The Generation of all q-Markov Covers

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Abstract —**Recently developed model reduction procedures lead (for the single input–output case) to qth degree systems with first q-Markov parameters and first q covariance parameters (Markov parameters of the causal part of the power spectrum) taking prescribed values. We parameterize the class of all such systems. For continuous-time systems the class includes an infinite set, whereas for discrete-time the class has only two members. The result is important in that it illustrates the freedom available in a model reduction procedure.**

I. Introduction

THERE HAVE BEEN many approaches advanced in the literature for solving the problem of model reduction, including aggregation methods [1], and balanced and Hankel norm approximation [2]–[6]. A potential drawback of these methods is that the reduced-order models are not guaranteed to match any of the rms values of the model output. Such matching may be important for systems with performance requirements stated in terms of rms values of outputs (e.g. antenna pointing, vibration control in flexible structures). For linear stationary systems, the output autocorrelation is

$$Ey(t+k)y^*(k)=\sum_{i=0}^{\infty}R_i\left(\frac{t^i}{i!}\right)$$

and the outpout power spectrum

$$\sum_{i=0}^{\infty}\left\{R_is^{-(i+1)}+R_i'(-s)^{-(i+1)}\right\}.$$

If the impulse response is

$$\sum_{i=0}^{\infty}M_i\left(\frac{t^i}{i!}\right)$$

then the transfer function matrix is $\sum_{i=0}^{\infty}M_is^{-(i+1)}$. Similar notation explains the discrete-time situations where s is replaced by the z transform and t,k take on integer values. Matching of *both* a subset of the M_i and the R_i is clearly a prima facie attractive basis for model reduction. The tasks of matching *only* the M_i or *only* the R_i have been considered separately in [11], [12].

Recently, methods have been evolved for achieving such matching. More precisely, these methods have taken a full order model, and shown how to approximate it with a lower order model with the same $M_0,\cdots,M_{q-1}$ and $R_0,\cdots,R_{q-1}$. Such approximations have been named q-Markov covers (COVER = COVariance Equivalent Realization) in [7]. Similar approximations appear in [9] for single input–output systems and in [10] for the multi-input–output case. These derivations are much less direct than ours and there is no attempt in [9], [10] to characterize the class of all q-Markov covers, as offered in this paper. In case the original system is single-input, single-output, the reduced-order model is of degree q, (at least generically). The situation is more complicated for multivariable system approximation.

The algorithms of [7], [9], [10], find a q-Markov cover. There is no attempt to characterize all q-Markov covers, or even to consider the question of whether a q Markov cover is unique. (It is not.) Our purpose in this paper is to show how given one q-Markov cover one can construct all (with the same $M_0,\cdots,M_{q-1}$ and $R_0,\cdots,R_{q-1}$). The parameterization in this paper when combined with the algorithm of [7] then provides a solution to the problem of finding all q-Markov cover approximations of a high-order system.

An important tool in our analysis is a specialized Hessenberg form state-variable realization, which we term the normalized Hessenberg form. Any transfer function (matrix) has a unique normalized Hessenberg state-variable realization. Parameterization of q-Markov covers is easy, in terms of the entries of this form. After defining the form in Section II, we describe in Section III all realizations which correspond to the same set of $M_0,\cdots,M_{q-1}$ (but not necessarily $R_0,\cdots,R_{q-1}$). In Section IV, we explain how some of the R_i are automatically defined by the M_j, and then in Section V, we indicate how, when the necessary R_i are specified, the collection of all scalar system q-Markov covers can be parameterized. In Section VI we consider the multivariable problem, and in Section VII we solve the corresponding discrete-time problem. Some concluding remarks are offered in Section VIII.

II. Normalized Hessenberg Form

Let $W(s)$ be a degree q strictly proper stable transfer function. In this section, we shall describe the construction of a particular state variable realization of $W(s)$ which we term the normalized Hessenberg form, and we shall establish a uniqueness result concerning this form.

Given an arbitrary minimal realization $\{A_1,B_1,C_1\}$ of $W(s)$ one can find a coordinate basis transformation generating a second realization $\{A,B,C\}$ of so-called Hessen-

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berg form, [8], i.e.,

$$C=[x \quad 0 \quad \cdots \quad 0]$$

$$A=\begin{bmatrix} x & x & 0 & \cdots & 0 \\ x & x & x & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & 0 \\ x & x & x & \cdots & x \\ x & x & x & \cdots & x \end{bmatrix} \quad B=\begin{bmatrix} x \\ x \\ \cdot \\ \cdot \\ \cdot \\ x \\ x \end{bmatrix}. \tag{2.1}$$

Minimality ensures that $c_1 \neq 0$, $a_{i,i+1} \neq 0$, for $i=1,\cdots,q-1$.

The normalized Hessenberg form is constructed as follows. Let X be the positive definite solution of

$$AX+XA'=-BB'. \tag{2.2}$$

(Recall that A is stable, and $\{A,B,C\}$ is minimal). Now let T be a lower triangular matrix such that

$$TT'=X. \tag{2.3}$$

Then set

$$\bar{A}=T^{-1}AT \qquad \bar{B}=T^{-1}B \qquad \bar{C}=CT. \tag{2.4}$$

The lower triangular nature of T ensures that $\bar{A}$, $\bar{B}$, $\bar{C}$ inherit the structural constraints of A,B,C including the nonzero nature of $\bar{c}_1$, $\bar{a}_{i,i+1}$ while also

$$\bar{A}+\bar{A}'=-\bar{B}\bar{B}'. \tag{2.5}$$

A triple $\{\bar{A},\bar{B},\bar{C}\}$ of this form has been described in [7] as being a cost-decoupled Hessenberg form.

To obtain the normalized Hessenberg form, one further coordinate basis transformation is made. The transforming matrix is $\operatorname{diag}\{\pm 1, \pm 1, \cdots, \pm 1\}$ where the signs are chosen so that after transformation the first entry c_1 of the new C vector and the superdiagonal entries $a_{i,i+1}$, $i=1,\cdots,q-1$ of the new A matrix are all positive. The transformation leaves undisturbed the zero pattern and (2.5) holds with the transformed matrices. Summing up, we have

Definition: A minimal realization $\{A,B,C\}$ of a stable qth degree transfer function $W(s)$ is a normalized Hessenberg form when it is a triple with the zero pattern of (2.1), with

$$A+A'=-BB' \tag{2.6}$$

and with

$$c_1>0, \quad a_{i,i+1}>0, \qquad i=1,\cdots,q-1$$

and

Lemma 2.1: Given an arbitrary minimal state-variable realization of a stable $W(s)$, one can construct from it a realization in normalized Hessenberg form.

