# Model reduction with time delay

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**Abstract:** The paper presents a procedure for approximation of a high-order system with rational transfer function by a low-order system with rational transfer function together with a pure time delay. The procedure introduces a delay into the system output and computes the low-order transfer function using truncation of a certain balanced realisation. Error bounds are obtained for both continuous-time and discrete-time cases and one part of the error bound is independent of the reduced-order rational transfer function so that it can be evaluated in advance, thereby aiding the selection of suitable delays. Some examples are presented to illustrate the method, which can be applied to multivariable as well as scalar plants.

## 1 Introduction

Many real physical systems contain pure time delays, although the delay values can be very small. It is, of course, common to model continuous-time linear systems with rational transfer functions. When delay is present in the underlying physical system, this often leads to a high order for the rational model, and it could be that far fewer parameters would appear in a model which permitted a time delay, with, at the same time, the model being a more accurate reflection of physical reality. Note that, in continuous time, the introduction of an irrational quantity $e^{-sT}$ into a transfer function may cause substantial analytical problems in design, but in discrete time this problem simply does not arise. This thinking motivates the problem considered in this paper, of approximating a high-order rational transfer function by a product of a low-order rational transfer function and $e^{-sT}$ (continuous time) or $z^{-k}$ (discrete time).

The idea of introducing time delay in simplifying a complex model is not a new one, and has appeared in some classical control textbooks, see e.g. Reference 1. Reference 2 proposes a procedure which approximates a high-order system by introducing a pure time delay into a low-order system to minimise a quality index. As shown in Reference 3, the computing effort of the method is highly dependent on the order of the original systems, and the reduced model is dependent on both the input signal and the weighting matrix in the quality index. Finally, no error bound is available for the method.

The purpose of this paper is to develop a new procedure based on truncation of balanced state-variable realisations, to approximate a high-order system by a low-order one with a pure time delay. This procedure only depends on the original system and the pure time delay chosen (although, as noted in the conclusions, we can if desired reflect input spectrum properties into the procedure). Also, error bounds for both the continuous-time case and the discrete-time case are derived in Section 4. Some examples are used to illustrate the application of the procedure to SISO (single input, single output) and SIMO (single input, multiple output) systems in Section 5. Finally, remarks and conclusions are given in Section 6.

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## 2 Continuous-time system approximation

We assume a linear time-variant and stable system with proper real rational transfer-function matrix $G(s)$. We restrict our consideration to a strictly proper $G(s)$ with distinct poles (we will discuss other cases in the final Section).

Let $\{\beta_1, \beta_2, \ldots, \beta_n\}$ be the poles of $G(s)$, and first suppose $G(s)$ is scalar. Then $G(s)$ can be represented in a partial fraction form:

$$G(s) = \sum_{i=1}^{n} \frac{\alpha_i}{s - \beta_i}, \quad \beta_i \neq \beta_j \text{ for } i \neq j, \tag{1}$$

where $\alpha_i (i = 1, 2, \ldots, n)$ are constant, and $\beta_i = \beta_j^*$ implies $\alpha_i = \alpha_j^*$.

Suppose we already know from other sources that the system with transfer function $G(s)$ is like a system with a certain time delay at the output, say $T$. We try to approximate $G(s)$ by an $r$th ($r < n$) order system comprising a cascade of a system with rational transfer function $\tilde{G}(s)$ together with a pure time delay $T$, i.e. we seek a $\tilde{G}(s)$ which makes

$$\|G(j\omega) - e^{-j\omega T}\tilde{G}(j\omega)\|_\infty = \|e^{j\omega T}G(j\omega) - \tilde{G}(j\omega)\|_\infty \tag{2}$$

small.

Now methods are available for reducing a high-order stable rational transfer function, e.g. truncation of balanced realisation [4–6], or Hankel norm reduction [7–11]. To exploit these methods, we first could find a (high-order) stable rational approximation of $e^{sT}G(s)$, and then approximate with a low-order transfer function this high-order approximation, to obtain what we want.

From eqn. 1, we see that

$$e^{sT}G(s) = \sum_{i=1}^{n} \frac{\alpha_i e^{sT}}{s - \beta_i} \tag{3}$$

For every $\alpha_i e^{sT}/(s - \beta_i)$, $i = 1, 2, \ldots, n$, we can construct a rational approximation by expanding $e^{sT}$ as a Taylor series around the point $s = \beta_i$ and using its first term, i.e.

$\alpha_i e^{\beta_i T}/(s - \beta_i)$. In this way, we can have a rational strictly proper approximation (we will discuss nonstrictly proper approximation in the final Section) of $e^{sT}G(s)$ as

$$\bar{G}(s) = \sum_{i=1}^{n} \frac{\alpha_i e^{\beta_i T}}{s - \beta_i}. \tag{4}$$

There is a second sense in which eqn. 4 can be thought of as an approximation of eqn. 3. Suppose one forms the inverse two-sided Laplace transform of eqn. 3, takes its causal ($t \geqslant 0$) part, and then takes the Laplace transform of this causal part. One obtains $\bar{G}(s)$, which can thus be thought of as a causal approximation to $G(s)$ which exactly matches its causal part.

We could also contemplate using other approximations of $e^{sT}$. Higher order Taylor series approximation will yield approximations of $e^{sT}G(s)$ which are not strictly proper. Padé approximations of $e^{sT}$ using all-pass function will have all unstable poles, just as Padé approximations of $e^{-sT}$ have all stable poles [12]. If we were to form a rational approximation of $e^{sT}G(s)$ using a Padé approximation of $e^{sT}$, and then take the strictly proper and stable part of the resulting transfer function, we would obtain a transfer function as $\bar{G}(s)$.

Now $\bar{G}(s)$ is stable, of $n$th order. We can now use a model reduction technique to reduce $\bar{G}(s)$ to an $r$th-order stable rational transfer function, say $\tilde{G}(s)$.

For the sake of easy implementation of the calculations and because we can easily obtain a meaningful error bound, we prefer to use the technique of truncating a balanced realisation [4–6]. Certainly, there are many other reduction methods that can be used [13–17]. But some of those may not be good for MIMO systems. Some of them have no meaningful error bounds. Overall then, the procedure advocated is as follows:

*Step 1:* Represent using partial fractions the transfer function $G(s)$ of the given system as

$$G(s) = \sum_{i=1}^{n} \frac{\alpha_i}{s - \beta_i}$$

*Step 2:* For the rational, stable and strictly proper approximation of $e^{sT}G(s)$,

$$\bar{G}(s) = \sum_{i=1}^{n} \frac{\alpha_i e^{\beta_i T}}{s - \beta_i}$$

find a balanced state-variable realisation and an $r$th-order truncated approximation of it, with transfer function $\tilde{G}(s)$. Finally, form the approximation of $G(s)$ with time delay as $e^{-sT}\tilde{G}(s)$.

For the MIMO case, we have a similar procedure. Suppose the system $G(s)$ has $l$ inputs and $m$ outputs. Then the partial fraction expression of $G(s)$ becomes

$$G(s) = \sum_{i=1}^{n} \frac{A_i}{s - \beta_i} \tag{5}$$

where $A_i$ ($i = 1, 2, \ldots, n$) are $m \times l$ constant matrices. If we assume $T_k$ as the time delay of the $k$th ($k = 1, 2, \ldots, m$) output of the system with transfer function $G(s)$, then eqn. 4 becomes

$$\bar{G}(s) = \sum_{i=1}^{n} \frac{\text{diag}\,\{e^{\beta_i T_1}, e^{\beta_i T_2}, \ldots, e^{\beta_i T_m}\}A_i}{s - \beta_i} \tag{6}$$

and we say

$$\text{diag}\,\{e^{-sT_1}, e^{-sT_2}, \ldots, e^{-sT_m}\}\tilde{G}(s) \tag{7}$$

is the approximation of the $G(s)$ with time delays, where $\tilde{G}(s)$ is the $r$th-order approximation of $\bar{G}(s)$ in eqn. 6, obtained by truncating a balanced state-variable realisation of $\bar{G}(s)$.

## 3 Discrete-time approach

We can easily carry over the same idea to discrete-time systems. We assume, given a linear time-invariant and stable system (SISO) with rational and causal transfer function $G(z) = D + C(zI - A)^{-1}B$, and a time delay $k\Delta$ (with $\Delta$ being the underlying sample period). As before, we need a rational strictly causal and stable (high-order) approximation $\bar{G}(z)$ of $z^kG(z)$ (we will discuss nonstrictly causal approximation in the final Section), and then a low-order approximation $\tilde{G}(z)$ of $\bar{G}(z)$, so that $G(z)$ is approximated by $z^{-k}\tilde{G}(z)$. We have the following algorithm:

*Step 1:* Calculate the Markov parameters of the system with transfer function $G(z)$:

$$M_o = D,$$
$$M_i = CA^{i-1}B, \quad i = 1, 2, 3, \ldots$$

As we know

$$G(z) = \sum_{i=0}^{\infty} M_i z^{-i}$$

*Step 2:* Find the strictly causal part of $z^kG(z)$, say $\bar{G}(z)$. Let $d(z)$ be the characteristic polynomial of $G(z)$, then $G(z) = N(z)/d(z)$, $N(z)$ is a polynomial, and let

$$F(z) = \sum_{i=0}^{k} M_i z^{k-i} \tag{8}$$

then

$$\bar{G}(z) = z^kG(z) - F(z) = \frac{z^kN(z) - d(z)F(z)}{d(z)}$$

It is easy to see that $\bar{G}(z)$ is strictly causal and stable (The use of a causal, but not strictly causal, approximation will be discussed later.)

