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RELATIONS BETWEEN FREQUENCY-DEPENDENT CONTROL AND STATE WEIGHTING IN LQG PROBLEMS

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SUMMARY

This paper considers the relation between controller designs for the same linear system achieved with two different quadratic performance indices. These indices differ only to the extent that one has frequency weighting on the control whereas the other has the inverse frequency weighting on the state. The feedback controllers are constructed by augmenting the given plant and by solving a modified LQG problem for the augmented system. The main result is that whereas the optimal controls as time functions can be the same, the optimal controllers are different. Filtering results are also obtained.

KEY WORDS Linear–quadratic controller design Frequency-shaped cost functionals State and control frequency weighting Robust design Optimal filtering

1. INTRODUCTION

Linear–quadratic controller design for time-invariant linear systems of the type

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{1}$$

has recently evolved from using performance indices of the form

$$V[\mathbf{x}_0, \mathbf{u}(\cdot)] = \int_0^\infty [\mathbf{u}^T\mathbf{R}\mathbf{u} + \mathbf{x}^T\mathbf{Q}\mathbf{x}]\ dt \tag{2}$$

(with the usual conditions on **A**, **B**, **Q** and **R** to guarantee existence of a unique, constant, stabilizing feedback control law) to the use of indices of the form

$$V[\mathbf{x}_0, \mathbf{u}(\cdot)] = \int_0^\infty [\mathbf{u}^T\mathbf{R}(j\omega)\mathbf{u} + \mathbf{x}^T\mathbf{Q}(j\omega)\mathbf{x}]\ dt \tag{3}$$

These developments are discussed, for example, in References 1–6. Different methods may be used to develop controllers with frequency weighting. For example, Reference 4 uses an approach based on augmentation of the given plant and solution of a modified time-domain problem and Reference 6 uses an approach based on spectral factorization and solution of a frequency-domain problem. The present paper employs the method of References 4 and 5 that

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is referred to here as the *augmented equations* approach. If **R** is constant in equation (3) and only $\mathbf{Q}(j\omega)$ is frequency dependent, the augmented equations approach and the *spectral factorization* approach of Reference 6 produce the same controllers. However, if $\mathbf{R}(j\omega)$ is frequency dependent, the controllers will differ. Hence the results of the present work apply only to controllers developed by using the augmented equations approach except in the case where **R** is constant.

If we apply the augmented equations approach, restrictions are imposed on **R** and **Q**: **R** must be rational and uniformly positive definite Hermitian for all real ω, and **Q** must be rational, bounded and non-negative definite Hermitian for all real ω. (The notation which mixes frequency-domain and time-domain quantities is to be criticized, but it is suggestive.)

Suppose now that the plant for which a control design is being made has associated with it high-frequency uncertainty. Then one can argue that by choosing $\mathbf{R}(j\omega)$ to be bigger for large ω than for small ω, one can penalize high-frequency controls more than low-frequency controls. In this way, one can reduce the loop gain at high frequencies and thus accommodate the high-frequency uncertainties. Equally, one might argue that by leaving **R** constant, one could achieve much the same goal by choosing $\mathbf{Q}(j\omega)$ to be smaller for large ω than small ω. This will mean that high-frequency components of the state will be more readily tolerated and thus less subject to corrective control action than the low-frequency components. Again, one would expect that the loop gain at high frequencies would be reduced.

This thinking at once gives rise to the question; is it better to put extra high-frequency weighting on $\mathbf{R}(j\omega)$ or reduced high-frequency weighting on $\mathbf{Q}(j\omega)$? This paper considers this question and in fact the more precise question of what is the difference in using the two performance indices for the same system (1)

$$V_1[\mathbf{x}_0, \mathbf{u}(\cdot)] = \int_0^\infty \left[\left| \frac{\alpha(j\omega)}{\beta(j\omega)} \right|^2 \mathbf{u}^T\mathbf{R}\mathbf{u} + \mathbf{x}^T\mathbf{Q}\mathbf{x} \right] \mathrm{d}t \tag{4}$$

$$V_2[\mathbf{x}_0, \mathbf{u}(\cdot)] = \int_0^\infty \left[\mathbf{u}^T\mathbf{R}\mathbf{u} + \left| \frac{\beta(j\omega)}{\alpha(j\omega)} \right|^2 \mathbf{x}^T\mathbf{Q}\mathbf{x} \right] \mathrm{d}t \tag{5}$$

As it turns out, the two indices yield the same closed-loop poles and the same optimal control expressed as a time function (assuming a matching of initial conditions in the now dynamic controller), but they yield different control laws when these are obtained via an intuitively reasonable augmentation procedure. These points are studied in Sections 2 and 3 of the paper, and various additional points are given in Section 4.

Related questions are considered in Sections 5 and 6 for filtering problems. There we find that if two filtering problems differ only in that one has coloured input noise and the other coloured measurement noise with the inverse spectrum, the two filters have the same poles (but are otherwise different). We are unable however to establish any equality between two time-domain quantities associated with the two filters.

2. EXAMPLE

Consider for the system $\dot{x} = u$ the minimization of

$$V_1[x(0), u(\cdot)] = \int_0^\infty \left[x^2 + \left| \frac{1 + \beta j\omega}{1 + \alpha j\omega} \right|^2 u^2 \right] \mathrm{d}t \tag{6}$$

When $\beta > \alpha$, this has the effect of introducing the classical notion of a lag compensation into the linear–quadratic problem; a key effect is to increase the high-frequency roll-off of the closed-loop system.[7] We should clarify two elements of imprecision about (6). First, the mixture of frequency-domain and time-domain notation, although now semistandard, means that

$$\alpha\dot{v} + v = \beta\dot{u} + u \tag{7}$$

and the integrand in (6) should be interpreted as $x^2 + v^2$. We can think in fact of the underlying system equations as

$$\begin{aligned} \dot{z} + \frac{1}{\beta} z &= v \\ \dot{x} &= \left(\frac{1}{\beta} - \frac{\alpha}{\beta^2}\right) z + \frac{\alpha}{\beta} v \end{aligned} \tag{8}$$

[Notice that $u = (\beta^{-1} - \alpha\beta^{-2})z + \alpha\beta^{-1}v$].

The second imprecise aspect of (6) is that the left side of (6) fails to display dependence on the initial condition associated with the frequency-dependent operation in the integrand. When (8) is viewed as the defining equation, (6) becomes dependent on $x(0)$, $z(0)$ and $v(\cdot)$ or, equivalently, $x(0)$, $z(0)$ and $u(\cdot)$.

The optimal control law can be computed as

$$v = -[\sqrt{(\alpha^2\beta^{-2} + \beta^{-2} + 2\beta^{-1})} - \beta^{-1} - \alpha\beta^{-1} \;\; 1] \begin{bmatrix} z \\ x \end{bmatrix} \tag{9}$$

(see Figure 1). Block diagram manipulation yields, with

$$\gamma \triangleq \sqrt{(\alpha^2\beta^{-2} + \beta^{-2} + 2\beta^{-1})}$$

and, in Laplace transform notation,

$$U_1^*(s) = \frac{-\beta^{-1}(\alpha s + 1)}{s + \gamma - \alpha\beta^{-1}} X(s) + \frac{\alpha^2\beta^{-2} + \beta^{-1} - \alpha\beta^{-1}\gamma}{s + \gamma - \alpha\beta^{-1}} z(0) \tag{10}$$

Next, consider, again for $\dot{x} = u$, the minimization of

$$V_2[x(0), u(\cdot)] = \int_0^\infty \left[\left| \frac{1 + \alpha j\omega}{1 + \beta j\omega} \right|^2 x^2 + u^2 \right] \mathrm{d}t \tag{11}$$

Define w by

$$\dot{w} + \beta^{-1} w = x \tag{12}$$

Then

$$V_2[x(0), u(\cdot)] = \int_0^\infty \{ [(\beta^{-1} - \alpha\beta^{-2})w + \alpha\beta^{-1}x]^2 + u^2 \} \, \mathrm{d}t$$

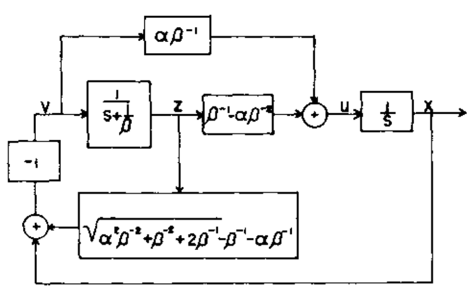

Figure 1. Controller structure with frequency weighting of input

(Of course, V_2 depends also on $w(0)$.) The optimal control is

$$u_2^* = -(\gamma - \beta^{-1})x - (\beta^{-1} + \beta^{-2} - \beta^{-1}\gamma)w \tag{13}$$

and so

$$U_2^*(s) = -(\gamma - \beta^{-1})X(s) - \frac{(\beta^{-1} + \beta^{-2} - \beta^{-1}\gamma)}{s + \beta^{-1}} X(s) - \frac{(\beta^{-1} + \beta^{-2} - \beta^{-1}\gamma)}{s + \beta^{-2}} w(0)$$

$$= \frac{-(\gamma - \beta^{-1})s - \beta^{-1}}{s + \beta^{-1}} X(s) - \frac{(\beta^{-1} + \beta^{-2} - \beta^{-1}\gamma)}{s + \beta^{-1}} w(0) \tag{14}$$

The controllers (10) and (14) are clearly quite different. The first controller has as a zero the stable pole of the frequency-domain function weighting u^2 in (6), and the second controller has as a pole the stable pole of the frequency-domain function weighting x^2 in (11).[4]