An important question to consider is the extent to which a given $W(s)$ might have more than one normalized form. As it turns out, normalized Hessenberg forms are unique:

Lemma 2.2: Let $W(s)$ be a degree q strictly proper stable transfer function. Then there is only one minimal state-variable realization of normalized Hessenberg form.

Proof: Let $\{A_i, B_i, C_i\}$ for $i=1,2$ be two realizations of normalized Hessenberg form. Hence $X_1=X_2=I$, and by minimality, there exists a T with $A_2=TA_1T^{-1}$, $B_2=TB_1$, $C_2=C_1T^{-1}$, $X_2=TX_1T'$. From $X_2=TX_1T'=TT'=I$ it follows that T must be unitary. Further, we have

$$\begin{bmatrix} C_1 \\ C_1A_1 \\ \vdots \\ C_1A_1^{q-1} \end{bmatrix} = \begin{bmatrix} C_2 \\ C_2A_2 \\ \vdots \\ C_2A_2^{q-1} \end{bmatrix} T$$

and the structure of C_i, A_i ensures that each observability matrix is lower triangular, with positive diagonal elements. Hence T is lower triangular with positive diagonal elements. So, therefore, is T^{-1}. On the other hand, $T^{-1}=T'$, and T' must be upper triangular with positive diagonal elements. Hence T^{-1} is diagonal with positive diagonal elements. Since T is orthogonal, it must be I.

III. Parameterization with Markov Parameter Specification

In preparation for the characterization of all q-covers, we here consider the following question. Suppose there is given the first q-Markov coefficients $M_0, M_1, \cdots, M_{q-1}$ of a qth degree stable transfer function. In a normalized Hessenberg state-variable realization of the transfer function, what parameters remain freely adjustable? The main result is the following:

Lemma 3.1: Consider the collection of all q-dimensional normalized Hessenberg realizations of qth degree stable transfer functions with prescribed Markov coefficients M_0, $M_1, \cdots, M_{q-1}$. Let ? denote a selectable parameter, and d denote a parameter determined by a combination of the data and the selectable parameter. Then the collection of such realizations is defined by

$$C=[? \quad 0 \quad \cdots \quad 0]$$

$$A=\begin{bmatrix} d & ? & 0 & \cdots & 0 & 0 \\ d & d & ? & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ d & d & \cdot & \cdots & d & ? \\ d & d & \cdot & \cdots & d & d \end{bmatrix} \qquad B=\begin{bmatrix} d \\ d \\ \cdot \\ \cdot \\ \cdot \\ d \\ d \end{bmatrix} \tag{3.1}$$

subject of course also to

$$A+A'=-BB'$$
$$c_1>0, \quad a_{i,i+1}>0, \qquad i=1,\cdots,q-1. \tag{3.2}$$

Remark 3.1: Note that the parameter count is correct. The class of qth degree strictly proper transfer functions can be parameterized by $2q$ parameters. When $M_0,\cdots,M_{q-1}$ are specified, q further parameters are left free, and these are taken up by the ? quantities in (3.1).

Proof: The proof is constructive. We must show that if c_1, $a_{i,i+1}$ take fixed positive values, and $M_0,\cdots,M_{q-1}$ are prescribed, all other nonzero entries of A and B can be computed.

From $M_0 = CB = c_1 b_1$, b_1 is obtained.

From $(A + A' + BB')_{11} = 0$, $a_{11} = -1/2b_1^2$ is obtained.

Since $CA = [c_1 a_{11} \;\; c_1 a_{12} \;\; 0 \;\; \cdots \;\; 0]$ which is now known, we have $M_1 = CAB = c_1 a_{11} b_1 + c_1 a_{12} b_2$ and obtain b_2. From $(A + A' + BB')_{i2} = 0$, for $i = 1, 2$ we have $a_{12} + a_{21} = -b_1 b_2$ and obtain a_{21} and $2a_{22} = -b_2^2$ and obtain a_{22}.

Now CA^2 is known, $M_2 = CA^2 B$ is used to determine b_3 and then the third row of A can be found. This process repeats.

Remark 3.2: A perusal of the above proof will show that an alternative assignment of selectable and determined parameters is provided by

$$C = [d \;\; 0 \;\; \cdots \;\; 0]$$

$$A = \begin{bmatrix} d & d & 0 & \cdots & 0 \\ d & d & d & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & d \\ d & d & d & \cdots & d \end{bmatrix} \qquad B = \begin{bmatrix} ? \\ ? \\ \cdot \\ \cdot \\ \cdot \\ ? \end{bmatrix}. \tag{3.3}$$

Remark 3.3: If fewer than q Markov parameters $M_0, \cdots, M_{r-1}$ are available where $r < q$, similar arguments show that the following parameter pattern is achievable:

$$C = [? \;\; 0 \;\; \cdots \;\; 0]$$

$$A = \begin{bmatrix} d & ? & 0 & \cdots & 0 \\ d & d & ? & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & d & ? \\ d & d & d & d & d \end{bmatrix} \qquad B = \begin{bmatrix} d \\ \cdot \\ \cdot \\ d \\ ? \\ \cdot \\ ? \end{bmatrix} \left.\vphantom{\begin{matrix} d \\ \cdot \\ \cdot \\ d \end{matrix}}\right\} r$$

Remark 3.4: If any two normalized Hessenberg realizations differ in any entry, the realizations necessarily correspond (by Lemma 2.2) to different transfer functions. It follows that all distinct choices of selectable parameters in (3.1) correspond to distinct transfer functions with the same Markov parameters.

Example. Consider a third order system in which $M_0 = 1$, $M_1 = -2$, $M_2 = -(1/2)$. Let us select $c_1 = \alpha$, $a_{12} = \beta$, $a_{23} = \gamma$. Thus

$$C = [\alpha \;\; 0 \;\; 0]$$

$$A = \begin{bmatrix} d & \beta & 0 \\ d & d & \gamma \\ d & d & d \end{bmatrix} \qquad B = \begin{bmatrix} d \\ d \\ d \end{bmatrix}$$

Because $M_0 = CB = 1$, $b_1 = \alpha^{-1}$. Then $a_{11} = -(1/2\alpha^2)$. Now $CA = [-(1/2\alpha), \alpha\beta, 0]$ and

$$M_1 = CAB = -2, \text{ or } -\frac{1}{2\alpha^2} + \alpha\beta b_2 = -2$$

whence

$$b_2 = -\frac{2}{\alpha\beta} + \frac{1}{2\alpha^3\beta}.$$

Then

$$a_{22} = -\frac{1}{2}b_2^2 = -\frac{1}{2\alpha^2\beta^2}\left[4 - \frac{2}{\alpha^2} + \frac{1}{4\alpha^4}\right].$$