*Step 3:* Find the approximation $\tilde{G}(z)$ of $\bar{G}(z)$ by truncation of a balanced state-variable realisation. Form the approximation of $G(z)$ with time delay as $z^{-k}\tilde{G}(z)$.

For the MIMO case (supposing we have $l$ inputs and $m$ outputs), we have the prescribed time delays for each output, say $k_i$ ($i = 1, 2, \ldots, m$). Then $F(z)$ becomes

$$F(z) = \{f_{ij}\}_{m \times l} \tag{9}$$

where

$$f_{ij} = \sum_{r=0}^{k_i} m_{ij}^{(r)} z^{k_i - r} \quad i = 1, 2, \ldots, m; j = 1, 2, \ldots, l \tag{10}$$

and $\{m_{ij}^{(r)}\}_{m \times l} = M_r$, $r = 0, 1, 2, \ldots$, the Markov parameter matrices of $G(z)$, i.e.

$$M_o = D$$
$$M_i = CA^{i-1}B, \quad i = 1, 2, \ldots.$$

The strictly causal part of diag $\{z^{k_1}, z^{k_2}, \ldots, z^{k_m}\}G(z)$ becomes

$$\bar{G}(z) = \text{diag}\,\{z^{k_1}, z^{k_2}, \ldots, z^{k_m}\}G(z) - F(z) = \frac{\text{diag}\,\{z^{k_1}, z^{k_2}, \ldots, z^{k_m}\}N(z) - d(z)F(z)}{d(z)} \tag{11}$$

where $N(z)$ is an $m \times l$ matrix polynomial, and $d(z)$ is the characteristic polynomial of $G(z)$, and $G(z) = N(z)/d(z)$. It is easy to see that $\bar{G}(z)$ is stable also. Then we obtain, as before, the approximation of $\bar{G}(z)$, say $\tilde{G}(z)$. Finally, we have the approximation of $G(z)$ with time delays as

$$\text{diag}\,\{z^{-k_1}, z^{-k_2}, \ldots, z^{-k_m}\}\tilde{G}(z) \tag{12}$$

We have now described the algorithm for approximating a high-order system with stable rational transfer function, by a low-order system with stable rational transfer function together with a pure time delay (or pure time delays) for both continuous-time and discrete-time cases and both SISO and MIMO system cases. However, two very important questions remain. When do these methods guarantee good approximations? More specifically, can we find error bounds for the approximations, using which we could judge the goodness of the approximations? Secondly, how should we choose the time delay $T$ for a continuous-time system, and the integer $k$ for a discrete-time system? We study the first question in the following Section, and obtain a number of results. Rather fewer results are obtained later in relation to the second question.

## 4 Error bounds

In this Section, we will derive several error bound expressions using the $L^\infty$ norm, which is defined for a continuous-time matrix transfer function $A(s)$ as

$$\|A(j\omega)\|_\infty = \sup_\omega \lambda_{max}^{1/2}\{A^*(j\omega)A(j\omega)\}$$

where $A^*(j\omega) = A'(-j\omega)$, and, for a discrete-time matrix transfer function $A(z)$, as

$$\|A(z)\|_\infty = \sup_{\omega \in [0, 2\pi]} \lambda_{max}^{1/2}\{A^*(z)A(z)\}$$

where $z = e^{j\omega}$, $A^*(e^{j\omega}) = A'(e^{-j\omega})$.

### *4.1 Continuous-time SISO system*

It is easy to see that

$$\|G(j\omega) - e^{-j\omega T}\tilde{G}(j\omega)\|_\infty \leqslant \|G(j\omega) - e^{-j\omega T}\bar{G}(j\omega)\|_\infty + \|e^{-j\omega T}\|_\infty \|\bar{G}(j\omega) - \tilde{G}(j\omega)\|_\infty$$

Error bounds known for the truncation of balanced realisation procedure yield [6, 9]

$$\|\bar{G}(j\omega) - \tilde{G}(j\omega)\|_\infty \leqslant 2\ \text{tr}\ \{\Sigma_2[\bar{G}(s)]\} \tag{13}$$

where $\Sigma_2[\bar{G}(s)]$ is the $(n-r) \times (n-r)$ diagonal matrix of the Hankel singular values corresponding to the least controllable and the least observable part of the system with transfer function $\bar{G}(s)$. As $\|e^{-j\omega T}\|_\infty \equiv 1$, we have

$$\|G(j\omega) - e^{-j\omega T}\tilde{G}(j\omega)\|_\infty \leqslant \|G(j\omega) - e^{-j\omega T}\bar{G}(j\omega)\|_\infty + 2\ \text{tr}\ \{\Sigma_2[\bar{G}(s)]\} \tag{14}$$

So we only need to find the error bound for the first term of the right-hand side of eqn. 14. Several characterisations are available.

*Theorem 4.1:* Suppose a continuous-time SISO linear time-invariant and stable system with rational strictly proper and stable transfer function $G(s)$ with distinct poles is given. Assume $T$ as the output time delay of an approximating system. Define the rational strictly proper approximation $\bar{G}(s)$ of $e^{sT}G(s)$ as in eqn. 4 and carry out the above reduction procedure. Then the first term of the reduction error bound in the right-hand side of eqn. 14 becomes

$$\|G(j\omega) - e^{-j\omega T}\bar{G}(j\omega)\|_\infty = \sup_\omega |R(\omega, T)|$$

where $R(\omega, T)$ is the time response of the original system with transfer function $G(s)$ evaluated at time $T$ due to the input signal $e^{j\omega t}$ applied from $t = 0$, assuming zero initial condition at $t = 0$.

*Proof:* From the assumption of theorem 4.1 and eqns. 1 and 3, we have

$$\begin{aligned}\|G(j\omega) &- e^{-j\omega T}\bar{G}(j\omega)\|_\infty \\ &= \left\| \sum_{i=1}^n \frac{\alpha_i}{j\omega - \beta_i}[1 - e^{(\beta_i - j\omega)T}] \right\|_\infty \\ &= \left\| \sum_{i=1}^n \int_0^T \alpha_i e^{\beta_i(t-T)} e^{j\omega(t-T)}\, dt \right\|_\infty \\ &= \left\| \int_0^T \left(\sum_{i=1}^n \alpha_i e^{\beta_i(T-t)}\right) e^{j\omega(t-T)}\, dt \right\|_\infty \\ &= \left\| \int_0^T g(T-t) e^{j\omega t}\, dt \right\|_\infty \end{aligned} \tag{15}$$

where $g(t) = \sum_{i=1}^n \alpha_i e^{\beta_i t}$ is just the impulse response associated with the transfer function $G(s)$. So for a SISO system with transfer function $G(s)$, we have

$$\begin{aligned}\|G(j\omega) - e^{-j\omega T}\bar{G}(j\omega)\|_\infty &= \left\| \int_0^T g(T-t) e^{j\omega t}\, dt \right\|_\infty \\ &= \sup_\omega |R(\omega, T)|\end{aligned}$$

with $R(\cdot, \cdot)$ as defined in the theorem statement.

Note that $\sup_\omega |R(\omega, T)|$ is an exact value for the first term of the error bound, for fixed $T$, determination of this exact value involves a one-dimensional searching problem. Cruder approximations which avoid one-dimensional searching can also be contemplated.

One direct corollary of Theorem 4.1 is the following:

*Corollary 4.1:* Assume the same hypotheses as theorem 4.1. Suppose also that the impulse response $g(t)$ of the original system with transfer function $G(s)$ does not change its sign on the interval $[0, T]$; then the first term of the error bound becomes

$$\|G(j\omega) - e^{-j\omega T}\bar{G}(j\omega)\|_\infty \leqslant |S(T)|$$

where $S(T)$ is the value of the step response of $G(s)$ at time $T$.

*Proof:* From eqn. 15, if $g(t)$ does not change its sign on the interval $[0, T]$, then

$$\left\| \int_0^T g(T-t) e^{j\omega t}\, dt \right\|_\infty \leqslant \left\| \int_0^T g(T-t)\, dt \right\|_\infty = |S(T)|$$

because $|e^{j\omega t}| \equiv 1$, and $S(0) = 0$.

