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Dynamic Errors-in-variables Systems with Three Variables*

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Errors-in-variables identification problems where there are three measured complex variables can be solved in the static case, while only partial progress can be made in the dynamic case.

Key Words—Identification; errors-in-variables; linear systems; transfer functions.

Abstract—Errors-in-variables problems are considered for the case of three variables, where the underlying relations among the noise-free variables are linear, and covariance information for the noisy variables is available. The static problem where the variables are complex (which can arise when narrowband filtering is used) is analysed in detail. Some results are also presented for the case when the linear relations among the variables are defined by dynamic transfer functions.

1. INTRODUCTION

ERRORS-IN-VARIABLES identification problems are problems where all observed variables are contaminated by noise errors. Given noisy measurements of an n-vector process, one can ask such questions as: "How many independent (possibly linear) relations exist among the non-noisy components of the process?", and "What is the set of such linear relations for which the data are compatible?".

In the main, such questions have been posed under a collection of standing assumptions: stationarity of the underlying processes, linearity of the underlying relations, and availability of covariance data. For discussion of static problems (i.e. the underlying processes are white, and the linear relations are memoryless), see e.g. Kalman (1982). For discussion of a dynamic problem where $n = 2$, see for example, Anderson (1985) and for dynamic problems where the measured vector of dimension $2m$ has a prescribed partitioning into an m-dimensional "input" vector and an n-dimensional "output" vector, see Green and Anderson (1986). Problems where higher order statistics are used are described in Deistler (1986).

In this paper, we focus on the case $n = 3$. Our first concern is with the case of static, complex data. Equivalently, we can consider that the collected data comprise noise-contaminated measurements in a very narrow frequency band of underlying broadband data. (This situation is typical in acoustic signal processing in a marine environment.) We provide a condition for the existence of just one linear relation among the noise-free variables. Our second concern is to apply these ideas to wideband information, i.e. we are given the power spectrum matrix of a 3-vector process, and we seek to model the underlying relation among the underlying noise-free components of the process by causal transfer functions. In this connection, we present a necessary condition for the existence of a causal model, and we suggest how such a causal model might be found. We cannot, however, give an exact algorithm for its construction.

2. THE STATIC PROBLEM OF DIMENSION THREE

We begin by describing the problem set up. Let x_{it}, $i = 1, 2, 3$, $t = 1, 2, 3, \ldots,$ comprise noisy measurements of underlying noise-free zero mean processes $\hat{x}_{it}$, with all variables complex. (Such measurements will arise if, for example, x_{it} is a phasor vector associated with the centre frequency of a narrowband dynamic process, i.e. $x_{it} = X_{0i} \exp(j\omega_0 t)$, ω_0 known and X_{0i} a constant complex vector.) By stacking the data together to form a vector $x_t = [x_{1t}x_{2t}x_{3t}]'$, we obtain the 3×3 covariance matrix of the x-process, viz. $\Sigma = E[x_t x_t^*]$. (Here, the superscript * denotes transpose with complex conjugation. For a scalar, it denotes simply complex conjugation.) This is related to the 3×3 covariance matrix $\hat{\Sigma}$ of the $\hat{x}$-process by

$$\Sigma = \tilde{\Sigma} + \hat{\Sigma} \tag{2.1}$$

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with $\tilde{\Sigma}$ the noise covariance matrix. The noises on the three components are assumed uncorrelated. Thus, whereas, Σ, $\hat{\Sigma}$ are in general complex hermitian, $\tilde{\Sigma}$, being diagonal, is real. Of course, Σ, $\tilde{\Sigma}$ and $\hat{\Sigma}$ are all nonnegative definite. To avoid pathological problems we shall assume Σ is positive definite.

The existence of a linear relation among the components of the $\hat{x}$-process, i.e.

$$\alpha_1 \hat{x}_{1t} + \alpha_2 \hat{x}_{2t} + \alpha_3 \hat{x}_{3t} = 0 \quad \forall t \tag{2.2}$$

(with α_i, $i = 1, 2, 3$ complex and not all zero) is equivalent to the existence to nonzero kernel for the $\hat{\Sigma}$ matrix. If there are precisely two independent linear relations among the components of the $\hat{x}$-process, $\hat{\Sigma}$ necessarily has corank 2. The case where there are three independent relations is equivalent to $\hat{\Sigma} = 0$, Σ = diagonal. To avoid trivialities, let us assume Σ is not diagonal and focus on the two questions: "Given Σ, under what conditions is there a decomposition of the form (2.1), with $\tilde{\Sigma}$, $\hat{\Sigma}$ nonnegative definite hermitian, $\tilde{\Sigma}$ diagonal, such that $\hat{\Sigma}$ has rank 2 (rank 1)?".