We now assert that if $w(0)$ and $z(0)$ are appropriately related, the actual time trajectories for u_1^* and u_2^* are the same. We use the fact that

$$sX(s) - x(0) = U(s)$$

$$X(s) = \frac{U(s)}{s} + \frac{x(0)}{s}$$

Using this result in (10) yields

$$[s^2 + \gamma s + \beta^{-1}]\, U_1^*(s) = -\beta^{-1}(\alpha s + 1)x(0) + s(\alpha^2\beta^{-2} + \beta^{-1} - \alpha\beta^{-1}\gamma)z(0)$$

Using it in (14) yields

$$[s^2 + \gamma s + \beta^{-1}]\, U_2^*(s) = [-(\gamma - \beta^{-1})s - \beta^{-1}]\, x(0) - s(\beta^{-1} + \beta^{-2} - \beta^{-1}\gamma)w(0)$$

Observe then that

$$U_1^*(s) \equiv U_2^*(s) \tag{15}$$

if and only if

$$-(\gamma - \beta^{-1})x(0) - (\beta^{-1} + \beta^{-2} - \beta^{-1}\gamma)w(0) = -\beta^{-1}\alpha x(0) + (\alpha^2\beta^{-2} + \beta^{-1} - \alpha\beta^{-1}\gamma)z(0) \tag{16}$$

To the extent that linear optimal control design is simply a vehicle for deriving a time-invariant feedback controller, the problems define the two different feedback controllers

$$U_1^*(s) = \frac{-\beta^{-1}(\alpha s + 1)}{s + (\gamma - \alpha\beta^{-1})} X(s) \tag{17}$$

and

$$U_2^*(s) = \frac{-\beta^{-1}[(\beta\gamma - 1)s + 1]}{s + \beta^{-1}} X(s) \tag{18}$$

despite the equality of $U_1^*(t)$ and $U_2^*(t)$. It is easily checked that the closed-loop characteristic polynomial with either controller is $s^2 + \gamma s + \beta^{-1}$. Likewise, (17) and (18) can be considered to be alternative controllers in the sense that with the same $x(0)$, the controllers yield $u_1^*(t) = u_2^*(t)$ provided that the initial conditions in the controllers are related.

Of course, the robustness properties, which bear on the effect of plant uncertainty, are different for the two controllers. Some discussion of this point appears in Reference 7, which shows that high-frequency robustness properties are most easily affected by putting frequency dependence in **R** rather than **Q**.

In the next section, we show that the conclusion in this example is a general one.

3. GENERAL ANALYSIS

We shall prove the following main result.

Theorem

Consider the system

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}\mathbf{u}, \ \mathbf{y} = \mathbf{H}^{\mathrm{T}}\mathbf{x} \tag{19}$$

and the two performance indices

$$V_1 = \int_0^\infty \left[\left| \frac{\beta(j\omega)}{\alpha(j\omega)} \right|^2 \mathbf{u}^{\mathrm{T}}\mathbf{u} + \mathbf{y}^{\mathrm{T}}\mathbf{y} \right] \mathrm{d}t \tag{20}$$

$$V_2 = \int_0^\infty \left[\mathbf{u}^{\mathrm{T}}\mathbf{u} + \left| \frac{\alpha(j\omega)}{\beta(j\omega)} \right|^2 \mathbf{y}^{\mathrm{T}}\mathbf{y} \right] \mathrm{d}t \tag{21}$$

Suppose further that

$$[\mathbf{F}, \mathbf{G}] \text{ is completely controllable and } [\mathbf{F}, \mathbf{H}] \text{ completely observable} \tag{22}$$

$$\frac{\alpha(s)}{\beta(s)} \text{ is proper} \tag{23}$$

$$\beta(s) \text{ and } \alpha(s) \text{ are coprime and have all zeros in } \mathrm{Re}[s] \leqslant 0 \tag{24}$$

There is no zero s_0 of $\mathbf{H}^{\mathrm{T}}(s\mathbf{I} - \mathbf{F})^{-1}\mathbf{G}$ such that

$$\alpha(s_0) = 0, \mathrm{Re}[s_0] = 0 \tag{25}$$

There is no eigenvalue s_0 of $\mathbf{F}$ with

$$\beta(s_0) = 0, \mathrm{Re}[s_0] = 0 \tag{26}$$

Suppose that optimal control laws $\mathbf{u}_1 = \mathbf{k}_1(s)\mathbf{x}$, $\mathbf{u}_2 = \mathbf{k}_2(s)\mathbf{x}$ are determined by following the state augmentation procedures illustrated in Section 2. Then if the initial states for the controllers are suitably related and the plant (19) has the same initial state for both optimization problems, the optimal controls are equal, not as laws, but as time functions:

$$\mathbf{u}_1^*(t) = \mathbf{u}_2^*(t) \tag{27}$$

Proof of Theorem. In outline, the proof will proceed as follows: (a) the structure of the optimal controllers will be established; (b) closed-loop systems will be defined by using these optimal controllers and an external input; (c) it is shown that if these two closed-loop systems have zero initial conditions and the same time functions for their inputs, then the inputs to the part of each closed-loop system which is the original plant are the same; (d) the external inputs will be viewed as setting an initial state, which will then allow us to conclude the main result. In step (c), we shall use matrix fraction descriptions of the various transfer-function matrices involved. For details, see the Appendix.

4. FURTHER ISSUES

Collection of controllers giving the same closed-loop modes

In Section 2 we exhibited for the system with transfer function $1/s$ two feedback controllers

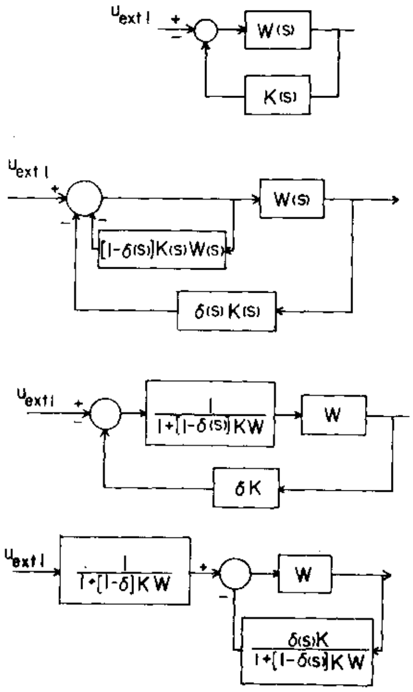

Figure 2. Manipulating one closed loop into another closed loop

with transfer functions

$$\frac{\beta^{-1}(\alpha s+1)}{s+(\gamma-\alpha\beta^{-1})} \text{ and } \frac{-\beta^{-1}[(\beta\gamma-1)s+1]}{s+\beta^{-1}}$$

and these controllers gave rise to the same closed-loop modes, in fact the same state trajectories, given the right initial conditions. Although the fact that such a result might be possible is perhaps counterintuitive, there is a simple explanation. (The external controls should be disregarded at this point.) As the block diagram manipulations in Figure 2 suggest, with zero external signal the plant $W(s)$ with feedback controllers

$$K(s) \text{ and } \frac{\delta(s)K(s)}{1+[1-\delta(s)]K(s)W(s)}$$

gives rise to two closed-loop systems with modes that are closely related. More precisely, if

$$W=\frac{b}{a},\ K=\frac{d}{c},\ \delta=\frac{f}{e} \quad (a, b, \dots, f \text{ polynomial})$$

the closed-loop modes for the first controller are the zeros of $ac+bd$. For the second controller, they are the zeros $ae(ac+bd)$ if we assume no pole–zero cancellations within $\delta K[1+(1-\delta)KW]^{-1}$. The selection of

$$\delta(s)=\frac{(\beta\gamma-1)s+1}{\alpha s+1} \tag{28}$$

provides the connection between the two controllers of the example of Section 2 (and the necessary pole–zero cancellations are guaranteed to occur so that the closed-loop modes are the same for both controllers).

Of course, there are an infinite number of possible selections of $\delta(s)$, and thus an infinite number of possible controllers giving the same closed-loop trajectories. The design method based on augmentation selects in each case a particular controller, whereas in the conventional regulator problem, with no frequency-dependent weighting, the design method selects the one controller that has no memory.

Robustness

The different controllers certainly give different robustness properties, and we have analysed the effect of the frequency-dependent weighting in (6) elsewhere.[7] To make the point clearly, suppose that $\alpha = 0$. Then the loop gains for the two controllers become

$$\frac{1}{\beta s(s+\gamma)} \text{ and } \frac{(\beta\gamma - 1)s + 1}{\beta s(s+\beta^{-1})}$$

Obviously as $\omega = s/j$ becomes very large, the first loop gain decreases far faster than the second, so that robustness in the face of high-frequency uncertainty will be different.