Two more estimates can be provided without the restriction on the sign of $g(t)$ on the interval $[0, T]$:

*Corollary 4.2:* Assume the same hypotheses as theorem 4.1, then the first term of the error bound

$$\|G(j\omega) - e^{-j\omega T}\bar{G}(j\omega)\|_\infty \leqslant \left[\int_0^T g^2(t)\, dt \cdot T\right]^{1/2} \tag{16}$$

$$\leqslant \max_{t \in [0, T]} |g(t)| \cdot T \tag{17}$$

*Proof:* From eqn. 15, we obtain

$$\|G(j\omega) - e^{-j\omega T}\bar{G}(j\omega)\|_\infty = \left\| \int_0^T g(T-t)e^{j\omega t}\, dt \right\|_\infty$$
$$\leqslant \left[ \int_0^T g^2(t)\, dt \cdot T \right]^{1/2}$$
(by the Schwarz inequality)
$$\leqslant \max_{t \in [0, T]} |g(t)| \cdot T$$

### *4.2 Continuous-time MIMO system*

In this case, from eqns. 5 and 6, the error bound becomes

$$\|G(j\omega) - \text{diag}\,\{e^{-j\omega T_1}, e^{-j\omega T_2}, \ldots, e^{-j\omega T_m}\}\bar{G}(j\omega)\|_\infty$$
$$\leqslant \|G(j\omega) - \text{diag}\,\{e^{-j\omega T_1}, e^{-j\omega T_2}, \ldots, e^{-j\omega T_m}\}\bar{G}(j\omega)\|_\infty$$
$$+ \|\text{diag}\,\{e^{-j\omega T_1}, e^{-j\omega T_2}, \ldots, e^{-j\omega T_m}\}\|_\infty \|\bar{G}(j\omega) - \tilde{G}(j\omega)\|_\infty$$
$$\leqslant \|G(j\omega) - \text{diag}\,\{e^{-j\omega T_1}, e^{-j\omega T_2}, \ldots, e^{-j\omega T_m}\}\bar{G}(j\omega)\|_\infty + 2\,\text{tr}\,\{\Sigma_2[\bar{G}(s)]\} \quad (18)$$

As in the SISO case, we need only to find a bound for the first term in eqn. 18. Let

$$A_r = \{a_{ij}^{(r)}\}_{m \times l}, \quad r = 1, 2, \ldots, n,$$

and

$$\xi \triangleq \|G(j\omega) - \text{diag}\,\{e^{-j\omega T_1}, e^{-j\omega T_2}, \ldots, e^{-j\omega T_m}\}\bar{G}(j\omega)\|_\infty \quad (19)$$

Then

$$\xi = \left\| \begin{bmatrix} \sum_{r=1}^{n} a_{11}^{(r)} \dfrac{(1 - e^{(\beta_r - j\omega)T_1})}{j\omega - \beta_r} & \cdots & \sum_{r=1}^{n} a_{1l}^{(r)} \dfrac{(1 - e^{(\beta_r - j\omega)T_1})}{j\omega - \beta_r} \\ \vdots & & \vdots \\ \sum_{r=1}^{n} a_{m1}^{(r)} \dfrac{(1 - e^{(\beta_r - j\omega)T_m})}{j\omega - \beta_r} & \cdots & \sum_{r=1}^{n} a_{ml}^{(r)} \dfrac{(1 - e^{(\beta_r - j\omega)T_m})}{j\omega - \beta_r} \end{bmatrix} \right\|_\infty$$
$$= \left\| \begin{bmatrix} \int_0^{T_1} g_{11}(T_1 - t)e^{j\omega t}\, dt & \cdots & \int_0^{T_1} g_{1l}(T_1 - t)e^{j\omega t}\, dt \\ \vdots & & \vdots \\ \int_0^{T_m} g_{m1}(T_m - t)e^{j\omega t}\, dt & \cdots & \int_0^{T_m} g_{ml}(T_m - t)e^{j\omega t}\, dt \end{bmatrix} \right\|_\infty \quad (20)$$

Before considering the general MIMO system, we first deal with two special cases: SIMO and MISO systems. Proofs are straightforward modifications of those applying in the SISO case.

*Lemma 4.1:* Assume a continuous-time SIMO (one input, $m$ outputs) linear time-invariant and stable system with rational strictly proper and stable transfer-function matrix $G(s)$ with distinct poles. Assume $T_k$ ($k = 1, 2, \ldots, m$) as the $k$th output time delay of the approximating system. Define the rational strictly proper approximation $\bar{G}(s)$ of diag $\{e^{sT_1}, e^{sT_2}, \ldots, e^{sT_m}\}G(s)$ as in eqn. 6 (with $l = 1$) and carry out the above reduction procedure. Then the first term of the reduction error bound $\xi$ defined in eqn. 19 satisfies

$$\xi \leqslant \left[ \sum_{r=1}^{m} \int_0^{T_r} g_{r1}^2(t)\, dt \cdot T_r \right]^{1/2} \quad (21)$$
$$\leqslant \max_{t \in [0, T]} [\Gamma'(t)\Gamma(t)]^{1/2} \cdot T \quad (22)$$

where

$$T = \max_{r = 1, \ldots, m} [T_r]$$

and

$$\Gamma'(t) = [g_{11}(t), \ldots, g_{m1}(t)] = \left[ \sum_{r=1}^{n} a_{11}^{(r)} e^{\beta_r t}, \ldots, \sum_{r=1}^{n} a_{m1}^{(r)} e^{\beta_r t} \right]$$

is the impulse response associated with the transfer-function matrix $G(s)$.

For MISO systems, we have:

*Lemma 4.2:* Assume a continuous-time MISO ($l$ inputs, one output) linear time-invariant and stable system with rational strictly proper and stable transfer-function matrix $G(s)$ with distinct poles. Assume $T$ as the output time delay of the approximating system. Define the rational strictly proper approximation $\bar{G}(s)$ of $e^{sT}G(s)$ as in eqn. 6 (with $m = 1$) and carry out the above reduction procedure. Then the first term of the reduction error bound $\xi$ defined in eqn. 19 satisfies

$$\xi \leqslant \left[ \int_0^T \Gamma(t)\Gamma'(t)\, dt \cdot T \right]^{1/2} \quad (23)$$
$$\leqslant \max_{t \in [0, T]} [\Gamma(t)\Gamma'(t)]^{1/2} \cdot T \quad (24)$$

where

$$\Gamma(t) = [g_{11}(t), \ldots, g_{1l}(t)] = \left[ \sum_{r=1}^{n} a_{11}^{(r)} e^{\beta_r t}, \ldots, \sum_{r=1}^{n} a_{1l}^{(r)} e^{\beta_r t} \right]$$

is the impulse response associated with the transfer-function matrix $G(s)$.

To find an estimate of $\xi$ for general MIMO systems, we need the following lemma, which is the complex version of the Wittmeyer theorem [18, 19]:

*Lemma 4.3:* For any complex matrix $X = \{x_{ij}\}_{n \times m}$, we have

$$\lambda_{max}^{1/2}\{X^*X\} \leqslant \max_i \{|x_{ii}|\} + \left( \sum_{i \neq j} \sum |x_{ij}|^2 \right)^{1/2}$$

*Proof:* It is easy to see that

$$\lambda_{max}^{1/2}\{X^*X\} \leqslant \left( \sum_{i=1}^{n} \lambda_i\{X^*X\} \right)^{1/2}$$

that is

$$\lambda_{max}^{1/2}\{X^*X\} \leqslant \left(\sum_{i=1}^{n}\sum_{j=1}^{m}|x_{ij}|^2\right)^{1/2} \tag{25}$$

Let $X = A + B$, where $A = \{a_{ij}\}_{n\times m}$, $B = \{b_{ij}\}_{n\times m}$,

$$a_{ij} = \begin{cases} x_{ij}, & i=j \\ 0, & i\neq j \end{cases} \qquad b_{ij} = \begin{cases} 0, & i=j \\ x_{ij}, & i\neq j \end{cases}$$

From the triangle inequality we have

$$\lambda_{max}^{1/2}\{X^*X\} \leqslant \lambda_{max}^{1/2}\{A^*A\} + \lambda_{max}^{1/2}\{B^*B\} \tag{26}$$

But

$$\lambda_{max}^{1/2}\{A^*A\} = \max_i |x_{ii}|$$

and

$$\lambda_{max}^{1/2}\{B^*B\} \leqslant \left(\sum\sum |b_{ij}|^2\right)^{1/2} \quad \text{(from eqn. 2)}$$
$$= \left(\sum_{i\neq j}\sum |x_{ij}|^2\right)^{1/2}$$

Hence, eqn. 26 becomes

$$\lambda_{max}^{1/2}\{X^*X\} \leqslant \max_i |x_{ii}| + \left(\sum_{i\neq j}\sum |x_{ij}|^2\right)^{1/2}$$

This completes the proof.

Certainly, eqn. 20 gives the following theorem:

*Theorem 4.2:* Assume a continuous-time MIMO ($l$ inputs, $m$ outputs) linear time-invariant and stable system with rational strictly proper and stable transfer-function matrix $G(s)$ with distinct poles. Assume $T_k$ ($k = 1, 2, \ldots, m$) as the $k$th output time delay of the approximating system. Define the rational strictly proper approximation $\bar{G}(s)$ of diag $\{e^{sT_1}, e^{sT_2}, \ldots, e^{sT_m}\}G(s)$ as in eqn. 6 and carry out the above reduction procedure. Then the first term of the reduction error bound $\xi$ defined in eqn. 19 is equal to

$$\|R(T_1, T_2, \ldots, T_m, \omega)\|_\infty$$

where

$$R(T_1, T_2, \ldots, T_m, \omega) = \{R_{ij}(T_i, \omega)\}_{m\times l}$$

and $R_{ij}(T_i, \omega)$ is the $i$th output response of the original system with transfer-function matrix $G(s)$, evaluated at time $T_i$, due to the input signal $e^{j\omega t}$ applied to the $j$th input from $t = 0$ ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, l$), assuming zero initial condition at $t = 0$.