It is at once clear that we can always find a decomposition in which $\hat{\Sigma}$ is singular: simply select $\tilde{\Sigma} = \lambda_{\min}(\Sigma) I$.

Let m^* denote the maximal number of linear relations among the $\hat{x}$-process entries, consistent with the data, i.e. the maximum corank which $\hat{\Sigma}$ can take among all decompositions. We have:

Theorem 2.1. Suppose $\Sigma = (\sigma_{ij})$ is positive definite, and $|\sigma_{ij}| \neq 0$ for every i,j pair. The following statements are equivalent:

(i) $m^* = 2$
(ii) $\sigma_{12}\sigma_{23}\sigma_{31}$ is real and positive, and $\sigma_{ii} \geq |\sigma_{ij}|\,|\sigma_{ik}|\,|\sigma_{jk}|^{-1}$ for all $i \neq j \neq k$
(iii) with $S = (s_{ij}) = \Sigma^{-1}$, $s_{12}s_{23}s_{31}$ is real and negative.

Remark. Before proving the result, we wish to indicate the connection between this result and the result of Kalman (1982) applicable when Σ is real. As shown in Kalman (1982), $m^* = 2$ corresponds to $s_{12}s_{23}s_{31} < 0$ and $m^* = 1$ corresponds to $s_{12}s_{23}s_{31} > 0$. Once one permits complex Σ, one then has to decide whether $\mathrm{Im}\,(s_{12}s_{23}s_{31}) \neq 0$ implies $m^* = 1$ or $m^* = 2$. The point of the theorem is that $\mathrm{Im}\,(s_{12}s_{23}s_{31}) \neq 0$ always implies $m^* = 1$.

Proof of theorem. (i)$\to$(ii) Suppose $m^* = 2$. Then there exists a decomposition (2.1) in which $\hat{\Sigma}$ has rank 1, so that $\hat{\sigma}_{ii}\hat{\sigma}_{jj} = |\sigma_{ij}|^2$, $i \neq j$. Consequently, $\hat{\Sigma}$ is uniquely determined by:

$$\hat{\sigma}_{ii} = \frac{|\sigma_{ij}|\,|\sigma_{ik}|}{|\sigma_{jk}|} \quad i \neq j \neq k \tag{2.3a}$$

$$\hat{\sigma}_{ij} = \sigma_{ij} \qquad i \neq j. \tag{2.3b}$$

The condition

$$\sigma_{ii} \geq \frac{|\sigma_{ij}|\,|\sigma_{ik}|}{|\sigma_{jk}|} \quad i \neq j \neq k \tag{2.4}$$

is then immediate. Further, since $\hat{\Sigma}$ is singular, we have

$$0 = \det \hat{\Sigma} = \hat{\sigma}_{11}\hat{\sigma}_{22}\hat{\sigma}_{33} - \hat{\sigma}_{11}|\sigma_{23}|^2 - \hat{\sigma}_{22}|\sigma_{13}|^2 - \hat{\sigma}_{33}|\sigma_{12}|^2 + 2\,\mathrm{Re}\,(\sigma_{12}\sigma_{23}\sigma_{31}). \tag{2.5}$$

Using (2.3) yields

$$|\sigma_{12}|\,|\sigma_{23}|\,|\sigma_{31}| = \mathrm{Re}\,(\sigma_{12}\sigma_{23}\sigma_{31}). \tag{2.6}$$