Identical weighting on control and state

Let us consider two optimization problems for $\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}\mathbf{u}$, $\mathbf{y} = \mathbf{H}^{\mathrm{T}}\mathbf{x}$, with $\{\mathbf{F}, \mathbf{G}, \mathbf{H}\}$ minimal. Problems 1 and 2 require minimization of

$$V_1 = \int_0^\infty [\mathbf{u}^{\mathrm{T}}\mathbf{u} + \mathbf{y}^{\mathrm{T}}\mathbf{y}]\, \mathrm{d}t \tag{29a}$$

and

$$V_2 = \int_0^\infty \left[\left| \frac{\beta(j\omega)}{\alpha(j\omega)} \right|^2 \mathbf{u}^{\mathrm{T}}\mathbf{u} + \left| \frac{\beta(j\omega)}{\alpha(j\omega)} \right|^2 \mathbf{y}^{\mathrm{T}}\mathbf{y} \right] \mathrm{d}t \tag{29b}$$

respectively, where β and α are coprime with all zeros in $\mathrm{Re}[s] \leqslant 0$, $\mathbf{H}^{\mathrm{T}}(s\mathbf{I} - \mathbf{F})^{-1}\mathbf{G}$ has no zero such that $\alpha(s_0) = 0$, $\mathrm{Re}[s_0] = 0$, and $\mathbf{F}$ has no eigenvalue s_0 with $\beta(s_0) = 0$, $\mathrm{Re}[s_0] = 0$. We shall exhibit a result on the equality of trajectories. Let $\bar{\mathbf{y}}$ be the output of a system with input $\mathbf{y}$, transfer function matrix $\beta\alpha^{-1}\mathbf{I}$, and zero initial state. Then if

$$V_2 = \int_0^\infty \left[\left| \frac{\beta(j\omega)}{\alpha(j\omega)} \right|^2 \mathbf{u}^{\mathrm{T}}\mathbf{u} + \bar{\mathbf{y}}^{\mathrm{T}}\bar{\mathbf{y}} \right] \mathrm{d}t$$

the main result of Section 3 implies that the same trajectories are associated with minimizing V_2 as are associated with minimizing

$$V_3 = \int_0^\infty \left[\mathbf{u}^{\mathrm{T}}\mathbf{u} + \left| \frac{\alpha(j\omega)}{\beta(j\omega)} \right|^2 \bar{\mathbf{y}}^{\mathrm{T}}\bar{\mathbf{y}} \right] \mathrm{d}t$$

provided that the initial conditions in the controllers are satisfactorily matched. However, $\mathbf{y} = \alpha\beta^{-1}\bar{\mathbf{y}}$, so that V_3 and V_1 take the same values for all $\mathbf{u}$ (given matching initial conditions). Consequently, (29a) and (29b) *give rise to the same optimal controls as time functions*. The optimal controller obtained for (29b) by using augmentation at the input and output will not be the same as that obtained for (29a).

Tracking

One might think that the non-uniqueness of the controllers is a consequence of the fact that the optimal control problem posed does not give rise to any external, as opposed to feedback, component of the optimal control. To show that this is not the case, we shall analyse a tracking problem. Consider, for $\dot{x} = u$,

$$V_1[x(0), u(\cdot)] = \int_0^\infty \left[(x - \bar{x})^2 + \left| \frac{1 + \beta j\omega}{1 + \alpha j\omega} u \right|^2 \right] \mathrm{d}t \tag{30}$$

and

$$V_2[x(0), u(\cdot)] = \int_0^\infty \left\{ \left[\left| \frac{1 + \alpha j\omega}{1 + \beta j\omega} \right| (x - \bar{x}) \right]^2 + u^2 \right\} \mathrm{d}t \tag{31}$$

where $\bar{x}$ is a reference trajectory. Let us suppose that $\bar{x}(\cdot)$ is square integrable. (If this is not the case, the integrals in (30) and (31) cannot be made finite, and one must look at very large but finite time intervals.)

For problem 1, we define an augmented system

$$\begin{bmatrix} \dot{z} \\ \dot{x} \end{bmatrix} = \mathbf{F}_1 \begin{bmatrix} z \\ x \end{bmatrix} + \mathbf{g}_1 v = \begin{bmatrix} -\beta^{-1} & 0 \\ \beta^{-1} - \alpha\beta^{-2} & 0 \end{bmatrix} \begin{bmatrix} z \\ x \end{bmatrix} + \begin{bmatrix} 1 \\ \alpha\beta^{-1} \end{bmatrix} v$$

and let $\mathbf{k}_1$ define the optimal control for the associated regulator problem (see section 2):

$$\mathbf{k}_1^{\mathrm{T}} = [\sqrt{(\alpha^2\beta^{-2} + \beta^{-2} + 2\beta^{-1})} - \beta^{-1} - \alpha\beta^{-1} \quad 1] = [\gamma - \beta^{-1} - \alpha\beta^{-1} \quad 1]$$

Then the solution for the tracking problem is[8]

$$v = -\mathbf{k}_1^{\mathrm{T}} \begin{bmatrix} z \\ x \end{bmatrix} + \int_t^\infty [\mathbf{g}_1^{\mathrm{T}}] \mathrm{e}^{(\mathbf{F}_1 - \mathbf{g}_1\mathbf{k}_1^{\mathrm{T}})^{\mathrm{T}}(\tau - t)} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \bar{\mathbf{x}}(\tau)\, \mathrm{d}\tau$$

The following Laplace transform is easily evaluated:

$$\mathscr{L}\left[\mathbf{g}_1^{\mathrm{T}} \exp(\mathbf{F}_1 - \mathbf{g}_1\mathbf{k}_1^{\mathrm{T}})^{\mathrm{T}} t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right] = \frac{\beta^{-1}(\alpha s + 1)}{s^2 + \gamma s + \beta^{-1}}$$

and so

$$V(s) = -\mathbf{k}_1^{\mathrm{T}} \begin{bmatrix} Z(s) \\ X(s) \end{bmatrix} + \left| \frac{\beta^{-1}(-\alpha s + 1)}{s^2 - \gamma s + \beta^{-1}} \bar{X}(s) \right|_+$$

where the subscript + denotes discarding of those components of a partial fraction expansion with poles in Re$[s] > 0$. Next, arguing as in Section 2, we obtain

$$U_1(s) = -\frac{\beta^{-1}(\alpha s + 1)}{s + \gamma - \alpha\beta^{-1}} X(s) + \frac{\alpha^2\beta^{-2} + \beta^{-1} - \alpha\beta^{-1}\gamma}{s + \gamma - \alpha\beta^{-1}} z(0)$$
$$+ \frac{\beta^{-1}(\alpha s + 1)}{s + \gamma - \alpha\beta^{-1}} \left| \frac{\beta^{-1}(-\alpha s + 1)}{s^2 - \gamma s + \beta^{-1}} \bar{X}(s) \right|_+ \tag{32}$$

For problem 2, the augmented system is

$$\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \mathbf{F}_2 \begin{bmatrix} x \\ w \end{bmatrix} + \mathbf{g}_2 u = \begin{bmatrix} 0 & 0 \\ 1 & -\beta^{-1} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

Also, with

$$\dot{\bar{w}} + \beta^{-1}\bar{w} = \bar{x}$$

the performance index is

$$V_2 = \int_0^\infty \left[\left\{ [\alpha\beta^{-1} \quad \beta^{-1} - \alpha\beta^{-2}] \begin{bmatrix} x - \bar{x} \\ w - \bar{w} \end{bmatrix} \right\}^2 + u^2 \right] \mathrm{d}t$$

If we denote the solution of the associated regulator problem by $u_2 = -\mathbf{k}_2^{\mathrm{T}} [x \ w]^{\mathrm{T}}$, where $\mathbf{k}_2^{\mathrm{T}} = [(\gamma - \beta^{-1}) \ (\beta^{-1} + \beta^{-2} - \beta^{-1}\gamma)]$, the solution of the tracking problem is

$$v_2 = -\mathbf{k}_2^{\mathrm{T}} \begin{bmatrix} x \\ w \end{bmatrix} + \int_t^\infty \mathbf{g}_2^{\mathrm{T}} \exp[(\mathbf{F}_2 - \mathbf{g}_2\mathbf{k}_2^{\mathrm{T}})^{\mathrm{T}}(\tau - t)] \begin{bmatrix} \alpha\beta^{-1} \\ \beta^{-1} - \alpha\beta^{-2} \end{bmatrix} \bar{y} \, \mathrm{d}\tau$$

where

$$\bar{y} = [\alpha\beta^{-1} \quad \beta^{-1} - \alpha\beta^{-2}] \begin{bmatrix} \bar{x} \\ \bar{w} \end{bmatrix}$$

Now

$$\mathscr{L}\{ [\alpha\beta^{-1} \quad \beta^{-1} - \alpha\beta^{-2}] [\exp(\mathbf{F}_2 - \mathbf{g}_2\mathbf{k}_2^{\mathrm{T}})t] \mathbf{g}_2 \} = \frac{\beta^{-1}(\alpha s + 1)}{s^2 + \gamma s + \beta^{-1}}$$

and

$$\bar{y}(s) = \frac{\alpha s + 1}{\beta s + 1} \bar{X}(s) + \frac{1 - \alpha\beta^{-1}}{\beta s + 1} \bar{w}(0)$$

so

$$U_2(s) = -\mathbf{k}_2^{\mathrm{T}} \begin{bmatrix} X(s) \\ W(s) \end{bmatrix} + \left[\frac{\beta^{-1}(-\alpha s + 1)}{s^2 - \gamma s + \beta^{-1}} \left\{ \frac{\alpha s + 1}{\beta s + 1} \bar{X}(s) + \frac{1 - \alpha\beta^{-1}}{\beta s + 1} \bar{w}(0) \right\} \right]_+$$

$$= -\frac{(\gamma - \beta^{-1})s - \beta^{-1}}{s + \beta^{-1}} X(s) - \frac{(\beta^{-1} + \beta^{-2} - \beta^{-1}\gamma)}{s + \beta^{-1}} w(0)$$

$$+ \left[\frac{\beta^{-1}(-\alpha s + 1)}{s^2 - \gamma s + \beta^{-1}} \left\{ \frac{\alpha s + 1}{\beta s + 1} \bar{X}(s) + \frac{1 - \alpha\beta^{-1}}{\beta s + 1} \bar{w}(0) \right\} \right]_+ \qquad (33)$$

Let us now show that as time functions, the two optimal controls are the same.