Further $a_{12} + a_{21} = -b_1 b_2$ and so

$$a_{21} = \frac{2}{\alpha^2\beta} - \frac{1}{2\alpha^4\beta} - \beta.$$

Then

$$CA^2 = \left[\frac{1}{4\alpha^3} + \frac{2}{\alpha} - \frac{1}{2\alpha^3} - \alpha\beta^2, \right.$$
$$\left. -\frac{\beta}{2\alpha} - \frac{1}{2\alpha\beta}\left[4 - \frac{2}{\alpha^2} + \frac{1}{4\alpha^4}\right], \alpha\beta\gamma\right].$$

Now $M_2 = CA^2 B = -(1/2)$, which yields

$$b_3 = \frac{-1}{2\alpha\beta\gamma} + \frac{1}{2\alpha^5\beta\gamma} - \frac{3}{\alpha^3\beta\gamma} + \frac{\beta}{\alpha\gamma} - \frac{4}{\alpha^3\beta^3\gamma}$$
$$+ \frac{3}{\alpha^5\beta^3\gamma} - \frac{3}{4\alpha^7\beta^3\gamma} + \frac{1}{16\alpha^9\beta^3\gamma}$$

$a_{31} = -b_1 b_3$, $a_{32} = -b_2 b_3 - \gamma$ and $a_{33} = -(1/2)b_3^2$ all follow easily.

IV. Markov and Covariance Parameter Relations

In preparation for the parameterization in the next section of all qth-order q-Markov covers, we shall indicate here relationships which must exist between Markov and covariance parameters. Let $W(s)$ be a stable strictly proper transfer function of order q and $\{A, B, C\}$ any minimal realization. Recall that

$$M_i = CA^i B, \; R_i = CA^i XC' \tag{4.1}$$

where for unit intensity noise inputs,

$$XA' + AX = -BB'. \tag{4.2}$$

Lemma 4.1: With quantities as defined above, for any integer m these hold

$$2R_{2m+1} = -M_0 M'_{2m} + M_1 M'_{2m-1}$$
$$- M_2 M'_{2m-2} \cdots - M_{2m} M'_0. \tag{4.3}$$

Proof: Let $Z(s) \triangleq C(sI - A)^{-1} XC'$. Observe that

$$\begin{aligned} Z(s) + Z'(-s) &= C\left[(sI - A)^{-1}X + X(-sI - A')^{-1}\right]C' \\ &= C(sI - A)^{-1}[X(-sI - A') \\ &\qquad + (sI - A)X](-sI - A')^{-1}C' \\ &= C(sI - A)^{-1}BB'(-sI - A')^{-1}C' \\ &= W(s)W'(-s). \end{aligned} \tag{4.4}$$

Because

$$Z(s) = R_0 s^{-1} + R_1 s^{-2} + R_2 s^{-3} + \cdots$$
$$W(s) = M_0 s^{-1} + M_1 s^{-2} + \cdots$$

we recover (4.3) by inverting these expansions in (4.4) and equating even powers of s^{-1} on each side of the equation.

An important conclusion flows from this lemma. Suppose that $M_0, M_1, \cdots, M_{q-1}$ and $R_0, R_1, \cdots, R_{q-1}$ are specified for some system. Then this is *not* a collection of $2q$ independent pieces of data. According as $q = 2p$ or

$2p+1$, it is a collection of $3p$ or $3p+2$ independent pieces of data, since $R_1, R_3, \cdots, R_{2p-1}$ are not free.

This means in turn that if one is seeking a qth degree system with prescribed $M_0, M_1, \cdots, M_{q-1}$ and $R_0, R_1, \cdots, R_{q-1}$ one cannot expect the system to be uniquely defined by the data. Rather, according as $q=2p$ or $2p+1$, i.e., the system requires $4p$ or $4p+2$ parameters to specify it completely, there will remain $p=(4p-3p)$ or $[(4p+2)-(3p+2)]$ parameters which can be adjusted.

If one is seeking an rth degree system with prescribed $M_0, M_1, \cdots, M_{q-1}$ and $R_0, R_1, \cdots, R_{q-1}$, there will remain $2r-q-\lceil q/2 \rceil$ parameters which can be adjusted. Here, $\lceil x \rceil$ denotes the least integer greater than or equal to x; subsequently $\lfloor x \rfloor$ will denote the greatest integer less than or equal to x.

V. Parameterization with Markov and Covariance Parameter Specification

The question we address in this section is the following: How can we parameterize all qth degree stable transfer functions with the same $M_0, M_1, \cdots, M_{q-1}$ and $R_0, \cdots, R_{q-1}$? In the light of the preceding material, an equivalent problem is the parameterization of qth-order normalized Hessenberg realizations achieving the prescribed M_i, R_i and we shall expect free parameters to occur.

Main Theorem: Consider the collection of all q-dimensional normalized Hessenberg realizations of qth degree stable strictly proper transfer functions with prescribed Markov and covariance coefficients $M_0, M_1, \cdots, M_{q-1}$ and $R_0, R_1, \cdots, R_{q-1}$ [these quantities in fact being computable from any realization as described in (4.1) and (4.2)] Let ? denote a selectable parameter, x a parameter determined by the given data M_i, R_i and d a parameter determined by the parameters ?, x and the data. Then the collection is defined by

$$C=[x \quad 0 \quad \cdots \quad 0]$$

$$A=\begin{bmatrix} x & x & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x & x & \cdots & x & \cdot & \cdot & \cdots & \cdot \\ x & x & \cdots & x & ? & \cdot & \cdots & \cdot \\ d & d & \cdots & d & d & ? & \cdots & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & ? \\ d & d & \cdots & \cdot & \cdot & \cdot & \cdots & d \end{bmatrix} \qquad B=\begin{bmatrix} x \\ \vdots \\ \vdots \\ x \\ d \\ \vdots \\ d \end{bmatrix}$$

with the number of ? entries being $\lfloor q/2 \rfloor$. The only constraint on the selectable parameters ? is that the stable (A, B, C) be controllable and observable, and $a_{i,i+1}>0$ as required by the normalized Hessenberg form. Stability of A is guaranteed under these conditions.

Remark 5.1: The covariance data contains $\lceil q/2 \rceil$ independent pieces of information. Relative to the parameterization in Section III, the first $\lceil q/2 \rceil$ parameters in the list $c_1, a_{12}, a_{23}, \cdots$ become determined by the data; as a consequence, the first $\lceil q/2 \rceil$ rows of A and B (with the exception of the right most nonzero entry in the bottom of these rows of A) become determined. The remaining parameters in the list $c_1, a_{12}, a_{23}, \cdots$, which are $\lfloor q/2 \rfloor = q - \lceil q/2 \rceil$ in number, remain selectable.