Equivalently, $\sigma_{12}\sigma_{23}\sigma_{31}$ is real and positive.
(ii)$\to$(i) Define $\hat{\Sigma}$ by (2.3). This ensures all 2×2 minors of $\hat{\Sigma}$ are zero. When $\det\hat{\Sigma}$ is evaluated and we use (2.6), there results $0 = \det\hat{\Sigma}$. Since $\hat{\sigma}_{ii} \geq 0$, $\hat{\Sigma}$ is nonnegative definite hermitian. Further, with $\sigma_{ii} \geq |\sigma_{ij}|\,|\sigma_{ik}|\,|\sigma_{jk}|^{-1}$ there holds $\sigma_{ii} \geq \hat{\sigma}_{ii}$, so that $\tilde{\Sigma} \geq 0$. Consequently, a decomposition (2.1) is exhibited with $m^* = 2$. (i)$\to$(iii) Observe that

$$\begin{aligned} s_{12}\det\Sigma &= \sigma_{13}\sigma_{32} - \sigma_{12}\sigma_{33} \\ &= \sigma_{13}\sigma_{32} - \sigma_{12}\hat{\sigma}_{33} - \sigma_{12}\tilde{\sigma}_{33} \\ &= -\sigma_{12}\tilde{\sigma}_{33} \quad \text{(because } \hat{\Sigma} \text{ has rank 1).} \end{aligned}$$

Similarly,

$$s_{23}\det\Sigma = -\sigma_{23}\tilde{\sigma}_{11}$$
$$s_{31}\det\Sigma = -\sigma_{31}\tilde{\sigma}_{22}$$

and so

$$s_{12}s_{23}s_{31}(\det\Sigma)^3 = -\sigma_{12}\sigma_{23}\sigma_{31}(\tilde{\sigma}_{11}\tilde{\sigma}_{22}\tilde{\sigma}_{33}).$$

Using (ii), it follows that $s_{12}s_{23}s_{31}$ is real and negative.
(iii)$\to$(i) Let φ_{ij} denote the phase of s_{ij}, and set

$$P = \mathrm{diag}\,[1, \exp(j\varphi_{12}), \exp(j\varphi_{13})].$$

Define $T = (t_{ij}) = PSP^*$. Observe that $t_{12} = s_{12}\exp(-j\varphi_{12}) = |s_{12}| > 0$, $t_{13} = s_{13}\exp(-i\varphi_{13}) = |s_{13}| > 0$. Also, (iii) implies $\varphi_{12} + \varphi_{23} + \varphi_{31} = \pi$ (mod 2π). Then $t_{23} = s_{23}\exp j(\varphi_{12} - \varphi_{13}) = s_{23}\exp j(-\varphi_{23} + \pi) = -|s_{23}|$. Hence T is a real positive definite matrix, in which t_{12}, t_{13} are positive, t_{23} is negative. By a main result of Kalman (1982), it follows that we can write $G = T^{-1}$ as

$$G = \tilde{G} + \hat{G}$$

where $\tilde{G}$ is diagonal, nonnegative definite, and $\hat{G}$ is real, symmetric, nonnegative definite, and of rank 1. Then

$$\Sigma = P^*GP = (P^*\tilde{G}P) + (P^*\hat{G}P)$$

gives a decomposition of Σ in the form of (2.1), where $\hat{\Sigma} = P^*\hat{G}P$ has rank 1, i.e. $m^* = 2$. This completes the proof.

We remark that in case $m^* = 2$, $\hat{\Sigma}$ is uniquely determinable, and as a consequence, the two independent linear relations among the data are essentially uniquely determined. On the other hand, when $m^* = 1$, there is no uniqueness concerning the linear relation. We can give a rather untidy parameterization in the following way.

Normalize the relation (2.2) by setting $\alpha_1 = 1$ (let us avoid consideration of nongeneric situations where $\alpha_1 = 0$). Then we have

$$(1 \ \alpha_2 \ \alpha_3)\begin{bmatrix} \sigma_{12} & \sigma_{13} \\ \hat{\sigma}_{22} & \sigma_{23} \\ \sigma_{32} & \hat{\sigma}_{33} \end{bmatrix} = 0. \tag{2.7}$$

In the generic situation, $\hat{\sigma}_{22}\hat{\sigma}_{33} - |\sigma_{23}|^2 > 0$, and so

$$\alpha_2 = \frac{-\sigma_{12}\hat{\sigma}_{33} + \sigma_{13}\sigma_{32}}{\hat{\sigma}_{22}\hat{\sigma}_{33} - |\sigma_{23}|^2} \tag{2.8a}$$

$$\alpha_3 = \frac{-\sigma_{13}\hat{\sigma}_{22} + \sigma_{12}\sigma_{23}}{\hat{\sigma}_{22}\hat{\sigma}_{33} - |\sigma_{23}|^2}. \tag{2.8b}$$

Thus the set of linear relations is defined by (2.8), when $\hat{\sigma}_{22}\hat{\sigma}_{33}$ satisfy the required constraints. These constraints are given in the following lemma.