The implementation of $U_1(s)$ is depicted in the first diagram of Figure 2, except that the initial condition terms are suppressed and $u_{\mathrm{ext}\ 1}$ is shorthand for the last term on the right of (32). In the Figure, we identify

$$K(s) = \frac{\beta^{-1}(\alpha s + 1)}{s + \gamma - \alpha\beta^{-1}}, \quad W(s) = 1/s$$

On selecting

$$\delta(s) = \frac{(\beta\gamma - 1)s + 1}{\alpha s + 1}$$

we obtain, as noted earlier,

$$\frac{\delta K}{1 + [1 - \delta]KW} = \frac{(\gamma - \beta^{-1})s - \beta^{-1}}{s + \beta^{-1}}$$

while also

$$\frac{1}{1 + [1 - \delta]KW} = \frac{s + \delta - \alpha\beta^{-1}}{s + \beta^{-1}}$$

By choosing the same relation between $w(0)$, $x(0)$ and $z(0)$ as used in Section 2, we see that $U_1(s)$ can be replaced by

$$U_3(s) = -\frac{(\gamma - \beta^{-1})s - \beta^{-1}}{s + \beta^{-1}} X(s) - \frac{(\beta^{-1} + \beta^{-2} - \beta^{-1}\gamma)}{s + \beta^{-1}} w(0)$$

$$+ \frac{\alpha s + 1}{\beta s + 1} \left[\frac{\beta^{-1}(-\alpha s + 1)}{s^2 - \gamma s + \beta^{-1}} \bar{X}(s) \right]_+ \quad (34)$$

Suppose that $\bar{X}(s)$ is rational, and that partial fraction expansions are obtained for each of

$$\left[\frac{\beta^{-1}(-\alpha s + 1)}{s^2 - \gamma s + \beta^{-1}} \frac{\alpha s + 1}{\beta s + 1} \bar{X}(s) \right]_+ \text{ and } \frac{\alpha s + 1}{\beta s + 1} \left[\frac{\beta^{-1}(-\alpha s + 1)}{s^2 - \gamma s + \beta^{-1}} \bar{X}(s) \right]_+$$

Then it is easily seen that the residues corresponding to poles of $\bar{X}(s)$ in each expression are equal; more generally

$$\left[\frac{\beta^{-1}(-\alpha s + 1)}{s^2 - \gamma s + \beta^{-1}} \frac{\alpha s + 1}{\beta s + 1} \bar{X}(s) \right]_+ - \frac{\alpha s + 1}{\beta s + 1} \left[\frac{\beta^{-1}(-\alpha s + 1)}{s^2 - \gamma s + \beta^{-1}} \bar{X}(s) \right]_+ = \frac{\phi}{\beta s + 1}$$

for some constant ϕ.

Since clearly

$$\left[\frac{\beta^{-1}(-\alpha s + 1)}{s^2 - \gamma s + \beta^{-1}} \frac{1 - \alpha\beta^{-1}}{\beta s + 1} \bar{w}(0) \right] = \frac{\phi}{\beta s + 1}$$

for some ϕ, if follows that (33) and (34) become identical for appropriate selection of $\bar{w}(0)$. Accordingly, the two optimal controls, as time functions, are still the same.

5. FILTERING EXAMPLE

Consider two problems, illustrated in Figure 3. The white noise sources are independent, zero mean and unit variance. We can ask: what are the relations between the steady-state filters 1 and 2 and the estimates $\hat{y}_1, \hat{y}_2$ on the basis of the measurements z_1, z_2? By analogy with the control result, one must expect some relationship between the filters.

Problem 1 is straightforward to answer. Observing that

$$1 + \frac{(-s+2)}{(-s+3)(-s+1)} \frac{(s+2)}{(s+3)(s+1)} = \left[1 + \frac{-\alpha s + \beta}{(-s+3)(-s+1)} \right] \left[1 + \frac{\alpha s + \beta}{(s+3)(s+1)} \right] \quad (35)$$

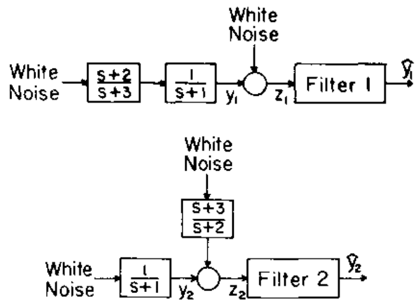

Figure 3. Augmented signal model for coloured (a) input noise (b) output noise

where

$$\alpha = \sqrt{(11 + 2\sqrt{13})} - 4 = 0\cdot 2674 \qquad (36)$$
$$\beta = \sqrt{13} - 3 = 0\cdot 6056$$

and $(s+3)(s+1) + \alpha s + \beta$ is Hurwitz, we conclude that filter 1 has the transfer function $\bar{w}_1(s) = w_1(s)/[1 + w_1(s)]$, where

$$w_1(s) = \frac{\alpha s + \beta}{(s+3)(s+1)} \qquad (37)$$

i.e.

$$\bar{w}_1(s) = \frac{\alpha s + \beta}{s^2 + (4+\alpha)s + 3 + \beta}$$

For problem 2, let us define

$$z_3 = \frac{s+2}{s+3} z_2 \qquad (38)$$

and seek the transfer function of the filter which processes z_3 to yield $\hat{y}_2$. It is clear that this is equivalent to searching for filter 3 in Figure 4, which estimates y_3 from z_3.

Let us construct a state-variable model for the signal process:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \qquad (39a)$$

$$z_3 = [1 \quad -1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + v \qquad (39b)$$

(Here, u and v are zero mean, unit intensity, independent, white noise processes and $x_1 = y_2$.)

Filter 3 is defined by

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \left\{ z_3 - [1 \quad -1] \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} \right\} \qquad (40a)$$

$$\hat{y}_2 = \hat{x}_1 \qquad (40b)$$

with k_1, k_2 determined either by using the standard Riccati equation approach, or by using the spectral factorization appropriate to the problem. As reference to Figure 3 and equation (35) shows, (35) is the correct spectral factorization for problem 2 as well as for problem 1. This means that

$$[1 \quad -1] \left\{ s\mathbf{I} - \begin{bmatrix} -1 & 0 \\ 1 & -3 \end{bmatrix} \right\}^{-1} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \frac{\alpha s + \beta}{(s+3)(s+1)}$$

i.e. $k_1 = \beta - \alpha$ and $k_2 = \beta - 2\alpha$.

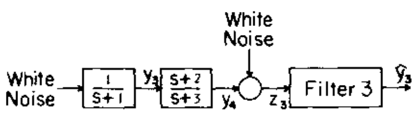

Figure 4. Alternative structure for studying coloured output noise

Then (40) defines a transfer function from z_3 to $\hat{y}_3$ of

$$\bar{w}_2 = [1 \quad 0]\left\{s\mathbf{I} - \begin{bmatrix} -1+\alpha-\beta & \beta-\alpha \\ 1+2\alpha-\beta & -3+\beta-2\alpha \end{bmatrix}\right\}^{-1}\begin{bmatrix}\beta-\alpha \\ \beta-2\alpha\end{bmatrix}$$

$$= \frac{(s+3)(\beta-\alpha)}{s^2+(4+\alpha)s+3+\beta}$$

and so filter 2 has transfer function

$$\frac{(\beta-\alpha)(s+2)}{s^2+(4+\alpha)s+3+\beta}$$

Notice that filter 2 has the same poles as filter 1, whereas the zero is different. (This fact will be exposed as a general property in the next section). The zero is in fact identical with the pole of the shaping filter producing the coloured measurement noise. The magnitude of the filter 2 gain is greater at all frequencies $s = j\omega$, ω real.

Had our task been to find a filter to estimate y_4 in Figure 4 from z_3, then we should have found the same filter as for problem 1. Herein lies the connection between the two problems. However, y_4 in Figure 4 does not correspond to any variable of interest in Figure 3(b).

6. FILTERING/CONTROL DUALITY

The result of Section 5 that two related filtering problems give rise to filters with the same poles but different zeros is a general one. We shall now demonstrate this point by setting up the duality between filtering and control problems with frequency-dependent weighting and then appeal to the earlier control results.

Frequency-weighted measurement noise

Suppose that the signal model is $\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}\mathbf{u}$, $\mathbf{y} = \mathbf{H}^T\mathbf{x}$ with $\mathbf{u}$ a zero mean, white noise process with $E[\mathbf{u}(t)\mathbf{u}^T(s)] = \mathbf{Q}\,\delta(t-s)$, and suppose that measurements $\mathbf{z} = \mathbf{y} + \mathbf{n}$ are available where $\mathbf{n}$ is a zero mean stationary process with power spectrum $\mathbf{R}(s) = \mathbf{W}_r(s)\mathbf{W}_r^T(-s)$ for some invertible stable, minimum phase $\mathbf{W}_r(s)$ with $\mathbf{W}_r^{-1}(s) = \mathbf{J}_r + \mathbf{H}_r^T(s\mathbf{I} - \mathbf{F}_r)^{-1}\mathbf{G}_r$. Figure 5 illustrates the arrangement; Figure 5(b) strongly suggests that, in order to obtain a filter for $\mathbf{x}$, $\mathbf{z}_1$ be regarded as the measurement for an augmented model defined by

$$\mathbf{F}_1 = \begin{bmatrix}\mathbf{F} & \mathbf{0} \\ \mathbf{G}_r\mathbf{H}^T & \mathbf{F}_r\end{bmatrix},\ \mathbf{G}_1 = \begin{bmatrix}\mathbf{G} \\ \mathbf{0}\end{bmatrix},\ \mathbf{H}_1^T = [\mathbf{J}_r\mathbf{H}^T \quad \mathbf{H}_r^T],\ \mathbf{Q}_1 = \mathbf{Q},\ \mathbf{R}_1 = \mathbf{I} \tag{41}$$