The proof of the main result depends on the following lemma.

Lemma 5.1: Let $A+A'=-BB'$ and $i \geqslant j \geqslant 1$ integers. Then

$$A^i(A')^j=(-1)^j\Big[A^{i+j}+A^{i+j-1}BB'-A^{i+j-2}BB'A' + A^{i+j-3}BB'(A')^2 \cdots +(-1)^{j-1}A^iBB'(A')^{j-1}\Big]. \quad (5.2)$$

The lemma is proved by replacing each occurrence of BB' on the right side of (5.2) by $-(A+A')$. Pre- and post-multiplication of (5.2) by C and C', respectively, leads to

$$CA^i(A')^jC'=(-1)^j\Big[R_{i+j}+M_{i+j-1}M_0-M_{i+j-2}M_1 \cdots +(-1)^{j-1}M_iM_{j-1}\Big], \qquad i \geqslant j \geqslant 1.$$

This fact will be used below.

Proof of Main Theorem: Incorporating knowledge of the Markov parameters gives us realizations of the form of (3.1). Next, the data at our disposal ensures that we can calculate the matrix

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^p \end{bmatrix}\begin{bmatrix} C' & A'C' & \cdots & (A')^pC' \end{bmatrix}$$

where $q=2p+1$ and the matrix

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^p \end{bmatrix}\begin{bmatrix} C' & A'C' & \cdots & (A')^{p-1}C' \end{bmatrix}$$

where $q=2p$.

Forget for the moment the last row in the first of these two matrices. Then we are dealing with square matrices of dimension $\lceil q/2 \rceil \times \lceil q/2 \rceil$. Now the structure of C and A ensure that

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^r \end{bmatrix}$$

is lower triangular, with positive diagonal elements, for any $r \leqslant q$. In fact, the diagonal elements are c_1, $c_1a_{12}, c_1a_{12}a_{23}, \cdots$. Identify r with $\lceil q/2 \rceil$ and notice that the lower triangular matrix is computable as the square root of a known matrix, uniquely because of the positivity of the diagonal elements. Consequently c_1, c_1a_{12}, $c_1a_{12}a_{23}, \cdots$ (there being $\lceil q/2 \rceil$ terms) are known. In order, one can then find b_1, a_{11}, b_2, a_{21}, $a_{22}, b_3, a_{31}, a_{32}, a_{33}, \cdots$, and finally row $\lceil q/2 \rceil$ of A and B except for the last nonzero entry of the row of A.

Now above we observed that when $q=2p$, the matrix $CA^p[C'A'C'\cdots(A')^{p-1}C']$ was computable from the data at our disposal. This matrix was *not* yet used to identify parameters in A, B, C, individually, and the question arises as to whether we can learn more about A, B, C from this matrix. A direct calculation shows that every entry of this matrix is expressible in terms of those entries of A, B, C for which values have been determined by the process outlined in the preceding paragraph. This means that the matrix $CA^p[C'A'C'\cdots(A')^{p-1}C']$, contains no additional information to that contained in the matrices $CA^r[C'A'C'\cdots(A')^{p-1}C']$, $r=0,1,\cdots,p-1$. Finally, the controllability constraint guarantees stability of A since $X=I>0$ and controllability assures that A is the unique stable solution of $A+A'+BB'=0$. Observability is assured by making the superdiagonal elements of A positive. This is guaranteed by the normalized Hessenberg form existence Lemma 2.2. ∇∇∇

Remark: If (A, B) is not controllable, then the uncontrollable modes may be deleted while maintaining a match of all the data. This reduced A will be asymptotically stable.

Example: Find all third-order systems with $M_0=1$, $M_1=-2$, $M_2=-(1/2)$, $R_0=1$, $R_1=-(1/2)$, $R_2=-(1/2)$.

We know in advance that the structure of a normalized Hessenberg form will be

$$C=[x\quad 0\quad 0]$$

$$A=\begin{bmatrix} x & x & 0\\ x & x & \alpha\\ d & d & d\end{bmatrix}\qquad B=\begin{bmatrix} x\\ x\\ d\end{bmatrix}$$

where α is a selectable parameter, and all d values are determined by the data and α. We shall find entries sequentially. $R_0=CC'=1$ and $c_1>0$ gives $c_1=1$. $M_0=CB=c_1b_1$ gives $b_1=1$ whence $a_{11}=-(1/2)CAC'$ can be computed from what has been already found, and is $-(1/2)$, the value of R_1. Recall from (4.3) that $2R_1=-M_0^2$ and hence the value of R_1 is not independently selectable. Now $CA=[-(1/2)\quad a_{12}\quad 0]$ and $CAA'C'=-CA^2C'-CABB'C'=-R_2-M_0M_1=5/2$. Also, $CAA'C'=1/4+a_{12}^2$. So $a_{12}^2=9/4$ or $a_{12}=3/2$. Now $CAB=[-1/2\quad 3/2\quad 0][b_1\quad b_2\quad b_3]'=-1/2+3/2b_2=M_1$. So $b_2=-1$. Then $a_{22}=-1/2$. $a_{12}+a_{21}=-b_1b_2$, so $a_{21}=-1/2$. At this stage, we have

$$C=[1\quad 0\quad 0]$$

$$A=\begin{bmatrix} -\frac{1}{2} & \frac{3}{2} & 0\\ -\frac{1}{2} & -\frac{1}{2} & \alpha\\ d & d & d\end{bmatrix}\qquad B=\begin{bmatrix} 1\\ -1\\ d\end{bmatrix}$$

Now $CA^2=[-1/2\quad -3/2\quad 3\alpha/2]$ and $M_2=CA^2B=-1/2$. These results $b_3=-\alpha^{-1}$. Then $a_{31}=-b_1b_3=\alpha^{-1}$, $a_{32}=-b_2b_3-a_{23}=\alpha^{-1}-\alpha$, $a_{33}=-(1/2)\alpha^{-2}$. Finally then,

$$C=[1\quad 0\quad 0] \tag{5.4a}$$

$$A=\begin{bmatrix} -\frac{1}{2} & \frac{3}{2} & 0\\ -\frac{1}{2} & -\frac{1}{2} & \alpha\\ \alpha^{-1} & -\alpha^{-1}-\alpha & -\frac{1}{2}\alpha^{-2}\end{bmatrix}\qquad B=\begin{bmatrix} 1\\ -1\\ -\alpha^{-1}\end{bmatrix}. \tag{5.4b}$$