Lemma 2.1. Let $0 \leqslant \hat{\Sigma} \leqslant \Sigma$, $\hat{\Sigma}$ singular, Σ nonsingular. Then

$$0 \leqslant \hat{\sigma}_{22} \leqslant \sigma_{22} \tag{2.9a}$$

$$0 \leqslant \hat{\sigma}_{33} \leqslant \sigma_{33} \tag{2.9b}$$

$$\hat{\sigma}_{22}\hat{\sigma}_{33} - |\sigma_{23}|^2 \geqslant 0 \tag{2.9c}$$

$$\left|\hat{\sigma}_{22} - \frac{|\sigma_{12}|^2}{\sigma_{11}}\right| \left|\hat{\sigma}_{33} - \frac{|\sigma_{13}|^2}{\sigma_{11}}\right| \geqslant \left|\sigma_{23} - \frac{\sigma_{12}^*\sigma_{31}^*}{\sigma_{11}}\right|^2 \tag{2.9d}$$

$$\hat{\sigma}_{22} \geqslant \frac{|\sigma_{12}|^2}{\sigma_{11}}, \quad \hat{\sigma}_{33} \geqslant \frac{|\sigma_{13}|^2}{\sigma_{11}}. \tag{2.9e}$$

Conversely, if these inequalities holds, we can find $\hat{\sigma}_{11}$ with $0 \leqslant \hat{\sigma}_{11} \leqslant \sigma_{11}$ such that $0 \leqslant \hat{\Sigma} \leqslant \Sigma$, $\hat{\Sigma}$ singular.

Proof. From $0 \leqslant \hat{\Sigma} \leqslant \Sigma$, (2.9a) through (2.9c) are immediate. Also, given $\hat{\Sigma} \geqslant 0$, it follows that $\bar{\Sigma} = \hat{\Sigma} + \operatorname{diag}(\sigma_{11} - \hat{\sigma}_{11}, 0, 0) \geqslant 0$, and so $\det \bar{\Sigma} \geqslant 0$. It can be verified that this is equivalent to (2.9d). Equations (2.9c) and (2.9e) express the fact that all 2×2 principal minors of $\bar{\Sigma}$ are nonnegative.

Conversely, suppose that (2.9) hold. These equations imply $\hat{\Sigma} + \operatorname{diag}(\sigma_{11} - \hat{\sigma}_{11}, 0, 0) \geqslant 0$ for all $\hat{\sigma}_{11}$ with $0 \leqslant \hat{\sigma}_{11} \leqslant \sigma_{11}$. If $\hat{\sigma}_{11} = \sigma_{11}$ would make $\hat{\Sigma}$ nonnegative definite and singular, make this choice of $\hat{\sigma}_{11}$. If $\hat{\sigma}_{11} = \sigma_{11}$ does not make $\hat{\Sigma}$ nonnegative definite, it must make $\hat{\Sigma}$ positive definite. Then there will be a choice of $\hat{\sigma}_{11} < \sigma_{11}$ for which $\hat{\Sigma}$ will be nonnegative definite and singular. Adopt this choice as the actual value of $\hat{\sigma}_{11}$. Then the requirements of the lemma are met.

There is an interesting corollary of these conditions, which ties back to the earlier theorem.

Corollary 2.1. Suppose $\Sigma = (\sigma_{ij})$ is positive definite and $|\sigma_{ij}| \neq 0$ for any i,j pair. Suppose $m^* = 1$. Then there is no linear relation defined by (2.8), (2.9) in which $\alpha_2 = 0$ or $\alpha_3 = 0$.

