *Given an arbitrary minimal state-variable realization of a stable W(s), one can construct from it a realization in normalized Hessenberg form.*

An important question to consider is the extent to which a given W(s) might have more than one normalized form. As it turns out, normalized Hessenberg forms are unique:

Lemma 2.2. *Let W(s) be a degree q strictly proper stable transfer function. Then there is only one minimal state variable realization of normalized Hessenberg form.*

Proof. Let $\{A_i, B_i, C_i\}$ for $i = 1, 2$ be two realizations of normalized Hessenberg form. Hence $X_1 = X_2 = I$, and by minimality, there exists a T with $A_2 = TA_1T^{-1}$ $B_2 = TB_1$, $C_2 = C_1T^{-1}$, $X_2 = TX_1T'$. From $X_2 = TX_1T' = TT' = I$ it follows that T must be unitary. Further, we have

$$\begin{bmatrix} C_1 \\ C_1A_1 \\ \cdot \\ \cdot \\ \cdot \\ C_1A_1^{q-1} \end{bmatrix} = \begin{bmatrix} C_2 \\ C_2A_2 \\ \cdot \\ \cdot \\ \cdot \\ C_2A_2^{q-1} \end{bmatrix} T$$

and the structure of C_i, A_i ensures that each observability matrix is lower triangular, with positive diagonal elements. Hence T is lower triangular with positive diagonal elements. So therefore is T^{-1}. On the other hand, $T^{-1} = T'$, and T' must be upper triangular with positive diagonal elements. Hence T^{-1} is diagonal with positive diagonal elements. Since T is orthogonal, it must be I.

3. PARAMETERIZATION WITH MARKOV PARAMETER SPECIFICATION

In preparation for the characterization of all q-covers, we here consider the following question. Suppose there is given the first q Markov coefficients $M_0, M_1, \ldots, M_{q-1}$ of a q-th degree stable transfer function. In a normalized Hessenberg state variable realization of the transfer function, what parameters remain freely adjustable? The main result is the following

Lemma 3.1 *Consider the collection of all q dimensional normalized Hessenberg realizations of q-th degree stable transfer functions with prescribed Markov coefficients $M_0, M_1, \ldots, M_{q-1}$. Let ? denote a selectable parameter, and d denote a parameter determined by a combination of the data and the selectable parameter. Then the collection of such realizations is defined by*

$$C = [? \;\; 0 \;\; \ldots \;\; 0]$$

$$A = \begin{bmatrix} d & ? & 0 & \cdots & 0 & 0 \\ d & d & ? & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ d & d & \cdot & \cdots & d & ? \\ d & d & \cdot & \cdots & d & d \end{bmatrix} \quad B = \begin{bmatrix} d \\ d \\ \cdot \\ \cdot \\ \cdot \\ d \\ d \end{bmatrix} \qquad (3.1)$$

subject of course also to

$$A + A' = -BB'$$

$$c_1 > 0 \quad a_{i,i+1} > 0 \quad i = 1, \ldots, q-1 . \qquad (3.2)$$

Remark 3.1 Note that the parameter count is correct. The class of q-th degree strictly proper transfer functions can be parameterized by 2q parameters. When $M_0, \ldots, M_{q-1}$ are specified, q further parameters are left free, and these are taken up by the ? quantities in (3.1).

Proof. The proof is constructive. We must show that if c_1, $a_{i,i+1}$ take fixed positive values, and $M_0, \ldots, M_{q-1}$ are prescribed, all other nonzero entries of A and B can be computed.

From $M_0 = CB = c_1b_1$, b_1 is obtained.

From $(A+A'+BB')_{11} = 0$, $a_{11} = -\frac{1}{2} b_1^2$ is obtained.

Since $CA = [c_1a_{11} \;\; c_1a_{12} \;\; 0 \cdots 0]$ which is now known, we have $M_1 = CAB = c_1a_{11}b_1 + c_1a_{12}b_2$ and obtain b_2. From $(A+A'+BB')_{i2} = 0$ for $i = 1, 2$ we have

$a_{12}+a_{21} = -b_1 b_2$ and obtain a_{21} and $2a_{22} = -b_2^2$ and obtain a_{22}.