The associated family of transfer functions is

$$W(s,\alpha)=\frac{s^2+\left(\frac{1}{2}\alpha^{-2}-1\right)s+\left(\alpha^2-\frac{1}{2}-\frac{1}{2}\alpha^{-2}\right)}{s^3+\left(1+\frac{1}{2}\alpha^{-2}\right)s^2+\left(\alpha^2+2+\frac{1}{2}\alpha^{-2}\right)s+\left(\frac{1}{2}\alpha^2-1+\frac{1}{2}\alpha^{-2}\right)}$$

for all real values of α which preserve minimality of (A, B, C). There is an infinite number of such values. The only positive value of α which is excluded is $\alpha=1$, since this is the only positive real root of

$$\alpha^8+6\alpha^6-5\alpha^4-\frac{3}{2}\alpha^2-\frac{1}{2}=0.$$

(All these values cause loss of controllability.) All other finite values of α describe the entire class of third-order systems which match the data.

VI. Multivariable Results for Continuous-Time Systems

Multivariable cost-decoupled Hessenberg forms have been used in the literature [7]. They are structured as follows

$$C=[C_1\quad 0\quad\cdots\quad 0]$$

$$A=\begin{bmatrix} A_{11} & A_{12} & \cdot & \cdots & 0\\ A_{21} & A_{22} & A_{23} & \cdots & 0\\ \cdot & \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdot & \cdots & \cdot\\ A_{q-11} & \cdot & \cdot & \cdots & A_{q-1q}\\ A_{q1} & \cdot & \cdot & \cdots & A_{qq}\end{bmatrix}\qquad B=\begin{bmatrix} B_1\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ B_q\end{bmatrix} \tag{6.1}$$

$$A+A'=-BB'.$$

The block row and column dimensions in A are $n_1, n_2,\cdots,n_q$ where $n=\Sigma n_i$ is the McMillan degree of $C(sI-A)^{-1}B$ and

$$n_i=\operatorname{rank}\begin{bmatrix} C\\ CA\\ \vdots\\ CA^{i-1}\end{bmatrix}-\operatorname{rank}\begin{bmatrix} C\\ CA\\ \vdots\\ CA^{i-2}\end{bmatrix}. \tag{6.2}$$

Further, an argument based on minimality shows that

$$\operatorname{rank} C_1=n_1,\quad \operatorname{rank} A_{i,i+1}=n_{i+1}. \tag{6.3}$$

We obtain a *normalized* Hessenberg form CT, $T^{-1}AT$, TB by taking $T=\text{diag}\{T_1,T_2,\cdots,T_q\}$. Here, the T_i are $n_i\times n_i$ orthogonal matrices, chosen sequentially so that C_1T_1 forms a lower triangular matrix with the first nonzero entry in each column being positive, $C_1A_{12}T_2$ forms a lower triangular matrix with the first nonzero entry in each column being positive, and so on. Note that (6.3) together with the property $n_i\geqslant n_{i+1}$ (proved in [8]) ensures that $C_1A_{12}A_{23}\cdots A_{r-1,r}$ is an $n_1\times n_r$ matrix of rank n_r, so after transformation by T_r, every column has at least one nonzero entry. If $C_1A_{12}\cdots A_{r-1,r}$ is 4×2, the possible patterns after transformation are

$$\begin{bmatrix}+&0\\x&+\\x&x\\x&x\end{bmatrix},\begin{bmatrix}+&0\\x&0\\x&+\\x&x\end{bmatrix},\begin{bmatrix}+&0\\x&0\\x&0\\x&+\end{bmatrix},\begin{bmatrix}0&0\\+&0\\x&+\\x&x\end{bmatrix},$$

$$\begin{bmatrix}0&0\\+&0\\x&0\\x&+\end{bmatrix},\begin{bmatrix}0&0\\0&0\\+&0\\x&+\end{bmatrix}$$

where $+$ denotes a positive number and x denotes unconstrained values. This procedure is easily seen to be equivalent for $r>1$ to selecting T_r so that $T'_{r-1}A_{r-1,r}T_r$ is lower triangular, with the first nonzero entry of each column being positive; notice that $T'_{r-1}A_{r-1,r}T_r$ is the transformed $A_{r-1,r}$.

Constructability of the normalized Hessenberg form is straightforward. Then uniqueness is established by a variation on the proof of Lemma 2.2.

Lemma 6.1: Let $W(s)$ be a strictly proper stable transfer function matrix. From an arbitrary minimal state-variable realization of $W(s)$, one can construct a realization in normalized Hessenberg form which is unique.

Parameterization with Markov parameter specification involves a variation in proceeding from the scalar to the matrix case. Of course, we are interested in the case where q-Markov parameters are given where q is not (normally) the degree of $W(s)$, but now its observability index.

Lemma 6.2: Consider the collection of normalized Hessenberg realizations of nth degree stable transfer function matrices with block sizes in the A matrix defined by integers $n_1,n_2,\cdots,n_q$, and with prescribed Markov coefficients $M_0,M_1,\cdots,M_{q-1}$. Selectable parameters are the nonzero entries of C_1, $A_{i,i+1}$ and the skew part of the matrices A_{ii}, $i=1,\cdots,q$, subject only to the observability, controllability constraints on the selectable parameters.

Proof: This follows largely as for the scalar case. From $M_0=CB=C_1B_1$ and rank $C_1=n_1=$ number of columns of C_1, B_1 is obtained. From $(A+A'+BB')_{11}=0$, the symmetric part of A_{11} is obtained. The skew part being assumed selectable, all of A_{11} is thus known. Then $M_1=CAB=C_1A_{11}B_1+C_1A_{12}B_2$. Now C_1A_{12} has full column rank and B_2 is thus determined, etc.

Relations between Markov and covariance parameters are derivable following the argument in Section IV. With $W(s)=C(sI-A)^{-1}B$,

$$M_i=CA^iB \quad R_i=CA^iXC' \quad XA'+AX=-BB' \tag{6.4}$$

one finds

$$R_{2m+1}+R'_{2m+1}=-M_0M'_{2m}+M_1M'_{2m-1}-M_2M'_{2m-2}\cdots-M_{2m}M'_0 \tag{6.5a}$$

and

$$R_{2m}-R'_{2m}=M_0M'_{2m-1}-M_1M'_{2m-2}+M_2M'_{2m-3}\cdots-M_{2m-1}M'_0. \tag{6.5b}$$

Lemma 5.1 was used to express these quantities $CA^i(A'^j)C'$ for $i\geqslant j\geqslant 1$ in terms of Covariance and Markov parameters. In the matrix case, one has

$$CA^i(A')^jC'=(-1)^j\Big[R_{i+j}+M_{i+j-1}M'_0-M_{i+j-2}M'_1\cdots+(-1)^{j-1}M_iM'_{j-1}\Big]. \tag{6.6}$$

Now let us discuss how knowledge of covariance parameters as well as Markov parameters determines some parameters that were previously free in a normalized Hessenberg representation.