Proof. Suppose there were a linear relation with $\alpha_2 = 0$. Then (2.8) shows that $\hat{\sigma}_{33} = \sigma_{13}\sigma_{32}\sigma_{12}^{-1}$ whence $\sigma_{12}\sigma_{23}\sigma_{31}$ is real and positive and $\sigma_{33} \geqslant |\sigma_{13}||\sigma_{32}||\sigma_{12}|^{-1}$. Further, use of this value of $\hat{\sigma}_{33}$ and (2.9c) imply $\sigma_{22} \geqslant \hat{\sigma}_{22} \geqslant |\sigma_{23}|\,|\sigma_{21}|\,|\sigma_{23}|^{-1}$, while the second inequality of (2.9e) gives $\sigma_{11} \geqslant |\sigma_{13}|\,|\sigma_{12}|\,|\sigma_{23}|^{-1}$. Thus condition (ii) of the theorem holds, implying a contradiction to the assumption that $m^* = 1$. The case $\alpha_3 = 0$ is similar.

We shall conclude this section by making some remarks about the case of arbitrary n, m^*. Since an arbitrary $n \times n$ nonegative hermitian matrix $\hat{\Sigma}$ of rank $(n - m^*)$ has a unique factorization LL^*, where L is $n \times (n - m^*)$, and lower triangular with positive diagonal elements, the number of free parameters in $\hat{\Sigma}$, counting a complex number as two parameters, is the number of real nonzero elements in L, viz. $(n - m^*)$, plus twice the number of complex elements in L, viz. $2 \times \frac{1}{2}(n - 1 + m^*)(n - 1 - m^* + 1)$. The total number is $n^2 - m^{*2}$. Also, the number of free parameters in $\bar{\Sigma}$ is n. On the other hand, the total number of free parameters in $\Sigma = n + 2 \cdot \frac{1}{2}n(n-1) = n^2$ (adding the number of diagonal or real elements to the number of superdiagonal, complex elements). Hence the dimension of the solution set of $\hat{\Sigma}$, $\bar{\Sigma}$ such that $\Sigma = \hat{\Sigma} + \bar{\Sigma}$ is

$$\begin{aligned} d(n, m^*) &= (n^2 - m^{*2}) + n - n^2 \\ &= n - m^{*2}. \end{aligned} \tag{2.10}$$

In case $n = 3$, $m^* = 2$, $d(n, m^*)$ becomes negative. The interpretation is that this situation is not robust in the face of arbitrary small perturbations in the data, and indeed, an arbitrarily small perturbation in any of σ_{12}, σ_{23} or σ_{31} will change the argument of $\sigma_{12}\sigma_{23}\sigma_{31}$. If initially $m^*=2$, this will not be preserved under data perturbation.

Because $d(4,2)=0$, one should expect a finite number of decompositions. Also, one should expect robustness in the face of perturbations of data. Therefore, it is *not* likely that $n=4$, $m^*=2$ will be characterized by a requirement that certain quantities which are generically complex should take a real value.

3. THE DYNAMIC PROBLEM OF DIMENSION THREE

Let us now postulate that we have available the hermitian positive definite spectral matrix $\Sigma(e^{j\omega})$ for $\omega \in [0, 2\pi]$ of x_t, with the noise-free $\hat{x}_{1t}$ process resulting as

$$\hat{x}_{1t} = -w_2(z)\hat{x}_{2t} - w_3(z)\hat{x}_{3t}. \tag{3.1}$$

Here

$$w_i(z) = \sum_{j=-\infty}^{+\infty} W_{ij} z^j$$

and z can be thought of as a backward shift operator as well as a complex variable. In effect, $-\hat{x}_{1t}$ is the sum of the outputs of two dynamical systems, with inputs $\hat{x}_{2t}$, $\hat{x}_{3t}$ and transfer functions $w_2(z)$, $w_3(z)$. Moreover, the processes $\hat{x}_{it}$ and x_{it} are now real.

The asymmetric treatment of the three variables arises because we are postulating (or we are given the *a priori* information) that the $\hat{x}_{it}$ *process results from processing of the* $\hat{x}_{2t}$, $\hat{x}_{3t}$ processes, i.e. that the last two processes "cause" the first. Without *a priori* information this can be but a hypothesis to be considered along with other hypotheses; there are many possibilities: besides permuting 1, 2 and 3, we could hypothesize that all three processes were "caused" by a fourth, or that one process "caused" the other two.