By applying lemma 4.3 to eqn. 20, we have

$$\xi \leqslant \max_i \left|\int_0^{T_i} g_{ii}(T_i - t)e^{j\omega t}\,dt\right| + \left[\sum_{i\neq k}\sum \left|\int_0^{T_i} g_{ik}(T_i - t)e^{j\omega t}\,dt\right|^2\right]^{1/2}$$

By using the Schwarz inequality, we obtain:

*Corollary 4.3:* Under the same hypotheses as theorem 4.2, the following holds

$$\xi \leqslant \max_i \left[\int_0^{T_i} g_{ii}^2(T_i - t)\,dt \cdot T_i\right]^{1/2} + \left[\sum_{i\neq k}\sum \int_0^{T_i} g_{ik}^2(T_i - t)\,dt \cdot T_i\right]^{1/2}$$
$$\leqslant \max_i \max_{t\in[0,T_i]} [|g_{ii}(t)|\,T_i] + \sum_{i\neq k}\sum \max_{t\in[0,T_i]} [|g_{ik}(t)| \quad T_i] \tag{27}$$

### 4.3 Discrete-time case

Now we consider the error bound for the discrete-time approach, general MIMO ($l$ inputs, $m$ outputs) case. We know that

$$\|G(z) - \text{diag}\{z^{-k_1}, z^{-k_2}, \ldots, z^{-k_m}\}\tilde{G}(z)\|_\infty$$
$$\leqslant \|G(z) - \text{diag}\{z^{-k_1}, \ldots, z^{-k_m}\}\bar{G}(z)\|_\infty + \|\text{diag}\{z^{-k_1}, \ldots, z^{-k_m}\}\|_\infty \|\bar{G}(z) - \tilde{G}(z)\|_\infty$$
$$\leqslant \|G(z) - \text{diag}\{z^{-k_1}, z^{-k_2}, \ldots, z^{-k_m}\}\bar{G}(z)\|_\infty + 2\,\text{tr}\{\Sigma_2[\bar{G}(z)]\} \tag{28a}$$

where $\bar{G}(z)$ was defined as in eqn. 11 and $\Sigma_2[\bar{G}(z)]$ is obtained similarly to the continuous-time case.

As in the continuous-time case, we only need to find estimates of the first term of eqn. 28*a*. From eqn. 11, we have

$$\|G(z) - \text{diag}\{z^{-k_1}, z^{-k_2}, \ldots, z^{-k_m}\}\bar{G}(z)\|_\infty$$
$$= \|G(z) - \text{diag}\{z^{-k_1}, \ldots, z^{-k_m}\} \times [\text{diag}\{z^{k_1}, \ldots, z^{k_m}\}G(z) - F(z)]\|_\infty$$
$$= \|F(z)\|_\infty = \sup_{\theta\in[0,2\pi]} \lambda_{max}^{1/2}[F^*(e^{j\theta})F(e^{j\theta})] \tag{28b}$$

where $F(z)$ was defined as in eqns. 9 and 10.

Approximations to this exact calculation can be obtained as in the continuous-time case. By using lemma 4.3, we have

$$\|F(z)\|_\infty \leqslant \sup_{\theta\in[0,2\pi]} \left\{\max_i |f_{ii}| + \left[\sum_{i\neq k}\sum |f_{ik}|^2\right]^{1/2}\right\}$$

By using the Schwarz inequality again with eqn. 10, we obtain:

*Theorem 4.3:* Assume a discrete-time MIMO ($l$ inputs, $m$ outputs) linear time-invariant and stable system with rational causal transfer-function matrix $G(z)$. Assume $k_iT$ ($i = 1, 2, \ldots, m$) as the $i$th output time delay of the approximating system (here $T$ is the sampling period, $k_i$ are integers). Define $F(z)$ as in eqns. 9 and 10 and carry out the reduction procedure in Section 3. Then the first term of the reduction error bound in eqn. 28*a* becomes

$$\|F(z)\|_\infty \leqslant \max_i \left[\sum_{r=0}^{k_i} (m_{ii}^{(r)})^2(k_i+1)\right]^{1/2} + \left\{\sum_{i\neq k}\sum \left[\sum_{r=0}^{k_i} (m_{ik}^{(r)})^2(k_i+1)\right]\right\}^{1/2} \tag{29a}$$

$$\|F(z)\|_\infty \leqslant \max_i \max_{r=0,\ldots,k_i} |m_{ii}^{(r)}|(k_i+1) + \sum_{i\neq k}\sum \max_{r=0,\ldots,k_i} |m_{ik}^{(r)}|(k_i+1) \tag{29b}$$

In the following Section, we shall examine how effective these bounds are with some numerical examples.

## 5 Examples

*Example 1:* This is an academic example from Reference 20. Consider a SISO continuous-time, 6th-order system whose state space matrices are given by

$$A = \begin{bmatrix} -0.5 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 10 & 0 & 0 & 0 \\ 0 & 0 & -20 & 10 & 0 & 0 \\ 0 & 0 & -18 & 0 & 10 & 0 \\ 0 & 0 & -8.4 & 0 & 0 & 10 \\ 0 & 0 & -1.68 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \\ -4 \\ 0 \\ -1.68 \\ 0 \end{bmatrix}$$
$$C = [1 \ \ 0 \ \ 0 \ \ 0 \ \ 0 \ \ 0] \qquad D = 0 \tag{30}$$

As explained in Reference 20, eqn. 30 is actually obtained as a rational approximation of the transfer function

$$G_0(s) = \frac{\exp(-s)}{(s + 0.5)(s + 2)} \qquad (31)$$

with a fourth-order Padé approximation being used for exp $(-s)$. We shall use our method to recover, from eqns. 30, a system comprising a cascade of a time delay of one second and a system of order less than 6. In the second order case, it is of some interest to compare the resulting approximation with eqn. 31.

We construct $\bar{G}(s)$ by using eqn. 3 with $T = 1$ and then

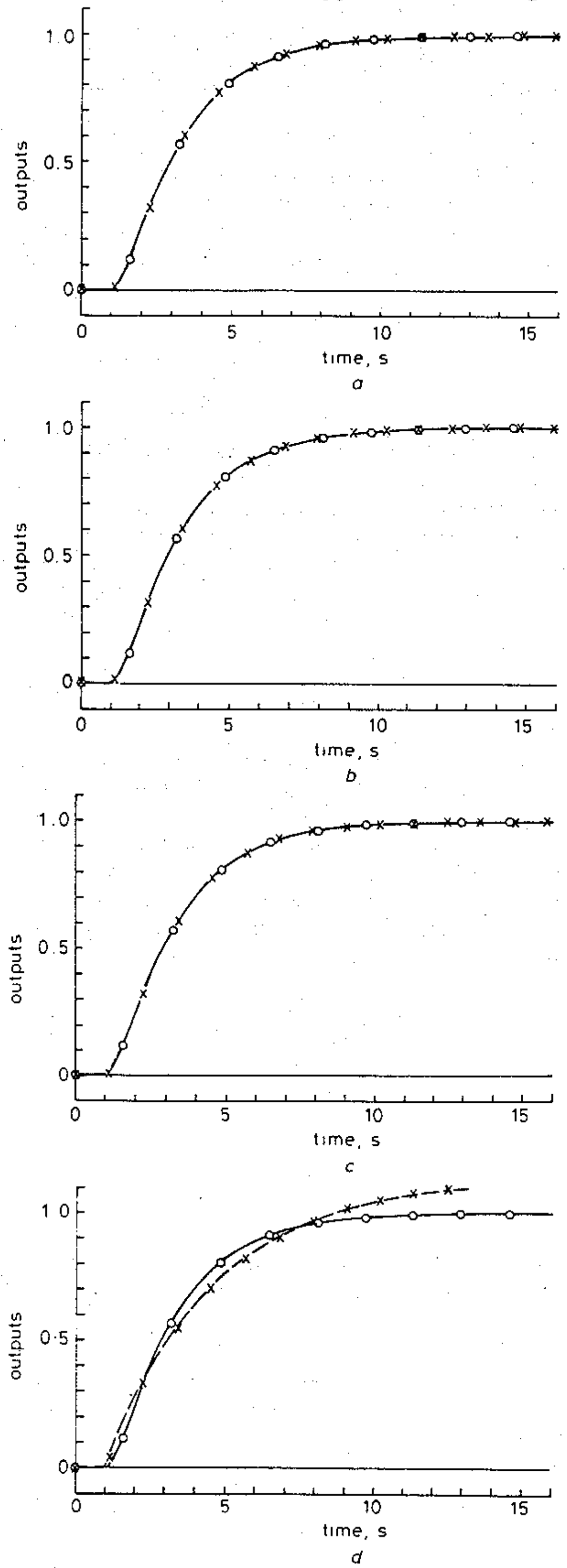


**Fig. 1** *Example 1 (delay T = 1.0) step responses*

—○— original
——— full order
- —×— reduced order
*a* 6th order to 4th
*b* 6th order to 3rd
*c* 6th order to 2nd
*d* 6th order to 1st