$R_0=CC'=C_1C'_1$ with C_1 possessing lower triangular structure and first nonzero element in each column positive uniquely identifies C_1. Then $M_0=C_1B_1$ gives B_1 as before. The symmetric part of A_{11} comes as before.

$R_1=CAC'=C_1A_{11}C'_1$ and the properties of C_1 then allow identification of A_{11}. Note that (6.5a) shows that the symmetric part of R_1 is already determined by Markov parameter data, a fact consistent with our determination of the symmetric part of A_{11} prior to working knowledge of R_1.

Now $CA=[C_1A_{11},C_1A_{12},0\cdots0]$ and $CAA'C'$ is known from R_2, M_0, M_1, by (6.6). Hence $C_1A_{12}A'_{12}C'_1$ is known and the structure of C_1A_{12} means that it too is known. From it, A_{12} follows.

Now $M_1=CAB=C_1A_{11}B_1+C_1A_{12}B_2$ and B_2 is determined, and so on.

The end result of these arguments is the following generalization of the Main Results of Section V.

Main Theorem (Matrix Case): Consider the collection of all normalized Hessenberg realizations (6.1) of those n-th degree stable, strictly proper transfer function matrices with prescribed observability degree q, block sizes n_1, $n_2,\cdots,n_q$ and Markov and covariance coefficients M_0, $M_1,\cdots,M_{q-1}$ and R_0, $R_1,\cdots,R_{q-1}$. Let $p=[q/2]$. Then C_1, and all matrices in the first p block row of A and B are determined except for $A_{p,p+1}$ and the skew part of A_{pp}. The skew part of A_{rr}, $r>p$ and $A_{r,r+1}$ may be selected, and all other entries of A and B are thereby determined via the data and these selections; in the selection of $A_{r,r+1}$, the controllability, observability and structural constraints (from the normalized Hessenberg definition) must be satisfied. Stability of A is guaranteed if and only if (A, B) is controllable.

Suppose in our algorithm (which realizes (A, B, C) from the data M_i, R_i $i=0, 1, \cdots, q-1$) that (A, B) is not controllable. Then the uncontrollable mode(s) is on the $j\omega$ axis and its deletion will give a reduced (A, B, C) which is asymptotically stable and matches all the data. (See [7] or [3] for proof that $X>0$, yields no right half plane eigenvalues for A).

VII. Discrete-Time q-Markov Covers

In this section, we shall examine the structure of discrete-time q-Markov covers of degree q (scalar case) and observability index q (matrix case). The results differ nontrivially from the continuous-time problem. For example, for scalar systems, the number of q covers of degree q is normally just two. The basic reason for this is that, in contrast to the continuous-time case, the covariance parameters are not in part determined by the Markov parameters. Now the $2q$ parameters (Markov and covariance) contain more information (and so allow less freedom in the $\{A, B, C\}$ tuple) in discrete time as opposed to continuous time.

The concepts of Hessenberg form, cost-decoupled Hessenberg form and normalized Hessenberg form carry over just as for continuous time, save that the covariance equation is

$$I = AA' + BB'. \tag{7.1}$$

There is a unique realization in normalized Hessenberg form of a prescribed transfer function matrix.

Now let us observe:

Lemma 7.1: With $\{A, B, C\}$ defining a normalized Hessenberg state variable realization of a stable discrete-time system there holds

$$A^iA'^j = A^{i-j} - \sum_{\alpha=1}^{j} A^{i-\alpha}BB'(A')^{j-\alpha}, \qquad 1 \geqslant j \geqslant 0 \tag{7.2}$$

and if

$$R_i = CA^iC' \quad M_i = CA^iB \tag{7.3}$$

$$CA^iA'^jC' = R_{i-j} - \sum_{\alpha=1}^{j} M_{i-\alpha}M'_{j-\alpha}, \qquad i \geqslant j \geqslant 0. \tag{7.4}$$

Proof: Observe that

$$A^iA'^j = A^{i-1}(I - BB')(A')^{j-1}$$
$$= A^{i-1}(A')^{j-1} - A^{i-1}BB'(A')^{j-1}$$

and repeatedly use $AA' = I - BB'$ until (7.2) follows. Equation (7.4) follows through premultiplication and postmultiplication of (7.2) by C and C', respectively.

Define

$$\Theta_q \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-1} \end{bmatrix}$$

$$\mathscr{R}_q \triangleq \begin{bmatrix} R_0 & R_1^* & R_2^* & \cdot & \cdot \\ R_1 & R_0 & R_1^* & \cdot & \cdot \\ R_2 & R_1 & R_0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ R_{q-1} & \vdots & R_2 & R_1 & R_0 \end{bmatrix}$$

$$\mathscr{M}_q = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & M_0 & 0 & \cdots & 0 \\ 0 & M_1 & M_0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & 0 \\ 0 & M_{q-2} & \cdot & M_1 & M_0 \end{bmatrix}$$

Then,

Corollary: For the normalized Hessenberg form $\{A, B, C\}$ the following relationship between Markov parameters M_i and covariances R_i and observability matrix Θ_q holds

$$\Theta_q\Theta_q^* = \mathscr{R}_q - \mathscr{M}_q\mathscr{M}'_q \tag{7.5}$$

and hence the Toeplitz matrix R_q obeys the inequality $\mathscr{R}_q \geqslant \mathscr{M}_q\mathscr{M}'_q$.

Proof of Corollary: The ij block element of (7.5) may readily be expanded to yield (7.4). This construction verifies (7.5).

Now suppose that we have available $M_0, \cdots, M_{q-1}$ and $R_0, \cdots, R_{q-1}$ for a qth degree system (scalar case) or system of observability index q (matrix case). Then the matrix

$$\Theta_q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-1} \end{bmatrix} \tag{7.6}$$

has full column rank (by minimality) and (7.4) shows that we know $\Theta_q\Theta_q'$. Further, the assumed structure guarantees that Θ_q is lower triangular, and the first nonzero entry in each column is positive. Hence Θ_q is uniquely determined by $\Theta_q\Theta_q'$.