Proceeding then with the assumption that (3.1) models the noiseless situation, the question of interest is: "To what extent can we identify the transfer functions from knowledge of $\Sigma(e^{j\omega})$, $\omega \in [0, 2\pi]$?". If we make no restriction at all about the causality of the $w_i(z)$ (in our notation, causality is equivalent to the requirement that $W_{ij}=0$ for all negative j), then in effect the ideas of the previous section give an answer to the problem. Provided that at each frequency $|\sigma_{ij}(e^{j\omega})| \neq 0$ for all i, j, and $s_{12}s_{23}s_{31}$ is not real and negative, we can choose values for $w_2(e^{j\omega})$, $w_3(e^{j\omega})$ in accordance with (2.8) and (2.9). (Of itself, this may not guarantee the existence of a Laurent series expansion characterizing the transfer function property. A continuity assumption will, however, guarantee the transfer function property.)

If, however, we wish to impose a causality constraint, then more can be said. Let us in fact require that

$$W_{ij} = 0 \text{ for } j<0, \quad \sum_{j=0}^{\infty} \rho^j |W_{ij}| < \infty \quad \text{for some } \rho > 1. \tag{3.2}$$

This ensures $w_i(z)$ is analytic in $|z|<\rho$, a very slightly stronger requirement than causality.

This analyticity then guarantees that $w_i(z)$ can only have a finite number of zeros in $|z| \leq 1$ (if there were an infinite number, there would be an accumulation point, at which analyticity would be lost). Moreover, since at each frequency $|\sigma_{ij}(e^{j\omega})| \neq 0$ and $s_{12}s_{23}s_{31}$ is not negative real, so that $m^*=1$ holds, Corollary 2.1 guarantees that $|w_2(e^{j\omega})|$ and $|w_3(e^{j\omega})|$ are never zero. Hence the principle of the argument applied to $w_2(e^{j\omega})$ and $w_3(e^{j\omega})$ shows that the change in argument as ω moves from 0 to 2π must be a nonnegative multiple of 2π. The first result of this section is that one can verify whether or not $w_2(e^{j\omega})$, $w_3(e^{j\omega})$ exist which have the correct change of argument, *without actually computing* $w_2(e^{j\omega})$, $w_3(e^{j\omega})$. Note that we are not verifying the existence of causal $w_2(e^{j\omega})$, $w_3(e^{j\omega})$, but only in effect checking a necessary condition for their existence.

Failure to satisfy the condition means that the hypothesis that (3.1) holds with causal $w_i(e^{j\omega})$ is a flawed hypothesis.

Theorem 3.1. Suppose that $\Sigma(e^{j\omega})$ is positive definite hermitian, $|\sigma_{ij}(e^{j\omega})| \neq 0$, $\forall\omega$, $\forall i \neq j$, and there exists no ω for which $S=\Sigma^{-1}$ has $s_{12}s_{23}s_{31}$ negative real (i.e. $m^*=1$). Consider any solution $w_2(e^{j\omega})$, $w_3(e^{j\omega})$ of the linear relation (2.8) for which $w_2(\cdot)$ and $w_3(\cdot)$ vary continuously in ω. Then the changes in $\arg w_2(e^{j\omega})$, $\arg w_3(e^{j\omega})$ over $[0, 2\pi]$ are computable from Σ, by a procedure set out in the proof of the theorem, and are independent of the particular $\tilde{\Sigma}$.

In order to prove this theorem, we require a preliminary lemma as follows.

Lemma 3.2. Assume the hypotheses of the theorem, and suppose that at some frequency, $\sigma_{12}\sigma_{23}\sigma_{31}$ is real. Then at this frequency, $\arg w_2 = \arg s_{12}$, $\arg w_3 = \arg s_{13}$.