Now CA^2 is known, $M_2 = CA^2B$ is used to determine b_3 and then the third row of A can be found. This process repeats.

Remark 3.2 A perusal of the above proof will show that an alternative assignment of selectable and determined parameters is provided by

$$C = [d \;\; 0 \;\; \ldots \;\; 0]$$

$$A = \begin{bmatrix} d & d & 0 & \cdots & 0 \\ d & d & d & \cdots & . \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ . & . & . & \cdots & d \\ d & d & d & \cdots & d \end{bmatrix} \qquad B = \begin{bmatrix} ? \\ ? \\ . \\ . \\ . \\ ? \end{bmatrix} \tag{3.3}$$

Remark 3.3 If fewer than r Markov parameters $M_0, \ldots, M_{r-1}$ are available where $r < q$, similar arguments show that the following parameter pattern is achievable:

$$C = [? \;\; 0 \;\; \ldots \;\; 0]$$

$$A = \begin{bmatrix} d & ? & 0 & \cdots & 0 \\ d & d & ? & \cdots & 0 \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ . & . & . & d & ? \\ d & d & d & d & d \end{bmatrix} \qquad B = \begin{bmatrix} d \\ . \\ . \\ d \\ ? \\ . \\ ? \end{bmatrix} \; r$$

Remark 3.4 If any two normalized Hessenberg realizations differ in any entry, the realizations necessarily correspond (by Lemma 2.2) to different transfer functions. It follows that all distinct choices of selectable parameters in (3.1) correspond to distinct transfer functions with the same Markov parameters.

Example. Consider a third order system in which $M_0 = 1$, $M_1 = -2$, $M_2 = -\frac{1}{2}$. Let us select $c_1 = \alpha$, $a_{12} = \beta$, $a_{23} = \gamma$. Thus

$$C = [\alpha \;\; 0 \;\; 0]$$

$$A = \begin{bmatrix} d & \beta & 0 \\ d & d & \gamma \\ d & d & d \end{bmatrix} \qquad B = \begin{bmatrix} d \\ d \\ d \end{bmatrix}$$

Because $M_0 = CB = 1$, $b_1 = \alpha^{-1}$. Then $a_{11} = -\frac{1}{2\alpha^2}$. Now $CA = [-\frac{1}{2\alpha}, \alpha\beta, 0]$ and $M_1 = CAB = -2$, or $-\frac{1}{2\alpha^2} + \alpha\beta b_2 = -2$, whence $b_2 = -\frac{2}{\alpha\beta} + \frac{1}{2\alpha^3\beta}$. Then $a_{22} = -\frac{1}{2} b_2^2 = -\frac{1}{2\alpha^2\beta^2}[4 - \frac{2}{\alpha^2} + \frac{1}{4\alpha^4}]$. Further $a_{12} + a_{21} = -b_1 b_2$ and so $a_{21} = \frac{2}{\alpha^2\beta} - \frac{1}{2\alpha^4\beta} - \beta$. Then

$$CA^2 = [\frac{1}{4\alpha^3} + \frac{2}{\alpha} - \frac{1}{2\alpha^3} - \alpha\beta^2,$$

$$-\frac{\beta}{2\alpha} - \frac{1}{2\alpha\beta}[4 - \frac{2}{\alpha^2} + \frac{1}{4\alpha^4}], \;\; \alpha\beta\gamma] .$$

Now $M_2 = CA^2B = -\frac{1}{2}$, which yields

$$b_3 = \frac{-1}{2\alpha\beta\gamma} + \frac{1}{2\alpha^5\beta\gamma} - \frac{3}{\alpha^3\beta\gamma} + \frac{\beta}{\alpha\gamma} - \frac{4}{\alpha^3\beta^3\gamma}$$

$$+ \frac{3}{\alpha^5\beta^3\gamma} - \frac{3}{4\alpha^7\beta^3\gamma} + \frac{1}{16\alpha^9\beta^3\gamma}$$

$a_{31} = -b_1 b_3$, $a_{32} = -b_2 b_3 - \gamma$ and $a_{33} = -\frac{1}{2} b_3^2$ all follow easily.