Let $\mathbf{\Pi}^{1f}$ be the solution of the associated steady-state Riccati equation. It is easy to see that the following control problem gives rise to the same steady-state Riccati equation if the approach

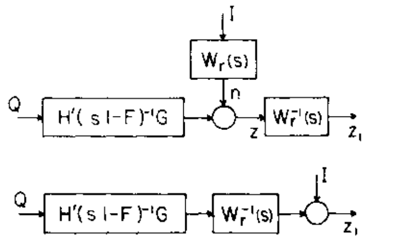

Figure 5. General structure for studying coloured output noise

of Section 3 to controller design is used; for the system $\dot{\mathbf{x}} = \mathbf{F}^{\mathrm{T}}\mathbf{x} + \mathbf{H}\mathbf{u}$, $\mathbf{y} = \mathbf{G}^{\mathrm{T}}\mathbf{x}$, minimize

$$\int_0^\infty [\| \mathbf{W}_r^{\mathrm{T}}(j\omega)u \|^2 + \mathbf{y}^{\mathrm{T}}\mathbf{Q}\mathbf{y}]\ \mathrm{d}t$$

Thus, the duality has $\mathbf{H}^{\mathrm{T}}(s\mathbf{I} - \mathbf{F})^{-1}\mathbf{G}$ and $\mathbf{R}(s)$ for the filtering problem becoming $\mathbf{G}^{\mathrm{T}}(s\mathbf{I} - \mathbf{F}^{\mathrm{T}})^{-1}\mathbf{H}$ and $\mathbf{R}^{\mathrm{T}}(s)$ for the control problem.

Assertion

The optimal filter can be represented in the form of Figure 6, where

$$\begin{bmatrix}\mathbf{K}_{f1}\\ \mathbf{K}_{f2}\end{bmatrix} = \mathbf{\Pi}^{1f}\mathbf{H}_1\mathbf{R}^{-1} \tag{42}$$

$$\mathbf{M}_f(s) = \mathbf{J}_r + \mathbf{H}_r^{\mathrm{T}}[s\mathbf{I} - \mathbf{F}_r + \mathbf{K}_{f2}\mathbf{H}_r^{\mathrm{T}}]^{-1}(\mathbf{G}_r - \mathbf{K}_{f2}\mathbf{J}_r) \tag{43}$$

Proof. Let $\mathbf{x}_1$ be the state vector of the augmented model, with $\mathbf{x}_1 = [\mathbf{x}^{\mathrm{T}}\quad \mathbf{x}_r^{\mathrm{T}}]^{\mathrm{T}}$. With $\mathbf{K}_{fi}$ as above, standard Kalman filter theory[8] yields

$$\dot{\hat{\mathbf{x}}} = \mathbf{F}\hat{\mathbf{x}} + \mathbf{K}_{f1}\mathbf{z}_1 - \mathbf{K}_{f1}\mathbf{J}_r\mathbf{H}^{\mathrm{T}}\hat{\mathbf{x}} - \mathbf{K}_{f1}\mathbf{H}_r^{\mathrm{T}}\hat{\mathbf{x}}_r$$
$$\dot{\hat{\mathbf{x}}}_r = \mathbf{F}_r\hat{\mathbf{x}}_r + \mathbf{G}_r\mathbf{H}^{\mathrm{T}}\hat{\mathbf{x}} + \mathbf{K}_{f2}\mathbf{z}_1 - \mathbf{K}_{f2}\mathbf{J}_r\mathbf{H}^{\mathrm{T}}\hat{\mathbf{x}} - \mathbf{K}_{f2}\mathbf{H}_r^{\mathrm{T}}\hat{\mathbf{x}}_r$$

Use the fact that

$$\mathbf{z}_1 = \mathbf{J}_r\mathbf{z} + \mathbf{H}_r^{\mathrm{T}}\mathbf{x}_r,\ \dot{\mathbf{x}}_r = \mathbf{F}_r\mathbf{x}_r + \mathbf{G}_r\mathbf{z} \tag{44}$$

to obtain

$$\dot{\hat{\mathbf{x}}} = \mathbf{F}\hat{\mathbf{x}} + \mathbf{K}_{f1}\mathbf{J}_r\mathbf{z} - \mathbf{K}_{f1}\mathbf{H}_r^{\mathrm{T}}(\hat{\mathbf{x}}_r - \mathbf{x}_r) - \mathbf{K}_{f1}\mathbf{J}_r\mathbf{H}^{\mathrm{T}}\hat{\mathbf{x}} \tag{45}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}(\hat{\mathbf{x}}_r - \mathbf{x}_r) = (\mathbf{F}_r - \mathbf{K}_{f2}\mathbf{H}_r^{\mathrm{T}})(\hat{\mathbf{x}}_r - \mathbf{x}_r) + (\mathbf{G}_r - \mathbf{K}_{f2}\mathbf{J}_r)\mathbf{H}^{\mathrm{T}}\hat{\mathbf{x}} + (\mathbf{K}_{f2}\mathbf{J}_r - \mathbf{G}_r)\mathbf{z}$$

so that

$$(\hat{\mathbf{x}}_r - \mathbf{x}_r) = -[s\mathbf{I} - \mathbf{F}_r + \mathbf{K}_{f2}\mathbf{H}_r^{\mathrm{T}}]^{-1}(\mathbf{G}_r - \mathbf{K}_{f2}\mathbf{J}_r)(\mathbf{z} - \mathbf{H}^{\mathrm{T}}\hat{\mathbf{x}})$$

Combining this with (45) yields the desired result.

Several points can be noted. First, $\mathbf{M}_f(s)$ is the transfer function obtained by applying output injection feedback of $-\mathbf{K}_{f2}\mathbf{z}_1$ to the state-variable realization of $\mathbf{W}_r^{-1}(s)$ in (44). Because of this feedback relationship, $\mathbf{M}_f(s)$ has the same zeros as $\mathbf{W}_r^{-1}(s)$ [which are also the poles of $\mathbf{W}_r(s)$], and these will also be zeros of the filter. Secondly, applying the ideas of Section 3 to the control problem yields for the controller

$$\mathbf{u}(t) = -[(\mathbf{G}_r^{\mathrm{T}} - \mathbf{J}_r^{\mathrm{T}}\mathbf{K}_{f2})(s\mathbf{I} - \mathbf{F}_r^{\mathrm{T}} + \mathbf{H}_r\mathbf{K}_{f2}^{\mathrm{T}})^{-1}\mathbf{H}_r + \mathbf{J}_r^{\mathrm{T}}]\mathbf{K}_{f1}^{\mathrm{T}}\mathbf{x}(t) \tag{46}$$

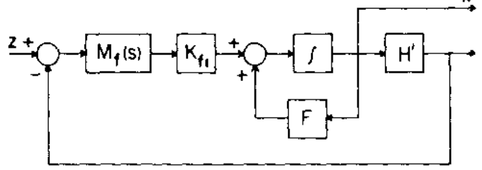

Figure 6. Filter structures with coloured output noise

and so the control/filtering duality is maintained. As the Assertion above makes clear, the effective filter gain is

$$\mathbf{K}_{f1}[\mathbf{J}_r + \mathbf{H}_r^T(s\mathbf{I} - \mathbf{F}_r + \mathbf{K}_{f2}\mathbf{H}_r^T)^{-1}(\mathbf{G}_r - \mathbf{K}_{f2}\mathbf{J}_r)]$$

The poles of the filter are obviously the same as the poles of the closed-loop system resulting from the above controller.

Frequency-weighted input noise

Suppose that the signal model is $\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}\mathbf{u}$, $\mathbf{y} = \mathbf{H}^T\mathbf{x}$, measurements $\mathbf{z} = \mathbf{y} + \mathbf{n}$ are available with $\mathbf{n}(\cdot)$ zero mean Gaussian white noise, $E[\mathbf{n}(t)\mathbf{n}^T(s)] = \mathbf{R}\delta(t-s)$. Suppose that the power spectrum of $\mathbf{u}$ is $\mathbf{Q}(j\omega)$, with $\mathbf{Q}(s) = \mathbf{W}_q(s)\mathbf{W}_q^T(-s)$ for some $\mathbf{W}_q(s) = \mathbf{J}_q + \mathbf{H}_q^T(s\mathbf{I} - \mathbf{F}_q)^{-1}\mathbf{G}_q$, with $\mathbf{W}_q(s)$ stable and minimum phase. To estimate $\mathbf{x}$, it is obvious that one should define an augmented system [as the cascade of $\mathbf{W}_q(s)$ and $\mathbf{H}^T(s\mathbf{I} - \mathbf{F})^{-1}\mathbf{G}$]. In obvious notation,

$$\mathbf{Q}_2 = \mathbf{I},\ \mathbf{R}_2 = \mathbf{R},\ \mathbf{F}_2 = \begin{bmatrix} \mathbf{F} & \mathbf{G}\mathbf{H}_q^T \\ \mathbf{0} & \mathbf{F}_q \end{bmatrix},\ \mathbf{G}_2 = \begin{bmatrix} \mathbf{G}\mathbf{J}_q \\ \mathbf{G}_q \end{bmatrix},\ \mathbf{H}_2^T = [\mathbf{H}^T \quad \mathbf{0}]$$

Suppose that the steady-state Riccati equation has a solution Π^{2f}.