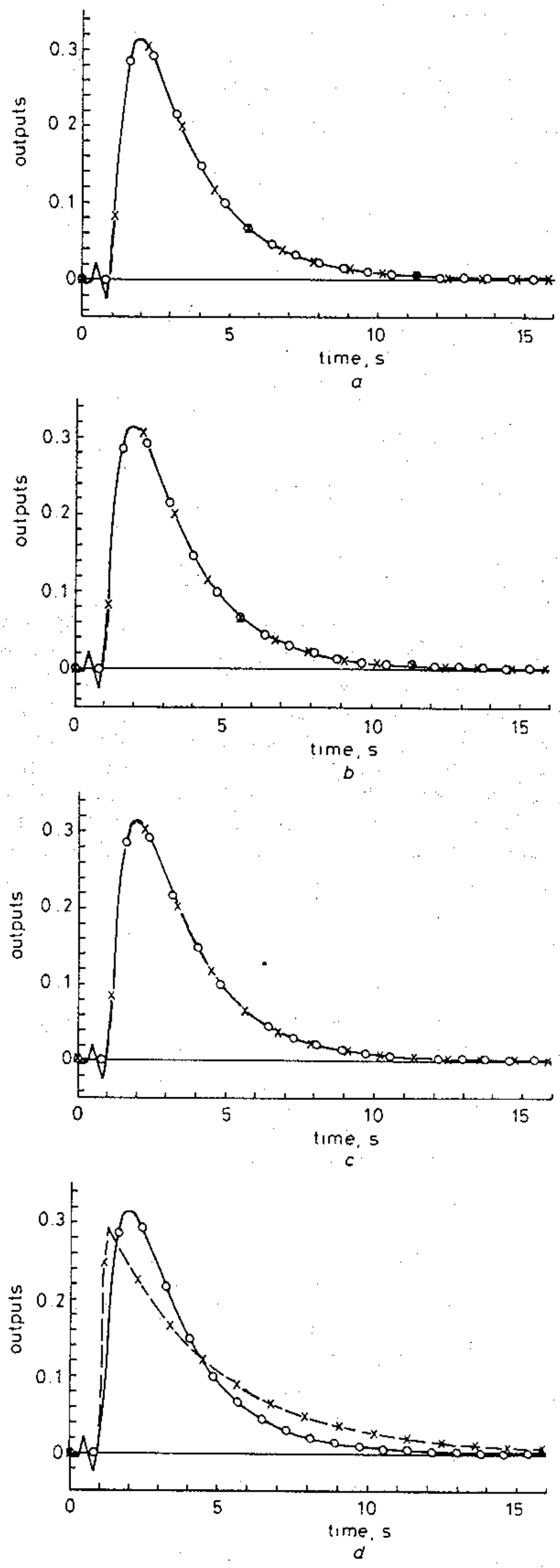


**Fig. 2** *Example 1 (delay T = 1.0) impulse responses*

—○— original
——— full order
- —×— reduced order
*a* 6th order to 4th
*b* 6th order to 3rd
*c* 6th order to 2nd
*d* 6th order to 1st

approximate $\bar{G}(s)$ with 4th-order, 3rd-order, 2nd-order, and 1st-order transfer functions. Then we introduce time delay $T = 1$ into these systems. We compare the step responses of these systems to the step responses of the full order eqn. 30 and of the original system eqn. 31 as shown in Fig. 1, the impulse responses as shown in Fig. 2, and the Bode plots as shown in Figs. 3 and 4.

For $T = 1$, we have

$$\sum[\bar{G}(s)] = \text{diag}\,\{0.569998,\ 0.706206 \times 10^{-1},\ 0.155776 \times 10^{-2},\ 0.435755 \times 10^{-3},\ 0.289636 \times 10^{-4},\ 0.935614 \times 10^{-6}\}$$

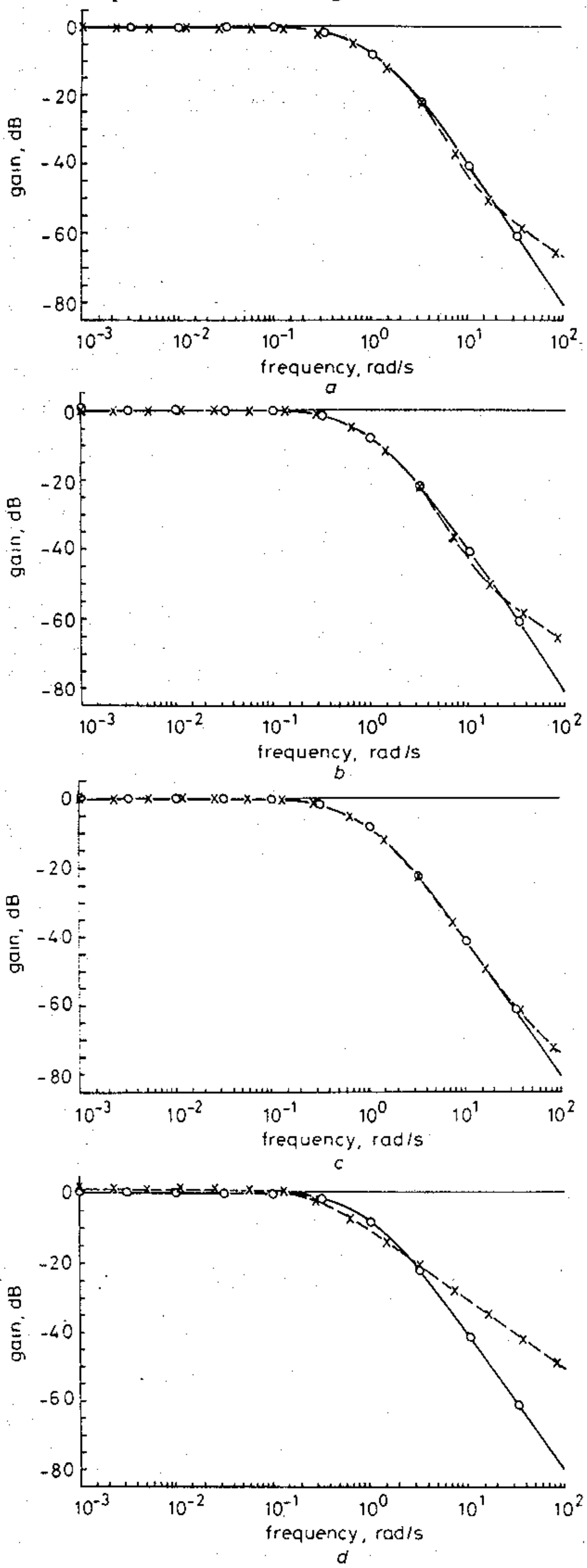


**Fig. 3** *Example 1 (delay T = 1.0) Bode plots*

—o— original
—— full order
- -x- reduced order
*a* 6th order to 4th
*b* 6th order to 3rd
*c* 6th order to 2nd
*d* 6th order to 1st

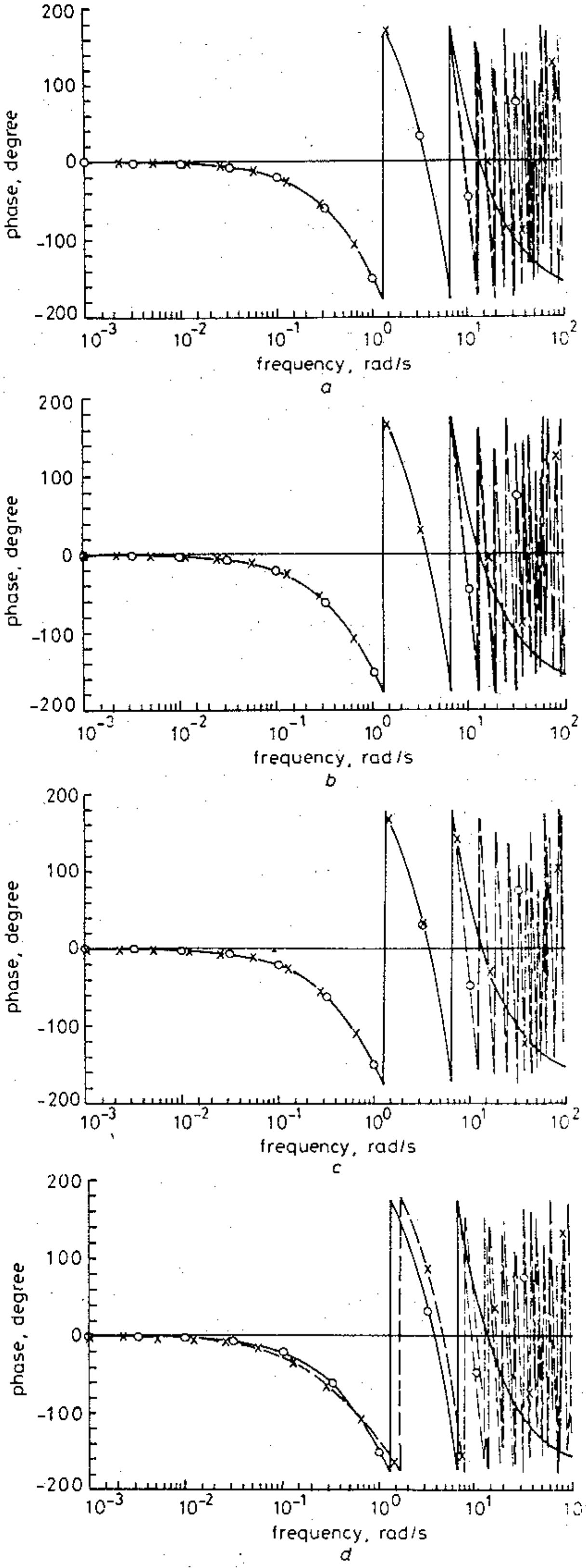


**Fig. 4** *Example 1 (delay T = 1.0) Bode plots*

---o--- original
—— full order
- -x- reduced order
*a* 6th order to 4th
*b* 6th order to 3rd
*c* 6th order to 2nd
*d* 6th order to 1st

From theorem 4.1, we obtain the first term of the error bound in condition 14 as 0.0113, which is the result of one-dimensional searching. By corollary 4.2, condition 17, we have an estimate of it, 0.0437. Hence, we obtained the whole error bounds (as in condition 14) for different reduced orders, as shown in Table 1. For comparison, the exact error of the reduction are also shown in Table 1.