4. MARKOV AND COVARIANCE PARAMETER RELATIONS

In preparation for the parameterization in the next section of all q-th order q-Markov covers, we shall indicate here relationships which must exist between Markov and covariance parameters. Let $W(s)$ be a stable strictly proper transfer function of order q and $\{A,B,C\}$ any minimal realization. Recall that

$$M_i = CA^iB , \quad R_i = CA^iXC' \tag{4.1}$$

where for unit intensity noise inputs,

$$XA' + AX = -BB' . \tag{4.2}$$

Lemma 4.1 *With quantities as defined above, for any integer m these hold*

$$2R_{2m+1} = -M_0M_{2m}' + M_1M_{2m-1}' \tag{4.3}$$

$$- M_2M_{2m-2}' \cdots - M_{2m}M_0' .$$

Proof Let $Z(s) \triangleq C(sI-A)^{-1}XC'$. Observe that

$$Z(s) + Z'(-s) = C[(sI-A)^{-1}X + X(-sI-A')^{-1}]C'$$

$$= C(sI-A)^{-1}[X(-sI-A') + (sI-A)X](-sI-A')^{-1}C'$$

$$= C(sI-A)^{-1}BB'(-sI-A')^{-1}C'$$

$$= W(s)W'(-s) \tag{4.4}$$

Because

$$Z(s) = R_0s^{-1} + R_1s^{-2} + R_2s^{-3} + \cdots$$

$$W(s) = M_0s^{-1} + M_1s^{-2} + \cdots$$

we recover (4.3) by inverting these expansions in (4.4) and equating even powers of s^{-1} on each side of the equation.

An important conclusion flows from this lemma. Suppose that $M_0, M_1, \ldots, M_{q-1}$ and $R_0, R_1, \ldots, R_{q-1}$ are specified for some system. Then this is not a collection of 2q independent pieces of data. According as $q = 2p$ or $2p+1$, it is a collection of 3p or 3p+2 independent pieces of data, since $R_1, R_3, \ldots, R_{2p-1}$ are not free.

This means in turn that if one is seeking a q-th degree system with prescribed $M_0, M_1, \ldots, M_{q-1}$ and $R_0, R_1, \ldots, R_{q-1}$ one cannot expect the system to be uniquely defined by the data. Rather, according as $q = 2p$ or $2p+1$, i.e. the system requires 4p or 4p+2 parameters to specify it completely, there will remain $p = (4p-3p)$ or $[(4p+2) - (3p+2)]$ parameters which can be adjusted.

If one is seeking an r-th degree system with prescribed $M_0, M_1, \ldots, M_{q-1}$ and $R_0, R_1, \ldots, R_{q-1}$, there will remain $2r - q - \lceil q/2 \rceil$ parameters which can be adjusted. Here, $\lceil x \rceil$ denotes the least integer greater than or equal to x; subsequently $\lfloor x \rfloor$ will denote the greatest integer less than or equal to x.

5. PARAMETERIZATION WITH MARKOV AND COVARIANCE PARAMETER SPECIFICATION

The question we address in this section is the following: how can we parameterize all q-th degree stable transfer functions with the same $M_0, M_1, \ldots, M_{q-1}$ and $R_0, \ldots, R_{q-1}$. In the light of the preceding material, an equivalent problem is the parameterization of q-th order normalized Hessenberg realizations achieving the prescribed M_i, R_i and we shall expect free parameters to occur.