It is readily checked that the following control problem gives rise to the same steady-state Riccati equation when the approach of Section 3 to controller design is used: for the system $\dot{\mathbf{x}} = \mathbf{F}^T\mathbf{x} + \mathbf{H}\mathbf{u}$, $\mathbf{y} = \mathbf{G}^T\mathbf{x}$, minimize

$$\int_0^\infty [\mathbf{u}^T\mathbf{R}\mathbf{u} + \| \mathbf{W}_q^T(j\omega)\mathbf{y} \|^2]\, dt$$

Thus the duality has $\mathbf{H}^T(s\mathbf{I} - \mathbf{F})^{-1}\mathbf{G}$ and $\mathbf{Q}(s)$ for the filtering problem becoming $\mathbf{G}^T(s\mathbf{I} - \mathbf{F}^T)^{-1}\mathbf{H}$ and $\mathbf{Q}^T(s)$ for the control problem.

Figure 7(a) illustrates the implementation of the filter and Figure 7(b) illustrates a rearrangement in which

$$\mathbf{K}_{f2}(s) = [\Pi_{11}^{2f} + \mathbf{G}\mathbf{H}_q^T(s\mathbf{I} - \mathbf{F}_q)^{-1}\Pi_{21}^{2f}]\mathbf{H}\mathbf{R}^{-1}$$

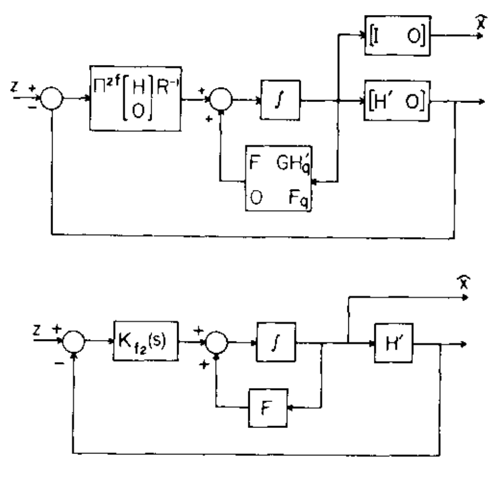

Figure 7. Filter structures with coloured input noise

If we use the method of Section 2, the control law for the control problem is

$$\mathbf{u} = \mathbf{K}_{c2}(s)\mathbf{x} = -\mathbf{R}^{-1}\mathbf{H}^{T}[\mathbf{\Pi}_{11}^{2f} + \mathbf{\Pi}_{21}^{2fT}(s\mathbf{I} - \mathbf{F}_{q}^{T})^{-1}\mathbf{H}_{q}\mathbf{G}^{T}]\mathbf{x}$$

which is consistent with the duality.

The poles of the filter are the zeros of $\det[s\mathbf{I} - \mathbf{F} - \mathbf{K}_{f2}(s)\mathbf{H}^{T}]$, and the poles of the closed-loop system with the above controller are the zeros of $\det[s\mathbf{I} - \mathbf{F}^{T} + \mathbf{H}\mathbf{K}_{c2}^{T}(s)]$. These are clearly the same.

Because of the duality, *we see that if two control problems give rise to the same closed-loop poles, then the two dual filtering problems must also give rise to the same closed-loop poles.* Duality does not however carry through to the point where equality of the time-trajectories in two related control problems implies an interesting equality in the filtering problem.

7. CONCLUSION

If the ultimate justification of a linear–quadratic design is really to minimize a performance index or to achieve certain closed-loop poles, we have shown that frequency dependence can be shuffled between the state and control weighting in the performance index. On the other hand, if the aim is in part to obtain a robust design that will offer some insensitivity to variation in the plant, then within the context of the augmented equations approach,[4,5] there is a very real difference in having frequency-dependent weighting on the control as opposed to the state. Frequency-dependent weighting on the control at least allows the moving in the frequency domain of robustness from one frequency band to another;[6] this only seems possible to a more limited extent with frequency-dependent weighting in the state term.

For the filter, we have seen that closed-loop poles can be retained when frequency-dependent weighting is shifted from the process noise to the measurement noise, but otherwise the filter changes.

APPENDIX: PROOF OF THE MAIN THEOREM

(a) Optimal controller structures

Suppose that $\alpha(s)/\beta(s) = j_{\alpha} + \mathbf{h}_{\alpha}^{T}(s\mathbf{I} - \mathbf{f}_{\alpha})^{-1}\mathbf{g}_{\alpha}$. Without loss of generality, suppose that $\mathbf{u}$ and $\mathbf{y}$ have the same dimension p (otherwise add columns of zeros to $\mathbf{G}$ or $\mathbf{H}$ as appropriate). Let

$$[\alpha(s)/\beta(s)]\mathbf{I}_{p} = \mathbf{J}_{\alpha} + \mathbf{H}_{\alpha}^{T}(s\mathbf{I} - \mathbf{F}_{\alpha})^{-1}\mathbf{G}_{\alpha} \tag{47}$$

with

$$\mathbf{J}_{\alpha} = j_{\alpha}\mathbf{I}_{p},\ \mathbf{H}_{\alpha}^{T} = \begin{bmatrix} \mathbf{h}_{\alpha}^{T} & \mathbf{0} & \mathbf{0} & . & . \\ \mathbf{0} & \mathbf{h}_{\alpha}^{T} & \mathbf{0} & . & . \\ . & . & . & . & . \end{bmatrix},\ \mathbf{F}_{\alpha} = \begin{bmatrix} \mathbf{f}_{\alpha} & \mathbf{0} & . & . \\ \mathbf{0} & \mathbf{f}_{\alpha} & . & . \\ . & . & . & . \end{bmatrix},\ \mathbf{G}_{\alpha} = \begin{bmatrix} \mathbf{g}_{\alpha} & \mathbf{0} & . & . \\ \mathbf{0} & \mathbf{g}_{\alpha} & . & . \\ . & . & . & . \end{bmatrix} \tag{48}$$

To study problem 1, we use Figure 8. Formally, we can construct an $\mathbf{F}_{1}$, $\mathbf{G}_{1}$, $\mathbf{H}_{1}$ triple as

$$\mathbf{F}_{1} = \begin{bmatrix} \mathbf{F}_{\alpha} & \mathbf{0} \\ \mathbf{G}\mathbf{H}_{\alpha}^{T} & \mathbf{F} \end{bmatrix},\ \mathbf{G}_{1} = \begin{bmatrix} \mathbf{G}_{\alpha} \\ \mathbf{G}\mathbf{J}_{\alpha} \end{bmatrix},\ \mathbf{H}_{1}^{T} = [\mathbf{0} \quad \mathbf{H}^{T}] \tag{49}$$

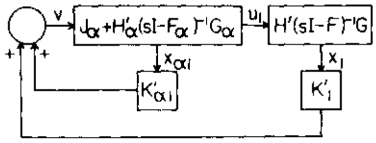

Figure 8. General controller structure with frequency weighting of input

and find $[\mathbf{K}_{\alpha 1}^{T} \quad \mathbf{K}_{1}^{T}]$ with

$$\mathbf{I}+\mathbf{G}_1^{T}(-s\mathbf{I}-\mathbf{F}_1^{T})^{-1}\mathbf{H}_1\mathbf{H}_1^{T}(s\mathbf{I}-\mathbf{F}_1)^{-1}\mathbf{G}_1$$
$$=\left[\mathbf{I}-\mathbf{G}_1^{T}(-s\mathbf{I}-\mathbf{F}_1^{T})^{-1}\begin{bmatrix}\mathbf{K}_{\alpha 1}\\ \mathbf{K}_1\end{bmatrix}\right][\mathbf{I}-[\mathbf{K}_{\alpha 1}^{T} \quad \mathbf{K}_1^{T}](s\mathbf{I}-\mathbf{F}_1)^{-1}\mathbf{G}_1] \tag{50}$$

and such that $\mathrm{Re}\lambda_i\{\mathbf{F}_1-\mathbf{G}_1[\mathbf{K}_{\alpha 1}^{T} \quad \mathbf{K}_1^{T}]\}<0$ for all i. (The existence of $\mathbf{K}_{\alpha 1}$, $\mathbf{K}_1$ with these properties is a consequence of the standard linear–quadratic theory;[8] note that (22) and (24)–(26) together guarantee that the triple $\{\mathbf{F}_1, \mathbf{G}_1, \mathbf{H}_1\}$ is detectable and stabilizable.) Now the optimal control instead of being obtained from the Riccati equation can be obtained from the above spectral factorization or return difference equality; we have[8]

$$\mathbf{v}(t)=\mathbf{K}_{\alpha 1}^{T}\mathbf{x}_{\alpha 1}(t)+\mathbf{K}_1^{T}\mathbf{x}_1(t) \tag{51}$$

and the optimal control is

$$\mathbf{u}_1(t)=\mathbf{J}_\alpha v(t)+\mathbf{H}_\alpha^{T}\mathbf{x}_{\alpha 1} \tag{52a}$$

Note that this can also be written as

$$\mathbf{u}_1(t)=[\mathbf{H}_\alpha^{T}(s\mathbf{I}-\mathbf{F}_\alpha-\mathbf{G}_\alpha\mathbf{K}_{\alpha 1}^{T})^{-1}\mathbf{G}_\alpha+\mathbf{J}_\alpha]\mathbf{K}_1^{T}\mathbf{x}_1(t) \tag{52b}$$

and in this form, correspondence with the ideas of Section 2 is obtained.