Proof. Observe first that $s_{12}s_{23}s_{31}$ must be real. For $s_{12}\det\Sigma = \sigma_{13}\sigma_{32} - \sigma_{12}\sigma_{33}$, so that $\sigma_{21}s_{12}$ is real. Similarly, $s_{23}\det\Sigma = \sigma_{21}\sigma_{13} - \sigma_{11}\sigma_{23}$ and

$s_{31} \det \Sigma = \sigma_{21}\sigma_{32} - \sigma_{31}\sigma_{22}$, so that $\sigma_{32}s_{23}$ and $\sigma_{13}s_{31}$ are real. Thus the product $(\sigma_{21}s_{12})(\sigma_{32}s_{23})(\sigma_{13}s_{31})$ is real, hence $s_{12}s_{23}s_{31}$ is real. Since $m^* = 1$, the product $s_{12}s_{23}s_{31}$ must be positive. Let $\varphi_{12} = \arg s_{12}$, $\varphi_{13} = \arg s_{13}$, and let $P = \mathrm{diag}\,[1, \exp j\varphi_{12}, \exp j\varphi_{13}]$. Then $PSP^* = (|s_{ij}|)$. (The fact that the 2–3 entry is $|s_{23}|$ follows from the positivity of $s_{12}s_{23}s_{31}$.) Now $[1, w_2, w_3]$ defines a linear relation for the variables $\hat{x}_{1t}$, $\hat{x}_{2t}$, $\hat{x}_{3t}$ if and only if $[1, w_2 \exp(-j\varphi_{12}), w_3 \exp(-j\varphi_{13})]$ defines a linear relation for the variables $[\hat{x}_{1t}, \exp(j\varphi_{12})\hat{x}_{2t}, \exp(j\varphi_{13})\hat{x}_{3t}]$. The inverse of the covariance matrix for the noisy version of these latter variables is, as just proved, $(|s_{ij}|)$, and we know then from Kalman (1982) that $[1, w_2 \exp(-j\varphi_{12}), w_3 \exp(-j\varphi_{13})]$ must be a convex linear combination of the rows of this matrix. This means that $w_2 \exp(-j\varphi_{12})$ and $w_3 \exp(-j\varphi_{13})$ are positive, as required.

We turn now to the proof of the theorem, which contains within it a procedure for calculating the change in argument of w_2. Of course, w_3 is treated similarly.

Proof of theorem. In the following, it is important to keep in mind that $\sigma_{ij} = \sigma_{ji}^*$. From (2.8a) we have

$$\sigma_{12}^* w_2(\hat{\sigma}_{22}\hat{\sigma}_{33} - |\sigma_{23}|^2) = \sigma_{13}\sigma_{32}\sigma_{21} - |\sigma_{12}|^2\hat{\sigma}_{33}.$$

The argument given in proving Corollary 2.1 will also prove that $\hat{\sigma}_{22}\hat{\sigma}_{33} - |\sigma_{23}|^2$ is never zero on the unit circle. So computation of the change in $\arg w_2$ can proceed by computation of the change in $\arg(\sigma_{13}\sigma_{32}\sigma_{21} - |\sigma_{12}|^2\,\hat{\sigma}_{33})$. However, $\hat{\sigma}_{33}$ is unknown. Whenever $\sigma_{12}\sigma_{23}\sigma_{31}$ is real, we know that $\arg w_2 = \arg s_{12}$, and so $\arg(\sigma_{12}\sigma_{23}\sigma_{31} - |\sigma_{12}|^2\,\hat{\sigma}_{33})$ is known. In the vicinity of any frequency ω_0 at which $\sigma_{12}\sigma_{23}\sigma_{31}$ is real, $\mathrm{Im}(\sigma_{13}\sigma_{32}\sigma_{21})$ can be easily determined, as so we can determine the way in which $\arg(\sigma_{13}\sigma_{32}\sigma_{21} - |\sigma_{12}|^2\,\hat{\sigma}_{33})$ is changing as ω changes from $\omega_0 - \varepsilon$ to $\omega_0 + \varepsilon$ (increasing, decreasing, at a turning point). Thus everywhere that $\sigma_{12}\sigma_{23}\sigma_{31}$ is real, we know both the argument of $\sigma_{13}\sigma_{32}\sigma_{21} - |\sigma_{12}|^2\,\hat{\sigma}_{33}$ (mod 2π of course) and whether it is increasing, decreasing or at a turning point. Also at $\omega = 0$, 2π we necessarily have $\sigma_{12}\sigma_{23}\sigma_{31}$ real. From this knowledge alone, it is not hard to see that we can evaluate the change in $\arg(\sigma_{13}\sigma_{32}\sigma_{21} - |\sigma_{12}|^2\,\hat{\sigma}_{33})$ around the unit circle. This completes the proof of the theorem.