Main Theorem Consider the collection of all q dimensional normalized Hessenberg realizations of q-th degree stable strictly proper transfer functions with prescribed Markov and covariance coefficients $M_0, M_1, \ldots, M_{q-1}$ and $R_0, R_1, \ldots R_{q-1}$ [these quantities in fact being computable from any realization as described in (4.1) and (4.2)] Let ? denote a selectable parameter; x a parameter determined by the given data M_i, R_i and d a parameter determined by the parameter ?, x and the data. Then the collection is defined by

$$C = [x \ \ 0 \ \ \ldots\ldots \ \ 0]$$

$$A = \begin{bmatrix} x & x & \cdots & . & . & . & \cdots & . \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x & x & \cdots & x & . & . & \cdots & . \\ x & x & \cdots & x & ? & . & \cdots & . \\ d & d & \cdots & d & d & ? & \cdots & . \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & ? \\ d & d & \cdots & . & . & . & \cdots & d \end{bmatrix} \quad B = \begin{bmatrix} x \\ \vdots \\ \vdots \\ . \\ x \\ d \\ \vdots \\ d \end{bmatrix}$$

with the number of ? entries being $\lfloor q/2 \rfloor$. The only constraint on the selectable parameters ? is that the (A,B,C) be controllable and observable, and $a_{i,i+1} > 0$ as required by the normalized Hessenberg form.

Remark 5.1 The covariance data contains $\lceil q/2 \rceil$ independent pieces of information. Relative to the parameterization in Section 3, the first $\lceil q/2 \rceil$ parameters in the list $c_1, a_{12}, a_{23}, \ldots$ become determined by the data; as a consequence, the first $\lceil q/2 \rceil$ rows of A and B (with the exception of the right most nonzero entry in the bottom of these rows of A) become determined. The remaining parameters in the list $c_1, a_{12}, a_{23}, \ldots$, which are $\lfloor q/2 \rfloor = q - \lceil q/2 \rceil$ in number, remain selectable.

The proof of the main result depends on the following lemma

Lemma 5.1 *Let* $A + A' = -BB'$ *and* $i \geq j \geq 1$ *integers. Then*

$$A^i(A')^j = (-1)^j[A^{i+j} + A^{i+j-1}BB' - A^{i+j-2}BB'A' + A^{i+j-3}BB'(A')^2 \cdots + (-1)^{j-1}A^iBB'(A')^{j-1}].$$

The lemma is proved by replacing each occurrence of BB' on the right side of (5.2) by $-(A+A')$.

Pre and post-multiplication of (5.2) by C and C' respectively leads to

$$CA^i(A')^jC' = (-1)^j[R_{i+j} + M_{i+j-1}M_0 - M_{i+j-2}M_1 \cdots + (-1)^{j-1}M_iM_{j-1}] \quad i \geq j \geq 1$$

This fact will be used below.

Proof of Main Theorem Incorporating knowledge of the Markov parameters gives us realizations of the form of (3.1). Next, the data at our disposal ensures that we can calculate the matrix

$$\begin{bmatrix} C \\ CA \\ . \\ . \\ . \\ CA^p \end{bmatrix} [C' \ \ A'C' \ \cdots \ (A')^pC']$$

where $q = 2p + 1$ and the matrix

$$\begin{bmatrix} C \\ CA \\ . \\ . \\ . \\ CA^p \end{bmatrix} [C' \ \ A'C' \ \cdots \ (A')^{p-1}C']$$

where $q = 2p$.

Forget for the moment the last row in the first of these two matrices. Then we are dealing with square matrices of dimension $\lceil q/2 \rceil \times \lceil q/2 \rceil$. Now the structure of C and A ensure that

$$\begin{bmatrix} C \\ CA \\ . \\ . \\ . \\ CA^r \end{bmatrix}$$

is lower triangular, with positive diagonal elements, for any $r \leq q$. In fact, the diagonal elements are $c_1, c_1a_{12}, c_1a_{12}a_{23}, \ldots$ Identify r with $\lceil q/2 \rceil$ and notice that the lower triangular matrix is computable as the square root of a known matrix, uniquely because of the positivity of the diagonal elements. Consequently $c_1, c_1a_{12}, c_1a_{12}a_{23}, \ldots$ (there being $\lceil q/2 \rceil$ terms) are known. In order, one can then find $b_1, a_{11}, b_2, a_{21}, a_{22}, b_3, a_{31}, a_{32}, a_{33}, \ldots$ and finally row $\lceil q/2 \rceil$ of A and B except for the last nonzero entry of the row of A.