**Table 1: Whole error bounds for example 1**

| Reduced order | Exact errors of the reduction with time delay | Whole error bounds using theorem 4.1 for the first term | Whole error bounds using corollary 4.2, condition 17, for the first term |
|---|---|---|---|
| 4th order | 0.0112433 | 0.0113598 | 0.0437598 |
| 3rd order | 0.0115345 | 0.0122313 | 0.0446313 |
| 2nd order | 0.0134479 | 0.0153468 | 0.0477468 |
| 1st order | 0.139999 | 0.156588 | 0.188988 |

For reduction leading to a 4th-, 3rd-, or 2nd-order rational part of the approximation, the maximum frequency domain error is about 1% of the DC gain.

In the 2nd-order case, we have the reduced-order system with time delay

$$G_2(s) = \exp(-s)\tilde{G}_2(s) = \frac{0.945886(0.0192753s + 1)\exp(-s)}{(s + 0.510075)(s + 1.856676)}.$$

Compare this with $G_o(s)$. Although one very far left stable zero ($z_1 = -51.8799$) has been introduced, the steady-state DC output error is less than 0.13%.

*Example 2:* This example is a continuous-time SIMO practical system. This 6th-order model represents the pitch plane dynamics of a flexible bodied rocket vehicle [21]. The physical significance of the state variables can be found in Reference 21. The model is as follows:

$$A = \begin{bmatrix} -0.21053 & -0.10526 & -0.0007378 & 0.0 & 0.0706 & 0.0 \\ 1.0 & -0.03537 & -0.000118 & 0.0 & 0.0004 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & -605.16 & -4.92 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & -3906.25 & -12.5 \end{bmatrix}$$

$$B = [-7.211, -0.05232, 0.0, 794.7, 0.0, -448.5]^T$$

$$C = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.000334 & 0.0 & -0.007728 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix} \quad D = \begin{bmatrix} 0.0 \\ 0.0 \end{bmatrix}$$

By studying the step response of this system, we choose no time delay ($T_1 = 0$) for the first output and time delay $T_2 = 0.31$ for the second output. By using our procedure, we reduce this system to 5th-order, 4th-order, 3rd-order and 2nd-order systems with time delays [0, 0.31], and compare the step responses, the impulse responses and the Bode plots of these systems, as shown in Figs. 5 to 12.

By using theorem 4.2, we have that the first term of the error bound is equal to 0.3541. Using corollary 4.3, condition 28*b*, we obtain an estimate of the first term of the error bound to be 0.6829. On the other hand, we have

$$\sum[\tilde{G}] = \{62.6091, 32.4137, 0.138713, 0.136868, 0.026611, 0.025156\}$$

Hence, we obtain the whole error bounds as shown in Table 2.

**Table 2: Whole error bounds for example 2**

| Reduced order | Exact errors of the reduction with time delay | Whole error bounds using theorem 4.2 for the first term | Whole error bounds using Corollary 4.3, condition 28*b*, for the first term |
|---|---|---|---|
| 5th order | 0.37816 | 0.404412 | 0.733212 |
| 4th order | 0.356006 | 0.457634 | 0.786434 |
| 3rd order | 0.578838 | 0.73137 | 1.06017 |
| 2nd order | 0.354549 | 1.00879 | 1.33759 |

**Table 3: Whole error bounds for example 3**

| Reduced order | Exact errors of the reduction with time delay | Error bounds using theorem 4.3 condition 29*a*, for the first term | Error bounds using theorem 4.3, condition 29*b*, for the first term term |
|---|---|---|---|
| 4th order | 0.0105734 | 0.0172945 | 0.0286719 |
| 3rd order | 0.0107476 | 0.0181531 | 0.0295305 |
| 2nd order | 0.0126851 | 0.0222473 | 0.0336247 |
| 1st order | 0.137148 | 0.177768 | 0.189145 |

*Example 3:* Now we consider the discrete-time case. For comparison, we discretise the full-order system eqns. 30 in example 1 by taking the sampling period as 0.1 s. Hence, the time delay is now $k = 10$ and the first 11 Markov parameters of the discretised full-order system are

$$M_0 = 0.0$$
$$M_1 = 7.92073 \times 10^{-4} \qquad M_6 = 5.21657 \times 10^{-4}$$
$$M_2 = -1.64536 \times 10^{-3} \qquad M_7 = -1.66341 \times 10^{-3}$$
$$M_3 = -8.87702 \times 10^{-4} \qquad M_8 = -2.60109 \times 10^{-3}$$
$$M_4 = 1.53347 \times 10^{-3} \qquad M_9 = -1.33466 \times 10^{-3}$$
$$M_5 = 2.13556 \times 10^{-3} \qquad M_{10} = 2.09335 \times 10^{-3}$$

The first term of the error bound of eqn. 29*a* of theorem 4.3 is 0.0172346, while the looser bound of eqn. 29*b* is 0.028612, and we have

$$\sum[\tilde{G}(z)] = \mathrm{diag}\{0.577714, 0.777601 \times 10^{-1}, 0.204711 \times 10^{-2}, 0.429298 \times 10^{-3}, 0.290753 \times 10^{-4}, 0.867521 \times 10^{-6}\}.$$

Hence, the error bounds of the discretised full-order system and the reduced-order systems are shown in Table 3. The comparisons of the performances are shown in Figs. 13 to 16.

*Example 4:* Consider the discrete-time transfer function [17]

$$G(z) = \frac{0.00484(z^4 - 0.492z^3 - 0.0261z^2 + 0.974z - 0.348)}{1.2184z^5 - 3.9926z^4 + 5.9024z^3 - 5.1692z^2 + 2.5876z - 0.5403}$$

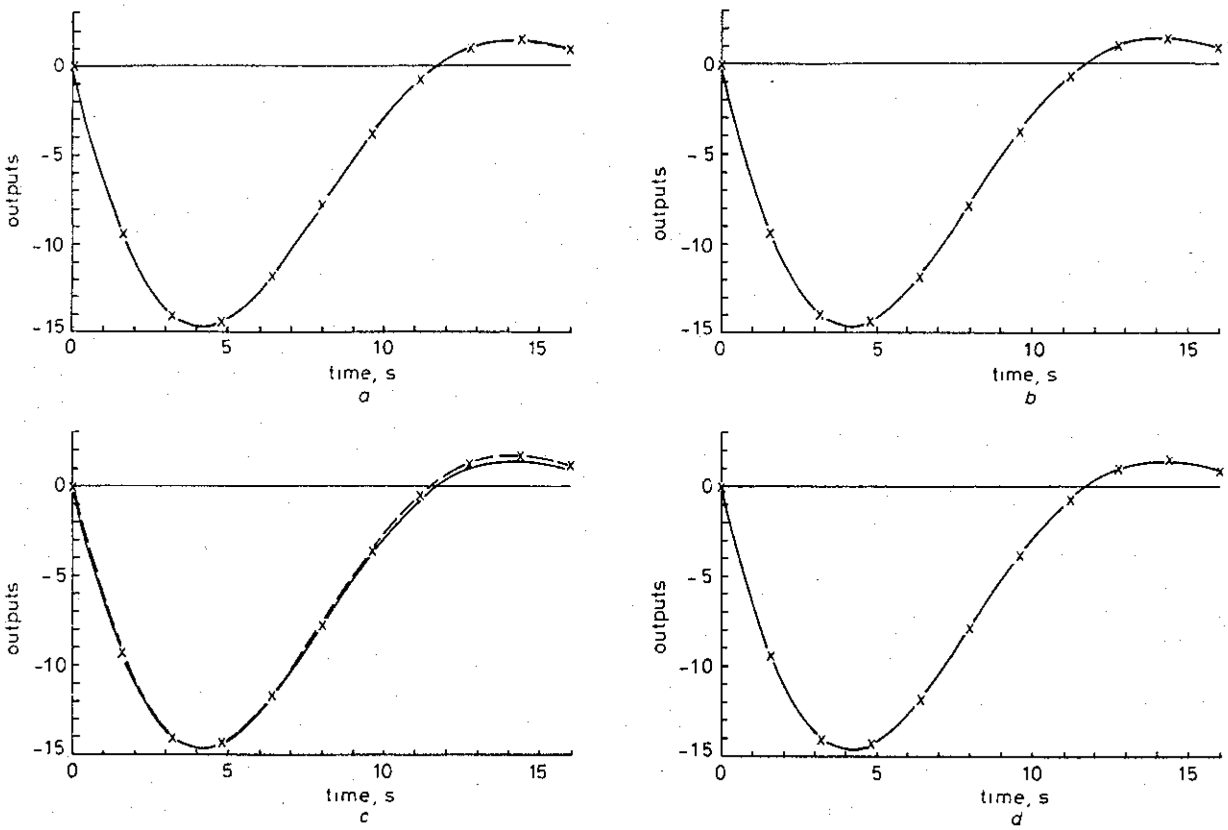


**Fig. 5** *Example 2 step responses (output 1, $T_1 = 0$)*
——— full order
– –×– reduced order
*a* 6th order to 5th
*b* 6th order to 4th
*c* 6th order to 3rd
*d* 6th order to 2nd