To study problem 2 (see Figure 9), we construct

$$\mathbf{F}_2=\begin{bmatrix}\mathbf{F} & \mathbf{0}\\ \mathbf{G}_\alpha\mathbf{H}^{T} & \mathbf{F}_\alpha\end{bmatrix},\ \mathbf{G}_2=\begin{bmatrix}\mathbf{G}\\ \mathbf{0}\end{bmatrix},\ \mathbf{H}_\alpha^{T}=[\mathbf{J}_\alpha\mathbf{H}^{T} \quad \mathbf{H}_\alpha^{T}] \tag{53}$$

and find $[\mathbf{K}_2^{T} \quad \mathbf{K}_{\alpha 2}^{T}]$ with

$$\mathbf{I}+\mathbf{G}_2^{T}(-s\mathbf{I}-\mathbf{F}_2^{T})^{-1}\mathbf{H}_2\mathbf{H}_2^{T}(s\mathbf{I}-\mathbf{F}_2)^{-1}\mathbf{G}_2$$
$$=\left[\mathbf{I}-\mathbf{G}_2^{T}(-s\mathbf{I}-\mathbf{F}_2^{T})^{-1}\begin{bmatrix}\mathbf{K}_2\\ \mathbf{K}_{\alpha 2}\end{bmatrix}\right][\mathbf{I}-[\mathbf{K}_2^{T} \quad \mathbf{K}_{\alpha 2}^{T}](s\mathbf{I}-\mathbf{F}_2)^{-1}\mathbf{G}_2] \tag{54}$$

and $\mathrm{Re}\gamma_i\{\mathbf{F}_2-\mathbf{G}_2[\mathbf{K}_2^{T} \quad \mathbf{K}_{\alpha 2}^{T}]\}<0$ for all i. Existence is guaranteed as before. The optimal control is

$$\mathbf{u}=\mathbf{K}_2^{T}\mathbf{x}_2+\mathbf{K}_{\alpha 2}^{T}\mathbf{x}_{\alpha 2} \tag{55}$$

Two observations at this stage are important. First, as argued below,

$$[\mathbf{K}_{\alpha 1}^{T} \quad \mathbf{K}_1^{T}](s\mathbf{I}-\mathbf{F}_1)^{-1}\mathbf{G}_1=[\mathbf{K}_2^{T} \quad \mathbf{K}_{\alpha 2}^{T}](s\mathbf{I}-\mathbf{F}_2)^{-1}\mathbf{G}_2 \tag{56}$$

To see this, observe first that

$$\begin{aligned}[\mathbf{H}^{T}(s\mathbf{I}-\mathbf{F})^{-1}\mathbf{G}][\mathbf{J}_\alpha+\mathbf{H}_\alpha^{T}(s\mathbf{I}-\mathbf{F}_\alpha)^{-1}\mathbf{G}_\alpha] &= \mathbf{H}^{T}(s\mathbf{I}-\mathbf{F})^{-1}\mathbf{G}[\alpha(s)/\beta(s)]\mathbf{I}_p\\ &= [\alpha(s)/\beta(s)]\mathbf{I}_p\mathbf{H}^{T}(s\mathbf{I}-\mathbf{F})^{-1}\mathbf{G}\\ &= [\mathbf{J}_\alpha+\mathbf{H}_\alpha^{T}(s\mathbf{I}-\mathbf{F}_\alpha)^{-1}\mathbf{G}_\alpha]\mathbf{H}^{T}(s\mathbf{I}-\mathbf{F})^{-1}\mathbf{G}\end{aligned}$$

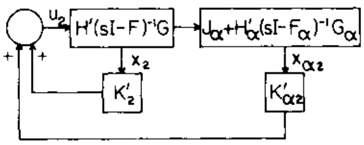

Figure 9. General controller structure with frequency weighting of output

Hence

$$\mathbf{H}_1^{\mathrm{T}}(s\mathbf{I}-\mathbf{F}_1)^{-1}\mathbf{G}_1=\mathbf{H}_2^{\mathrm{T}}(s\mathbf{I}-\mathbf{F}_2)^{-1}\mathbf{G}_2$$

In the case in which $\{\mathbf{F}_1,\mathbf{G}_1,\mathbf{H}_1\}$ is controllable and observable, it is an easy argument, based on the uniqueness of the optimal control law, to conclude (56). Extension to cope with detectability and stabilizability is intricate, but not difficult.

The second observation is that what is in effect important in Figure 9 is the control law

$$\mathbf{u}_2=[\mathbf{K}_2^{\mathrm{T}}+\mathbf{K}_{\alpha 2}^{\mathrm{T}}(s\mathbf{I}-\mathbf{F}_\alpha)^{-1}\mathbf{G}_\alpha\mathbf{H}^{\mathrm{T}}]\mathbf{x}_2 \tag{57}$$

Now because of the structure of $\mathbf{F}_\alpha$, $\mathbf{G}_\alpha$, we have

$$\mathbf{K}_{\alpha 2}^{\mathrm{T}}(s\mathbf{I}-\mathbf{F}_\alpha)^{-1}\mathbf{G}_\alpha=\begin{bmatrix}\mathbf{K}_{\alpha 2}^{11\mathrm{T}} & \mathbf{K}_{\alpha 2}^{12\mathrm{T}} & \cdots & \mathbf{K}_{\alpha 2}^{1p\mathrm{T}}\\ \mathbf{K}_{\alpha 2}^{21\mathrm{T}} & \mathbf{K}_{\alpha 2}^{22\mathrm{T}} & \cdots & \mathbf{K}_{\alpha 2}^{1p\mathrm{T}}\\ \vdots & & & \\ \mathbf{K}_{\alpha 2}^{p1\mathrm{T}} & \mathbf{K}_{\alpha 2}^{p2\mathrm{T}} & \cdots & \mathbf{K}_{\alpha 2}^{pp\mathrm{T}}\end{bmatrix}$$

$$\times\begin{bmatrix}(s\mathbf{I}-\mathbf{f}_\alpha)^{-1} & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{0} & (s\mathbf{I}-\mathbf{f}_\alpha)^{-1} & \cdots & \mathbf{0}\\ \vdots & & & \vdots\\ \mathbf{0} & \mathbf{0} & \cdots & (s\mathbf{I}-\mathbf{f}_\alpha)^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{g}_\alpha & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{0} & \mathbf{g}_\alpha & \cdots & \mathbf{0}\\ \vdots & & & \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{g}_\alpha\end{bmatrix}$$

where each $\mathbf{K}_{\alpha 2}^{ij\mathrm{T}}$ is a row vector. It is readily checked then that

$$\mathbf{K}_{\alpha 2}^{\mathrm{T}}(s\mathbf{I}-\mathbf{F}_\alpha)^{-1}\mathbf{G}_\alpha=\mathbf{G}_\alpha^{\mathrm{T}}(s\mathbf{I}-\mathbf{F}_\alpha^{\mathrm{T}})^{-1}\hat{\mathbf{K}}_{\alpha 2}$$

where

$$\hat{\mathbf{K}}_{\alpha 2}=\begin{bmatrix}\mathbf{K}_{\alpha 2}^{11} & \cdots & \mathbf{K}_{\alpha 2}^{1p}\\ \vdots & & \\ \mathbf{K}_{\alpha 2}^{p1} & & \mathbf{K}_{\alpha 2}^{p1}\end{bmatrix}$$

Therefore, we can regard the optimal control (57) as alternatively being generated by

$$\mathbf{u}_2=[\mathbf{K}_2^{\mathrm{T}}+\mathbf{G}_\alpha^{\mathrm{T}}(s\mathbf{I}-\mathbf{F}_\alpha^{\mathrm{T}})^{-1}\hat{\mathbf{K}}_{\alpha 2}\mathbf{H}^{\mathrm{T}}]\mathbf{x}_2 \tag{58}$$

Notice also that if $\mathbf{x}_{\alpha 2}(0)$ is an initial state for $\{\mathbf{F}_\alpha,\mathbf{G}_\alpha,\mathbf{K}_{\alpha 2}\}$, there exists a corresponding initial state $\hat{\mathbf{x}}_{\alpha 2}(0)$ for $\{\mathbf{F}_\alpha^{\mathrm{T}},\hat{\mathbf{K}}_{\alpha 2},\mathbf{G}_\alpha\}$ because controllability of $\mathbf{F}_\alpha$, $\mathbf{G}_\alpha$ guarantees that there exists a $\mathbf{z}(t)$, zero outside of $[t_0,0]$, taking $\mathbf{x}_{\alpha 2}(t_0)=0$ to $\mathbf{x}_{\alpha 2}(0)$. The initial state for $\{\mathbf{F}_\alpha^{\mathrm{T}},\hat{\mathbf{K}}_{\alpha 2},\mathbf{G}_\alpha\}$ is then that resulting at time 0 from application of $\mathbf{z}(t)$, assuming that $\hat{\mathbf{x}}_{\alpha 2}(t_0)=0$. This means that we can also regard the optimal control as being defined by

$$\mathbf{u}_2(t)=\mathbf{K}_2^{\mathrm{T}}\mathbf{x}_2(t)+\mathbf{G}^{\mathrm{T}}\hat{\mathbf{x}}_{\alpha 2}(t) \tag{59}$$

where $\hat{\mathbf{x}}_{\alpha 2}(t)$ is well defined.