Now above we observed that when $q = 2p$, the matrix $CA^p[C'A'C' \cdots (A')^{p-1}C']$ was computable from the data at our disposal. This matrix was not yet used to identify parameters in A,B,C, individually; and the question arises as to whether we can learn more about A,B,C from this matrix. A direct calculation shows that every entry of this matrix is expressible in terms of those entries of A,B,C for which values have been determined by the process outlined in the preceding paragraph. This means that the matrix $CA^p[C'A'C' \cdots (A')^{p-1}C']$, contains no additional information to that contained in the matrices $CA^r[C'A'C' \cdots (A')^{p-1}C']$, $r = 0, 1, \ldots, p-1$. Finally, controllability is required so that $A + A' + BB' = 0$ implies $X = I$, and observability is required to make the super-diagonal elements of A positive. ∇∇∇.

Example Find all third order systems with $M_0 = 1$, $M_1 = -2$, $M_2 = -\frac{1}{2}$, $R_0 = 1$, $R_1 = -\frac{1}{2}$, $R_2 = -\frac{1}{2}$.

We know in advance that the structure of a normalized Hessenberg form will be

$$C = [x \ \ 0 \ \ 0]$$

$$A = \begin{bmatrix} x & x & 0 \\ x & x & \alpha \\ d & d & d \end{bmatrix} \quad B = \begin{bmatrix} x \\ x \\ d \end{bmatrix}$$

Where α is a selectable parameter, and all d values are determined by the data and α. We shall find entries sequentially. $R_o = CC' = 1$ and $c_1 > 0$ gives $c_1 = 1$. $M_o = CB = c_1 b_1$ gives $b_1 = 1$ whence $a_{11} = -\frac{1}{2} CAC'$ can be computed from what has been already found, and is $-\frac{1}{2}$, the value of R_1. Recall from (4.3) that $2R_1 = -M_o^2$ and hence the value of R_1 is not independently selectable. Now $CA = [-\frac{1}{2}\ a_{12}\ 0]$ and $CAA'C' = -CA^2C' - CABB'C' = -R_2 - M_oM_1 = 5/2$. Also, $CAA'C' = \frac{1}{4} + a_{12}^2$. So $a_{12}^2 = 9/4$ or $a_{12} = 3/2$. Now $CAB = [-1/2\ 3/2\ 0]\ [b_1\ b_2\ b_3]' = -1/2 + 3/2\ b_2 = M_1$. So $b_2 = -1$. Then $a_{22} = -1/2$. $a_{12} + a_{21} = -b_1 b_2$, so $a_{21} = -1/2$. At this stage, we have

$$C = [1\ \ 0\ \ 0]$$

$$A = \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \alpha \\ d & d & d \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ -1 \\ d \end{bmatrix}$$

Now $CA^2 = [-\frac{1}{2}\ \ -\frac{3}{2}\ \ \frac{3\alpha}{2}]$ and $M_2 = CA^2B = -1/2$. These results $b_3 = -\alpha^{-1}$. Then $a_{31} = -b_1 b_3 = \alpha^{-1}$, $a_{32} = -b_2 b_3 - a_{23} = \alpha^{-1} - \alpha$, $a_{33} = -\frac{1}{2}\alpha^{-2}$. Finally then,

$$C = [1\ 0\ 0] \tag{5.4a}$$

$$A = \begin{bmatrix} -1/2 & 3/2 & 0 \\ -1/2 & -1/2 & \alpha \\ \alpha^{-1} & -\alpha^{-1}-\alpha & -\frac{1}{2}\alpha^{-2} \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ -1 \\ -\alpha^{-1} \end{bmatrix} \tag{5.4b}$$

The associated family of transfer functions is

$$W(s,\alpha) = \frac{s^2 + (\frac{1}{2}\alpha^{-2} - 1)s + (\alpha^2 - 1/2 - 1/2\alpha^{-2})}{s^3 + (1 + \frac{1}{2}\alpha^{-2})s^2 + (\alpha^2 + 2 + \frac{1}{2}\alpha^{-2})s + (\frac{1}{2}\alpha^2 - 1 + \frac{1}{2}\alpha^{-2})}$$

for all real values of α which preserve minimality of (A,B,C). There is an infinite number of such values. The only positive value of α which is excluded is $\alpha = 1$, since this is the only positive real root of

$$\alpha^8 + 6\alpha^6 - 5\alpha^4 - \frac{3}{2}\alpha^2 - \frac{1}{2} = 0$$

(All these values cause loss of controllability.) All other finite values of α describe the entire class of 3rd order systems which match the data, and the corresponding table of poles and zeros appear in TABLE 1 for $\alpha > 1$. The poles and zeros for $\alpha < 1$ appear in TABLE 2.