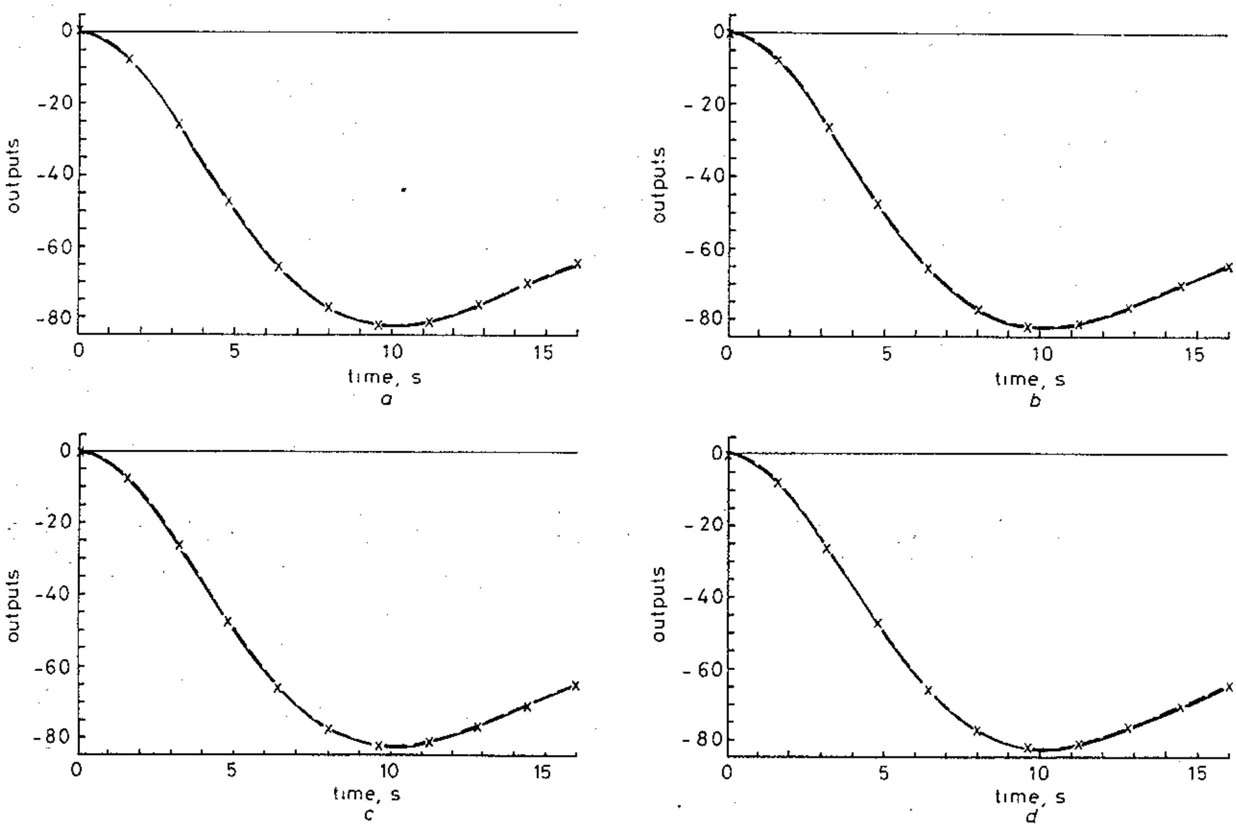


**Fig. 6** *Example 2 step responses (output 2, $T_2 = 0.31$)*
——— full order
- × - reduced order
*a* 6th order to 5th
*b* 6th order to 4th
*c* 6th order to 3rd
*d* 6th order to 2nd

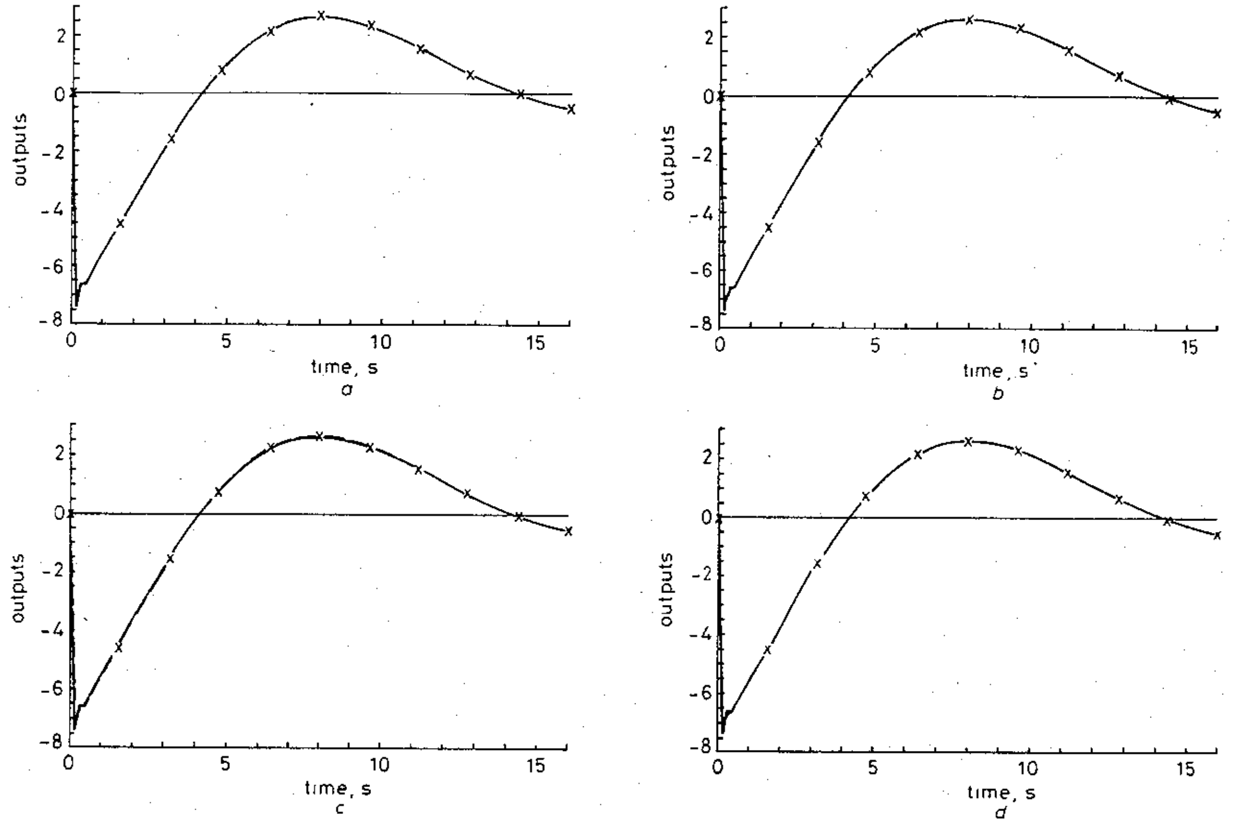


**Fig. 7** *Example 2 impulse responses (output 1, $T_1 = 0$)*
——— full order
- -×- reduced order
*a* 6th order to 5th
*b* 6th order to 4th
*c* 6th order to 3rd
*d* 6th order to 2nd