The actual determination of causal $w_2(e^{j\omega})$ and $w_3(e^{j\omega})$, assuming they exist, appears very difficult, and an exact solution escapes us. Nevertheless, we wish to indicate one possible approach. This approach incidentally suggests that the number of pairs of causal $w_2(e^{j\omega})$, $w_3(e^{j\omega})$ consistent with the data matrix Σ is likely to be finite.

As (2.7) suggests, our task is to solve the equation:

$$\begin{bmatrix} \hat{\sigma}_{22} & \sigma_{32} \\ \sigma_{23} & \hat{\sigma}_{33} \end{bmatrix} \begin{bmatrix} w_2 \\ w_3 \end{bmatrix} = - \begin{bmatrix} \sigma_{12} \\ \sigma_{13} \end{bmatrix} \tag{3.3}$$

given a causality constraint on w_2, w_3 and various constraints on $\hat{\sigma}_{22}$, $\hat{\sigma}_{33}$, as per (2.9). Choose a (large) integer N, and consider this equation at the frequencies $2\pi k/N$, $k = 0, 1, \ldots, N-1$. Set

$$w_i(e^{j\omega}) = u_i(e^{j\omega}) + jv_i(e^{j\omega}) \tag{3.4a}$$

$$\sigma_{ij}(e^{j\omega}) = g_{ij}(e^{j\omega}) + jh_{ij}(e^{j\omega}). \tag{3.4b}$$

Then (3.3) yields at each frequency

$$\begin{bmatrix} \hat{\sigma}_{22} & 0 & g_{23} & h_{23} \\ 0 & \hat{\sigma}_{33} & -h_{23} & g_{23} \\ g_{23} & -h_{23} & \hat{\sigma}_{22} & 0 \\ h_{23} & g_{23} & 0 & \hat{\sigma}_{33} \end{bmatrix} \begin{bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix} = - \begin{bmatrix} g_{12} \\ h_{12} \\ g_{13} \\ h_{13} \end{bmatrix} \tag{3.5}$$

and when we stack these equations together, for $\omega = 2\pi k/N$, $k = 0, 1, \ldots, N-1$ we obtain the real matrix equation

$$FU = G \tag{3.6}$$

with F a $4N \times 4N$ matrix that is block diagonal with N blocks of size 4×4, while U and G are $4N$ vectors.

Now let us invoke the constraint that w_2 is causal, by requiring that the sequence $v_i(e^{j2\pi k/N})$, $k = 0, 1, \ldots,$ be the Hilbert transform of the Discrete Fourier Transform Data $u_i(e^{j2\pi k/N})$, $k = 0, 1, \ldots$; see Oppenheim and Schafer (1975). This means that for some known matrix H, of dimension $2N \times N$, we can write:

$$U = \begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix} \begin{bmatrix} U_2 \\ U_3 \end{bmatrix} \tag{3.7}$$

where $U_2 = [u_2(e^{j0}) u_2(e^{j2\pi/N}) \cdots u_2(e^{j2\pi(n-1)/N})]'$ and U_3 similarly. Thus (3.6) becomes

$$F \begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix} \begin{bmatrix} U_2 \\ U_3 \end{bmatrix} = G. \tag{3.8}$$

In F, there are $2N$ adjustable quantities, viz. $\hat{\sigma}_{22}(e^{j2\pi k/N})$ and $\hat{\sigma}_{33}(e^{j2\pi k/N})$. Equation (3.8) is an equation for $4N$ unknowns in all, viz. the $2N$ adjustable quantities in F, and the $2N$ entries of U_2 and U_3. Other than by use of a standard nonlinear equation solver, it is not clear how (3.8) could be solved.

The fact that the number of scalar equations in (3.8) is the same as the number of unknowns suggest that there is likely to be only a finite number of solutions.

4. CONCLUSIONS

This paper has in a sense dealt with two different, though related, problems. The first is a complex version of a static errors-in-variables problem; a solution has been given for the case of three scalar variables. Attempts to extend the result to cope with more variables have so far been unsuccessful. In the second problem, we have searched for causal solutions of a dynamic three variable problem. A checkable necessary condition for a solution to exist has been exhibited, and an indication given of how a solution might (numerically) be obtained. A more explicit solution is not however available.

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