Observe next that

$$\Theta_q B = \begin{bmatrix} M_0 \\ \vdots \\ M_{q-1} \end{bmatrix} \triangleq m_q \tag{7.7}$$

and with Θ_q of full column rank, it follows that B is at once known by $B = \Theta_q^+ m_q$, where Θ_q^+ is the left inverse of Θ_q.

Next, let us observe that the first $q-1$ block rows of A are determined. We have

$$C=[C_1 \quad 0 \quad \cdots \quad 0]$$

$$A=\begin{bmatrix} A_{11} & A_{12} & 0 & \cdots & 0 \\ A_{21} & A_{22} & A_{23} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ A_{q-11} & \cdot & \cdot & \cdots & A_{q-1q} \\ A_{q1} & \cdot & \cdot & \cdots & A_{qq} \end{bmatrix}.$$

The first row of Θ_q yields C_1. The second row yields $[C_1A_{11}, C_1A_{12}, 0\cdots 0]$ and because C_1 has full column rank, A_{11} and A_{12} are known. Next,

$$CA^2=(CA)A=C_1A_{11}[A_{11} \quad A_{12} \quad 0\cdots 0] + C_1A_{12}[A_{21} \quad A_{22} \quad A_{23}\cdots 0]$$

from which the second block row of A follows. We can continue in this way up to utilization of CA^{q-1} to determine $A_{q-11}\cdots A_{q-1q}$.

It remains to determine the last row of A, and it is here (but only here) that choice enters the picture.

Let us write

$$A=\begin{bmatrix} \alpha & T \\ X & Y \end{bmatrix}$$

$$B=\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

where α has 1 block column and $(q-1)$ block rows, X has 1 block column and row, etc. Notice that T is lower triangular, with $(q-1)$ block rows and columns. The diagonal entries of T are $A_{12}, A_{23}, \cdots$ and because each of these has full column rank, T has a left inverse. Call it S.

Now use the covariance equation (7.1) to obtain

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}=\begin{bmatrix} \alpha\alpha'+TT' & \alpha X'+TY' \\ X\alpha'+YT' & XX'+YY' \end{bmatrix}+\begin{bmatrix} \beta_1\beta_1' & \beta_1\beta_2' \\ \beta_2\beta_1' & \beta_2\beta_2' \end{bmatrix}$$

whence

$$\alpha X'+TY'+\beta_1\beta_2'=0 \quad (7.8a)$$

$$XX'+YY'+\beta_2\beta_2'=I. \quad (7.8b)$$

We shall eliminate Y from these equations and solve for X. From (7.8a),

$$Y'=-S\alpha X'-S\beta_1\beta_2'. \quad (7.9)$$

So in (7.8b) we have

$$X[I+\alpha'S'S\alpha]X'+\beta_2\beta_1'S'S\alpha X'+X\alpha'S'S\beta_1\beta_2' + \beta_2(I+\beta_1'S'S\beta_1)\beta_2'-I=0.$$

Write this as

$$XMX'+NX'+XN'=P. \quad (7.9)$$

In case we are dealing with a scalar system, X is a scalar, and (7.8) is just a quadratic equation, yielding in general two solutions. Equation (7.8) then determines Y in terms of X and so A is fully determined (to within one of two possibilities). We proceed as follows when X is a matrix.

With $M^{1/2}$ the positive definite symmetric square root of M, set

$$\bar{X}\triangleq XM^{1/2} \qquad \bar{N}'\triangleq M^{-1/2}N'. \quad (7.10)$$

Then (7.9) yields

$$\bar{X}\bar{X}'-\bar{N}\bar{X}'+\bar{X}\bar{N}'=P$$

or

$$(\bar{X}+\bar{N})(\bar{X}'+\bar{N}')=P+\bar{N}\bar{N}'$$

or

$$(\bar{X}+\bar{N})=(P+\bar{N}\bar{N}')^{1/2}V \quad (7.11)$$

where V is any matrix with the same dimension as $\bar{X}$ and such that

$$VV'=I. \quad (7.12)$$

This main result allows the construction of the following algorithm for generating q-Markov covers of discrete-time systems.

The Discrete q-Markov Cover Algorithm:

Step I: Given the data $\{M_0, M_1, M_2, \cdots, M_{q-1}\}$ $\{R_0, R_1, R_2, \cdots, R_{q-1}\}$, construct the data matrix (from definitions in (7.5))

$$D\triangleq \mathscr{R}_q-\mathscr{M}_{q-1}\mathscr{M}'_{q-1},$$

$$(\text{block } D_{ji} \text{ has dim } n_1\times n_1). \quad (7.13)$$

Step II: Compute the $n_1\times n_i$ matrix (start with $i=1$, and ignore matrices with zero or negative subscripts)

$$\Theta_{ii}=\left[D_{ii}-\sum_{k=1}^{i-1}\Theta_{ik}\Theta_{ik}'\right]_{LTF} \quad (7.14)$$

where $[\cdot]_{LTF}$ means lower triangular factor with the first nonzero entry in each column positive. Θ_{ii} has rank n_i with linearly independent columns.

Step III: Compute the ith block column of Θ_q by computing the $n_1\times n_i$ matrices

$$\Theta_{ji}=\left[D_{ji}-\sum_{\alpha=1}^{i-1}\Theta_{j\alpha}\Theta_{i\alpha}'\right]\Theta_{ii}^{+\prime} \quad (7.15)$$

where Θ_{ii}^+ denotes the left inverse of Θ_{ii}, $\Theta_{ii}^+\Theta_{ii}=\mathrm{I}$. Set $i=i+1$ return to Step II. Repeat Step's II and III until and including $i=q-1$, and the process concludes with Step II with $i=q$. The matrix Θ_q is now determined.

Step IV: Define $C\triangleq[\Theta_{11} \quad 0]$. Set $j=1$ to continue.

Step V: Compute (ignore matrices with zero or negative subscripts)

$$A_{jk}=\Theta_{jj}^+\left[\Theta_{j+1,k}-\sum_{\alpha=1}^{j-1}\Theta_{j\alpha}A_{\alpha k}\right]. \quad (7.16)$$

Repeat for $k=1,2,\cdots,j+1$. ($A_{\alpha k}=0$ for $k>\alpha+1$). Set $j=j+1$ and return to Step V. Repeat until and including $j=q-1$. This generates all but the last block of rows of A.