(b) Definition of two closed-loop systems

We consider the two closed-loop systems defined in Figures 10 and 11. Zero initial conditions at time $t_0 < 0$ are assumed for both systems; the external inputs are $\mathbf{w}_1(t)$, $\mathbf{w}_2(t)$, assumed nonzero over $[t_0, 0]$. Notice that for $t > 0$, both schemes in effect become the optimal control implementation of Figure 8; the optimal control implementation of Figure 9 is modified by the replacement of $\mathbf{K}_{\alpha 2}^T(s\mathbf{I} - \mathbf{F}_\alpha)^{-1}\mathbf{G}_\alpha$ with $\mathbf{G}_\alpha^T(s\mathbf{I} - \mathbf{F}_\alpha^T)^{-1}\hat{\mathbf{K}}_{\alpha 2}$. In Figure 11, $\hat{\mathbf{K}}_{\alpha 1}$ is defined from $\mathbf{K}_{\alpha 1}$ just as $\hat{\mathbf{K}}_{\alpha 2}$ is defined from $\mathbf{K}_{\alpha 2}$. Hence

$$\mathbf{G}_\alpha^T(s\mathbf{I} - \mathbf{F}_\alpha^T)^{-1}\hat{\mathbf{K}}_{\alpha 1} = \mathbf{K}_{\alpha 1}^T(s\mathbf{I} - \mathbf{F}_\alpha)^{-1}\mathbf{G}_\alpha \tag{60}$$

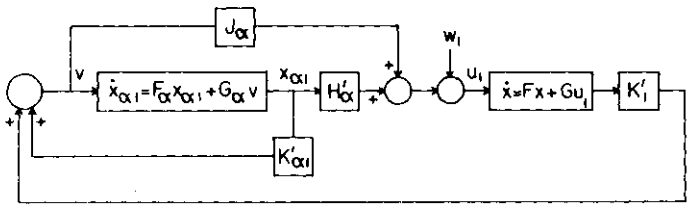

Figure 10. Scheme of Figure 10 with external input

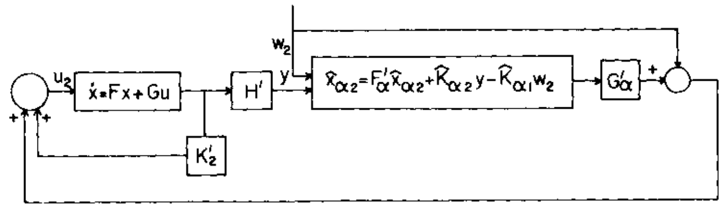

Figure 11. Scheme of Figure 11 with external input

(c) Analysis of the two closed-loop schemes

Let us introduce matrix fraction descriptions of various transfer-function matrices as follows:

$$\mathbf{H}^T(s\mathbf{I} - \mathbf{F})^{-1}\mathbf{G} = \mathbf{B}\mathbf{A}^{-1}, \ \mathbf{B}, \mathbf{A} \text{ coprime} \tag{61a}$$

$$\mathbf{K}_1^T(s\mathbf{I} - \mathbf{F})^{-1}\mathbf{G} = \mathbf{C}_1\mathbf{A}^{-1}, \quad \mathbf{K}_2^T(s\mathbf{I} - \mathbf{F})^{-1}\mathbf{G} = \mathbf{C}_2\mathbf{A}^{-1} \tag{61b}$$

$$\mathbf{J}_\alpha + \mathbf{H}_\alpha^T(s\mathbf{I} - \mathbf{F}_\alpha)^{-1}\mathbf{G}_\alpha = (\alpha\mathbf{I})(\beta\mathbf{I})^{-1} = (\beta\mathbf{I})^{-1}(\alpha\mathbf{I}) \tag{61c}$$

$$\mathbf{K}_{\alpha 1}^T(s\mathbf{I} - \mathbf{F}_\alpha)^{-1}\mathbf{G}_\alpha = \mathbf{C}_{\alpha 1}(\beta\mathbf{I})^{-1} = (\beta\mathbf{I})^{-1}\mathbf{C}_{\alpha 1} \tag{61d}$$

$$\mathbf{K}_{\alpha 2}^T(s\mathbf{I} - \mathbf{F}_\alpha)^{-1}\mathbf{G}_\alpha = \mathbf{C}_{\alpha 2}(\beta\mathbf{I})^{-1} = (\beta\mathbf{I})^{-1}\mathbf{C}_{\alpha 2} \tag{61e}$$

Notice that the definition (61a) assumes the existence of $\mathbf{C}_1$, $\mathbf{C}_2$ in (61b), and (61c) ensures the existence of $\mathbf{C}_{\alpha 1}$, $\mathbf{C}_{\alpha 2}$ in (61d) and (61e).

Now the scheme of Figure 10 (with zero initial conditions) implies that

$$\begin{aligned}\mathbf{v} &= \mathbf{C}_{\alpha 1}(\beta\mathbf{I})^{-1}\mathbf{v} + \mathbf{C}_1\mathbf{A}^{-1}\mathbf{u} \\ &= \mathbf{C}_{\alpha 1}(\beta\mathbf{I})^{-1}\mathbf{v} + \mathbf{C}_1\mathbf{A}^{-1}(\alpha\mathbf{I})(\beta\mathbf{I})^{-1}\mathbf{v} + \mathbf{C}_1\mathbf{A}^{-1}\mathbf{w}_1\end{aligned}$$

or

$$[\mathbf{I} - \mathbf{C}_{\alpha 1}(\beta\mathbf{I})^{-1} - \mathbf{C}_1\mathbf{A}^{-1}(\alpha\mathbf{I})(\beta\mathbf{I})^{-1}]\mathbf{v} = \mathbf{C}_1\mathbf{A}^{-1}\mathbf{w}_1$$

Since

$$\mathbf{u}_1 = (\alpha\mathbf{I})(\beta\mathbf{I})^{-1}\mathbf{v} + \mathbf{w}_1$$

it follows that

$$[\mathbf{I} - \mathbf{C}_{\alpha 1}(\beta\mathbf{I})^{-1} - \mathbf{C}_1\mathbf{A}^{-1}(\alpha\mathbf{I})(\beta\mathbf{I})^{-1}](\mathbf{u}_1 - \mathbf{w}_1) = (\alpha\mathbf{I})(\beta\mathbf{I})^{-1}\mathbf{C}_1\mathbf{A}^{-1}\mathbf{w}_1$$

or

$$[\mathbf{I} - \mathbf{C}_{\alpha 1}(\beta\mathbf{I})^{-1} - \mathbf{C}_1\mathbf{A}^{-1}(\alpha\mathbf{I})(\beta\mathbf{I})^{-1}]\mathbf{u}_1 = [\mathbf{I} - \mathbf{C}_{\alpha 1}(\beta\mathbf{I})^{-1}]\mathbf{w}_1 \quad (62)$$

For the scheme of Figure 11, we have

$$\begin{aligned}\mathbf{u}_2 &= \mathbf{C}_2\mathbf{A}^{-1}\mathbf{u}_2 + \mathbf{C}_{\alpha 2}(\beta\mathbf{I})^{-1}\mathbf{y} + [\mathbf{I} - \mathbf{C}_{\alpha 1}(\beta\mathbf{I})^{-1}]\mathbf{w}_2 \\ &= \mathbf{C}_2\mathbf{A}^{-1}\mathbf{u}_2 + \mathbf{C}_{\alpha 2}(\beta\mathbf{I})^{-1}\mathbf{B}\mathbf{A}^{-1}\mathbf{u}_2 + [\mathbf{I} - \mathbf{C}_{\alpha 1}(\beta\mathbf{I})^{-1}]\mathbf{w}_2\end{aligned}$$

or

$$[\mathbf{I} - \mathbf{C}_2\mathbf{A}^{-1} - \mathbf{C}_{\alpha 2}(\beta\mathbf{I})^{-1}\mathbf{B}\mathbf{A}^{-1}]\mathbf{u}_2 = [\mathbf{I} - \mathbf{C}_{\alpha 1}(\beta\mathbf{I})^{-1}]\mathbf{w}_2 \quad (63)$$

Now note that if one uses (49), (52) and (61), (56) is equivalent to

$$\mathbf{C}_{\alpha 1}(\beta\mathbf{I})^{-1} + \mathbf{C}_1\mathbf{A}^{-1}(\alpha\mathbf{I})(\beta\mathbf{I})^{-1} = \mathbf{C}_2\mathbf{A}^{-1} + \mathbf{C}_{\alpha 2}(\beta\mathbf{I})^{-1}\mathbf{B}\mathbf{A}^{-1} \quad (64)$$

We conclude from (62) and (63) that if $\mathbf{w}_1$ and $\mathbf{w}_2$ are identical and are zero outside $[t_0, 0]$, then $\mathbf{u}_1$ and $\mathbf{u}_2$ are identical.

(d) Equality of the optimal controls

Suppose the initial state of the plant is $\mathbf{x}(0)$. Let $\mathbf{z}(t)$ be such that the state of $\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}\mathbf{u}(t)$ is driven from $\mathbf{x}(t_0) = 0$ to $\mathbf{x}(0)$ with $\mathbf{u}(t) = \mathbf{z}(t)$ on $[t_0, 0]$. Clearly, there exists a $\mathbf{w}_1(t)$ which is zero outside of $[t_0, 0]$ such that in Figure 10, $\mathbf{u}_1(t) = \mathbf{z}(t)$ over $[t_0, 0]$. Hence, $\mathbf{w}_1(t)$ sets up the right initial state on the plant. To the extent that states within the controller are reachable using $\mathbf{u}_1$ in Figure 10, we can set up certain initial states in the controller also. (Actually, a sufficient condition for all $[\mathbf{x}^T(0) \quad \mathbf{x}_{\alpha 1}^T(0)]^T$ to be reachable is that $\mathbf{K}_1^T\mathbf{K}_1$ be non-singular.)

Now suppose that $\mathbf{w}_2(t)$ in Figure 11 is set equal to $\mathbf{w}_1(t)$. This ensures that $\mathbf{u}_2(t) = \mathbf{u}_1(t)$, first over $[t_0, 0]$, which means that $\mathbf{x}_2(0) = \mathbf{x}_1(0)$, or the initial plant states are the same, and secondly over $[0, \infty)$, which means that the optimal controls are the same.

Notice that the required correspondence between $\mathbf{x}_{\alpha 1}(0)$ and $\hat{\mathbf{x}}_{\alpha 2}(0)$ noted in the theorem statement is an automatic consequence of the equality of $\mathbf{w}_1(\cdot)$ and $\mathbf{w}_2(\cdot)$.

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