TABLE 1

Poles & Zeros of $W(s, \alpha)$ for $\alpha > 1$

α	Z1	Z2	P1	P2	P3
2.	0.43 - 1.78i	0.43 + 1.78i	-0.40 - 2.39i	-0.40 + 2.39i	-0.18 + 0.i
4.	0.48 - 3.90i	0.48 + 3.90i	-0.31 - 4.20i	-0.31 + 4.20i	-0.39 + 0.i
6.	0.49 - 5.93i	0.49 + 5.93i	-0.28 - 6.13i	-0.28 + 6.13i	-0.45 + 0.i
8.	0.49 - 7.95i	0.49 + 7.95i	-0.26 - 8.10i	-0.26 + 8.10i	-0.47 + 0.i
10.	0.49 - 9.96i	0.49 + 9.96i	-0.25 - 10.08i	-0.26 + 10.08i	-0.48 + 0.i
12.	0.49 - 11.96i	0.49 + 11.96i	-0.25 - 12.07i	-0.25 + 12.07i	-0.48 + 0.i
14.	0.49 - 13.97i	0.49 + 13.97i	-0.25 - 14.06i	-0.25 + 14.06i	-0.48 + 0.i
16.	0.49 - 15.97i	0.49 + 15.97i	-0.25 - 16.05i	-0.25 + 16.05i	-0.48 + 0.i
18.	0.49 - 17.97i	0.49 + 17.97i	-0.25 - 18.04i	-0.25 + 18.04i	-0.49 + 0.i
20.	0.49 - 19.98i	0.49 + 19.98i	-0.25 - 20.04i	-0.25 + 20.04i	-0.49 + 0.i
22.	0.49 - 21.98i	0.49 + 21.98i	-0.25 - 22.04i	-0.25 + 22.04i	-0.49 + 0.i
24.	0.49 - 23.98i	0.49 + 23.98i	-0.25 - 24.03i	-0.25 + 24.03i	-0.49 + 0.i
26.	0.49 - 25.98i	0.49 + 25.98i	-0.25 - 26.03i	-0.25 + 26.03i	-0.49 + 0.i
28.	0.49 - 27.98i	0.49 + 27.98i	-0.25 - 28.03i	-0.25 + 28.03i	-0.49 + 0.i
30.	0.49 - 29.98i	0.49 + 29.98i	-0.25 - 30.02i	-0.25 + 30.02i	-0.49 + 0.i
32.	0.49 - 31.98i	0.49 + 31.98i	-0.25 - 32.02i	-0.25 + 32.02i	-0.49 + 0.i
34.	0.49 - 33.98i	0.49 + 33.98i	-0.25 - 34.02i	-0.25 + 34.02i	-0.49 + 0.i
36.	0.49 - 35.98i	0.49 + 35.98i	-0.25 - 36.02i	-0.25 + 36.02i	-0.49 + 0.i
38.	0.49 - 37.99i	0.49 + 37.99i	-0.25 - 38.02i	-0.25 + 38.02i	-0.49 + 0.i
40.	0.49 - 39.99i	0.49 + 39.99i	-0.25 - 40.02i	-0.25 + 40.02i	-0.49 + 0.i

TABLE 2

Poles & Zeros of $W(s,\alpha)$ for $\alpha \leq 1$.