Step VI: Compute B from

$$\Theta_q B = \begin{bmatrix} M_0 \\ \vdots \\ M_{q-1} \end{bmatrix} = m_q,\ B = \Theta_q^+ m_q. \tag{7.17}$$

Define (note that Λ_2 has linearly independent cols.)

$$\Lambda_1 \triangleq \begin{bmatrix} A_{11} \\ \vdots \\ A_{q-1,1} \end{bmatrix}, \Lambda_2 \triangleq \begin{bmatrix} A_{12} & \cdots & 0 \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ A_{q-1,2} & \cdots & A_{q-1,q} \end{bmatrix}, B_1 \triangleq \begin{bmatrix} B_{11} \\ \vdots \\ B_{q-1,1} \end{bmatrix}$$

$$\Lambda_3 \triangleq [A_{q1}],\ \Lambda_4 \triangleq [A_{q2} \cdots A_{qq}],\ B_2 \triangleq B_{q1}$$
$$\Lambda_2^+ \triangleq \text{left inverse of } \Lambda_2$$
$$N_3 \triangleq [I + \Lambda_1' \Lambda_2^{+\prime} \Lambda_2^+ \Lambda_1]^{1/2}$$
$$N_2 \triangleq \Lambda_1' \Lambda_2^{+\prime} \Lambda_2^+ B_1 B_2'$$
$$N_1 \triangleq I - B_2 [I + B_1' \Lambda_2^{+\prime} \Lambda_2^+ B_1] B_2'$$
$$N_0 \triangleq N_2' [N_3 N_3']^{-1} N_2.$$

Step VII: Compute final A elements

$$\Lambda_3 = \left[[N_0 + N_1]^{1/2} V - N_2' N_3'^{-1} \right] N_3^{-1} \tag{7.18a}$$
$$\Lambda_4 = -[B_2 B_1' + \Lambda_3 \Lambda_1'] \Lambda_2^{+\prime} \tag{7.18b}$$

where V has the dimensions of Λ_3 and is arbitrary subject to $VV' = I$. This completes the computations of (A, B, C) from the data $\{M_0, \cdots, M_{q-1}, R_0, \cdots, R_{q-1}\}$. The resulting q-Markov cover is

$$A = \begin{bmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_3(V) & \Lambda_4(V) \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{7.19}$$
$$C = [\Theta_{11} \quad 0]$$
$$X = I.$$

Example: Find the class of all third-order discrete-time systems having Markov parameters

$$\{M_0, M_1, M_2\} = \left\{3, \frac{1}{4}, \frac{9}{16}\right\}$$

and covariances

$$\{R_0, R_1, R_2\} = \{9.3968, 0.9016, 1.7683\}.$$

The answer is provided by the above algorithm as any coordinate transformation of

$$C = [3.0654 \quad 0 \quad 0]$$

$$A_1 = \begin{bmatrix} 0.0959 & 0.1817 & 0 \\ 0.9848 & -0.1542 & 0.0414 \\ 0.1193 & 0.9258 & 0.3082 \end{bmatrix}, B = \begin{bmatrix} 0.9787 \\ -0.0679 \\ -0.1836 \end{bmatrix}$$

or with the same B, C and a second choice for A,

$$A_2 = \begin{bmatrix} 0.0959 & 0.1817 & 0 \\ 0.9848 & -0.1542 & 0.0414 \\ 0.1443 & 0.9126 & -0.3358 \end{bmatrix}.$$

Both of these choices are controllable and observable with identity state covariance $X = I$. Recall that the freedom in Step VII of the algorithm is $V = \pm 1$ since $A_{q1} = A_{31}$ is a scalar in this example. The transfer functions of the two 3-Markov covers are

$$C(zI - A_1)^{-1} B = \frac{3z^2 - 0.5z - 0.25}{z^3 - 0.25z^2 - 0.25z + 0.0625}$$

$$C(zI - A_2)^{-1} B = \frac{3z^2 + 1.4319z + 0.0251}{z^3 + 0.3940z^2 - 0.212z - 0.0625}.$$

Suppose that M_0, M_1, M_2 and R_0, R_1, R_2 are quantities associated with a certain third-order (rather than higher order) system. Then this system must have transfer function equal to one of the above two transfer functions, and of the two transfer functions above, one (and only one) will obviously match *all* Markov and covariance parameters. Indeed, to generate the data, we used the system

$$C_0 = [1 \quad 1 \quad 1]$$

$$A_0 = \begin{bmatrix} -1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}, B_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$X_0 = \begin{bmatrix} \frac{4}{3} & \frac{4}{5} & \frac{8}{9} \\ \frac{4}{5} & \frac{4}{3} & \frac{8}{7} \\ \frac{8}{9} & \frac{8}{7} & \frac{16}{15} \end{bmatrix}$$

and in the above example

$$C(sI - A_1)^{-1} B = C_0 (sI - A_0)^{-1} B_0.$$

VIII. Concluding Remarks

There may be many motivations for model reduction. A common one is to take a complex controller design (obtained for example by linear-quadratic methods) and simplify it, in order to more easily realize it. There is no single optimality criterion, L_2, L_1, Hankel norm, which reflects the many different and competing objectives of controller design, which may include transient performance, robustness in the face of structured and unstructured plant parameter variations and disturbance and noise suppression. It follows that it is helpful to designers to have available a collection of tools for controller simplification, since no one tool is likely to be universally preferable. Methods based on q-Markov covers have *a priori* appeal on the grounds that they may accurately retain steady state noise properties (as noted above) as well as some transient properties (through Markov coefficient matching).

In this paper, we have solved the problem of characterizing all qth degree systems which have the same first q-Markov and covariance parameters. By coupling this result with those of [7], which explain how to construct a qth degree q-Markov cover given a high-order system, we have a parameterization of all the q-th degree systems which are q-Markov covers of a high order system. The stability of these reduced order models is also established. The algorithm directly yields a stable A if and only if (A, B) is controllable. If it is not controllable a reduction in order, *deleting* the uncontrollable mode will yield a stable reduced order model that still matches the $2q$ pieces of data.

For single input/output there is an *infinite* number of continuous-time realizations of order q which match the first q-Markov and covariance parameters, whereas there are only *two* such systems for discrete time.

There are additional problems which might be considered. For example, could one show for the continuous-time case that knowledge of further Markov and/or covariance data would determine in a simple, perhaps sequential way, some of the free parameters in the collection of q-Markov covers of qth degrees? Connections with this question and the lattice filters, Levinson algorithm, and Hessenberg form should be investigated. Second, one could examine the parameterization of all weighted q-Markov covers. To explain what is meant, let $v(s)$ be a scalar stable system which is fixed (the weighting system). We have to parameterise all $w(s)$ such that all transfer functions $w(s)v(s)$ have the same $M_0, \cdots, M_{q-1}$ and $R_0, \cdots, R_{q-1}$. Such notions would be useful in controller reduction since the input to the controller is not white noise.

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