(note the factor of 100 for the poles)

α	Z1	Z2	P1	P2	P3
			x(100)		
0.05	-200.00	1.00	-3.99 - 0.00i	-0.00 - 0.00i	-0.00 + 0.00i
0.10	-50.00	1.00	-0.49 - 0.00i	-0.00 - 0.00i	-0.00 + 0.00i
0.15	-22.26	1.02	-0.22 - 0.00i	-0.00 - 0.00i	-0.00 + 0.00i
0.20	-12.53	1.03	-0.12 - 0.00i	-0.00 - 0.00i	-0.00 + 0.00i
0.25	-8.04	1.04	-0.07 + 0. i	-0.00 - 0.00i	-0.00 + 0.00i
0.30	-5.61	1.06	-0.05 + 0. i	-0.00 - 0.00i	-0.00 + 0.00i
0.35	-4.16	1.07	-0.03 + 0. i	-0.00 - 0.00i	-0.00 + 0.00i
0.40	-3.20	1.08	-0.02 - 0.00i	-0.01 - 0.00i	-0.01 + 0.00i
0.45	-2.53	1.08	-0.01 - 0.01i	-0.01 + 0.01i	-0.00 - 0.00i
0.50	-2.05	1.08	-0.01 - 0.01i	-0.01 + 0.01i	-0.00 + 0. i
0.55	-1.72	1.07	-0.01 - 0.01i	-0.01 + 0.01i	-0.00 + 0. i
0.60	-1.46	1.05	-0.01 - 0.01i	-0.01 + 0.01i	-0.00 + 0. i
0.65	-1.24	1.03	-0.01 - 0.01i	-0.01 + 0.01i	-0.00 + 0. i
0.70	-1.02	1.00	-0.00 - 0.01i	-0.00 + 0.01i	-0.00 + 0. i
0.75	0.96	-0.85	-0.00 - 0.01i	-0.00 + 0.01i	-0.00 + 0. i
0.80	0.91	-0.69	-0.00 - 0.01i	-0.00 + 0.01i	-0.00 + 0. i
0.85	0.85	-0.54	-0.00 - 0.01i	-0.00 + 0.01i	-0.00 + 0. i
0.90	0.77	-0.39	-0.00 - 0.01i	-0.00 + 0.01i	-0.00 + 0. i
0.95	0.67	-0.22	-0.00 - 0.01i	-0.00 + 0.01i	-0.00 + 0. i
*1.00	0.50	0.	-0.00 - 0.01i	-0.00 + 0.01i	0. + 0. i

* note that $\alpha = 1$ is not permitted due to loss of controllability.

CONCLUDING REMARKS

There may be many motivations for model reduction. A common one is to take a complex controller design (obtained for example by linear-quadratic methods) and simplify it, in order to more easily realize it. There is no single optimality criterion, L_2, L_1, Hankel norm, which reflects the many different and competing objectives of controller design, which may include transient performance, robustness in the face of structured and unstructured plant parameter variations and disturbance and noise suppression. It follows that it is helpful to designers to have available a collection of tools for controller simplification, since no one tool is likely to be universally preferable. Methods based on q-Markov covers have a priori appeal on the grounds that they may accurately retain steady state noise properties (as noted above) as well as some transient properties (through Markov coefficient matching).

In this paper, we have solved the problem of characterizing all q-th degree systems which have the same first q Markov and covariance parameters. By coupling this result with those of [7], which explain how to construct a q-th degree q-Markov cover given a high order system, we have a parameterization of all the q-th degree systems which are q-Markov covers of a high order system.

For single input/output there is an *infinite* number of continuous-time realizations of order q which match the first q Markov and covariance parameters, whereas there are only *two* such systems for discrete time.

There are additional problems which might be considered. For example, could one show for the continuous-time case that knowledge of further Markov and/or covariance data would determine in a simple, perhaps sequential way, some of the free parameters in the collection of q-Markov covers of q-th degrees? Connections with this question and the lattice filters, Levinson algorithm, and Hessenberg form should be investigated. Second, one could examine the parameterization of all weighted q-Markov covers. To explain what is mean, let v(s) be a scalar stable system which is fixed (the weighting system). We have to parameterise all w(s) such that all transfer functions w(s)v(s) have the same $M_o, \ldots, M_{q-1}$ and $R_o, \ldots, R_{q-1}$. Such notions would be useful in controller reduction since the input to the controller is not white noise.

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