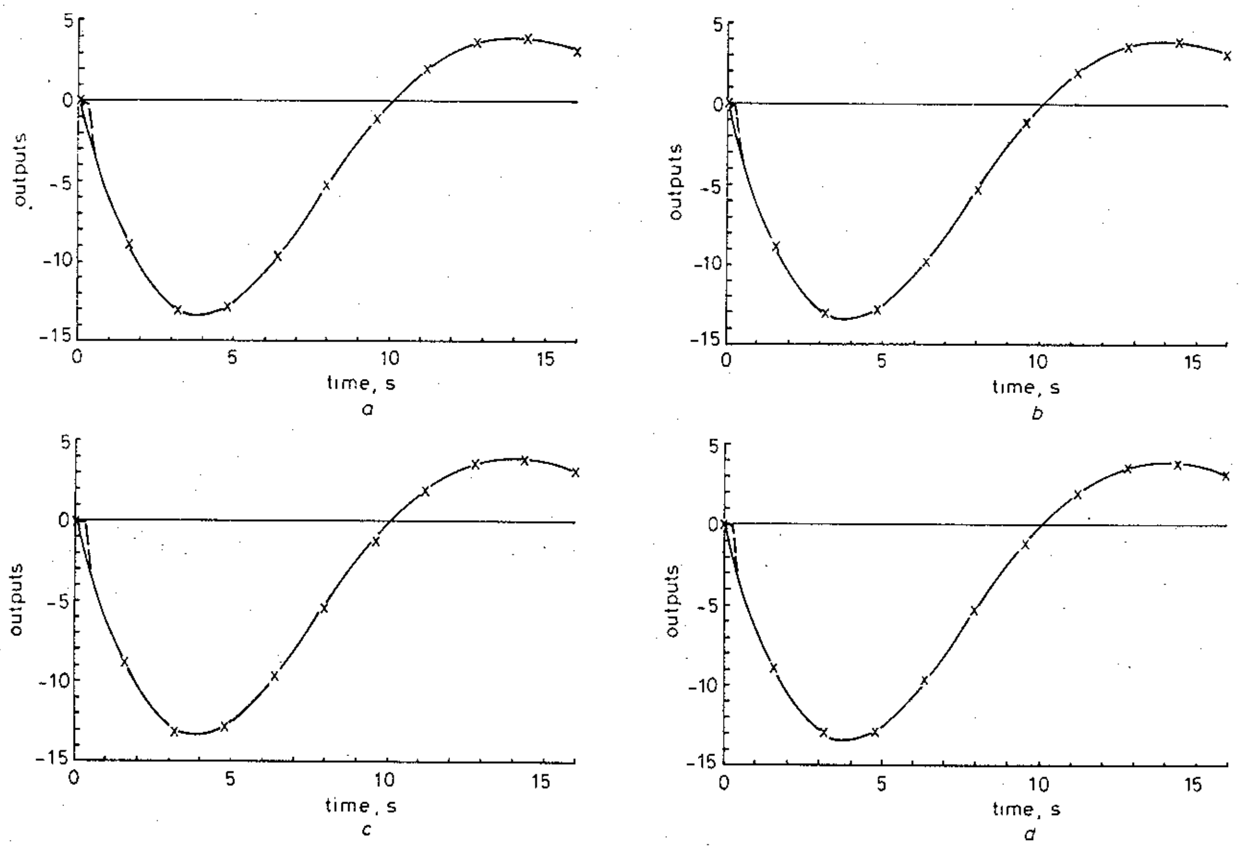


**Fig. 8** *Example 2 impulse responses (output 2, $T_2 = 0.31$)*
——— full order
- -×- reduced order
*a* 6th order to 5th
*b* 6th order to 4th
*c* 6th order to 3rd
*d* 6th order to 2nd

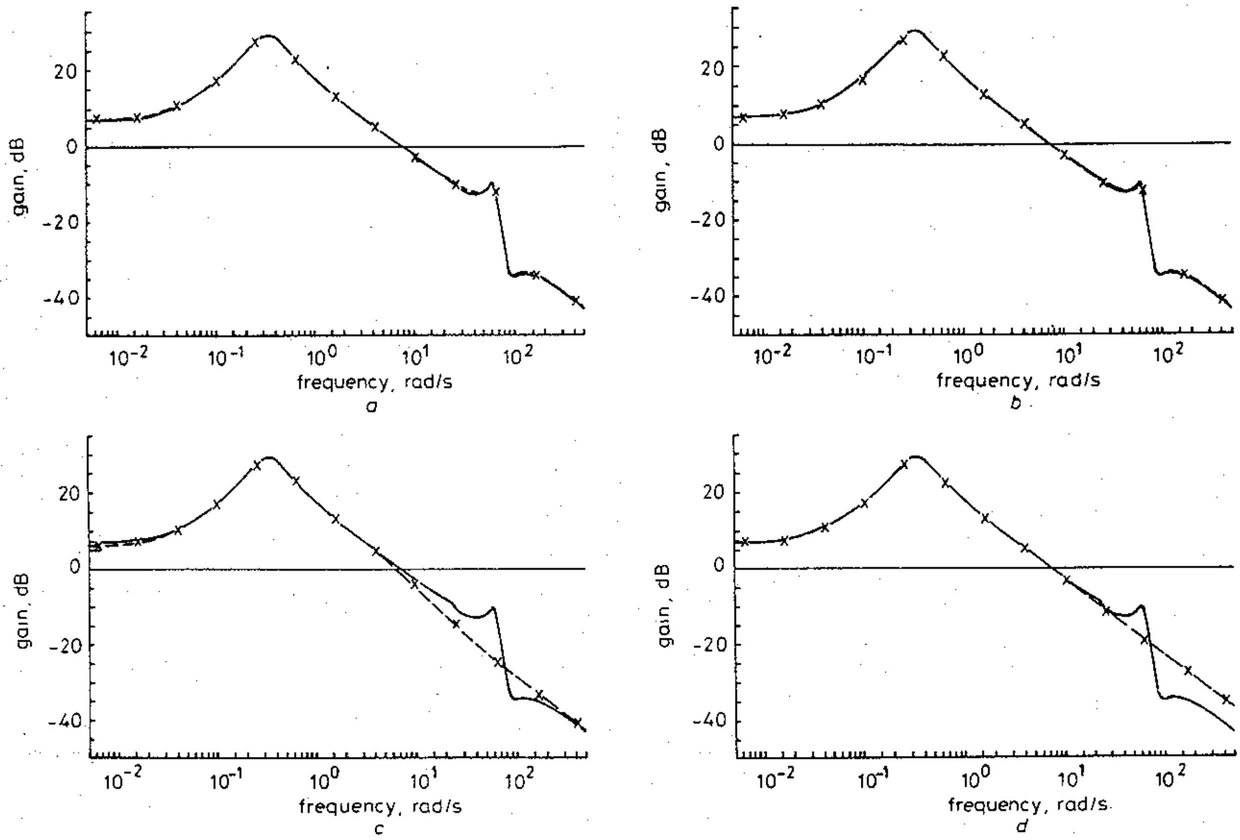


**Fig. 9** *Example 2 Bode plots (gain $-1$, $T_1 = 0$)*
——— full order
– –×– reduced order
*a* 6th order to 5th
*b* 6th order to 4th
*c* 6th order to 3rd
*d* 6th order to 2nd

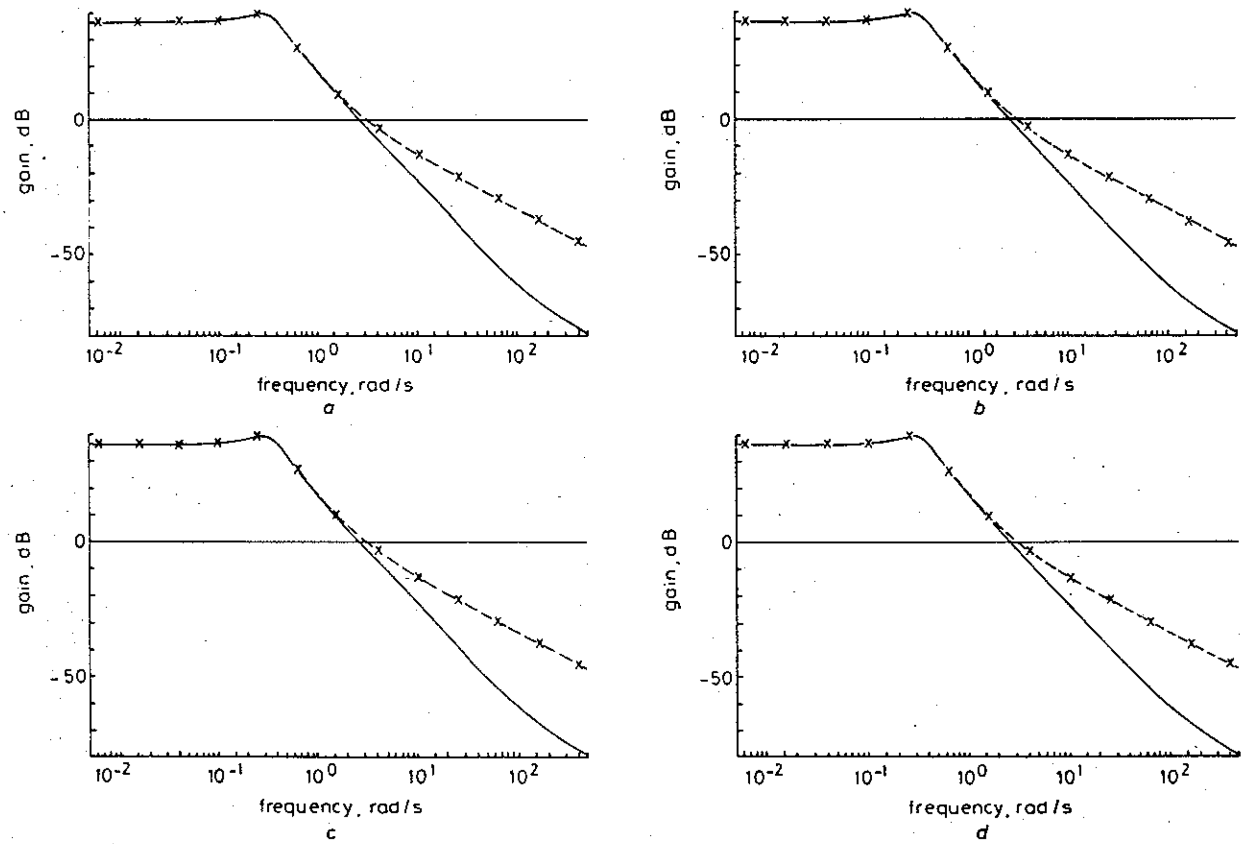


**Fig. 10** *Example 2 Bode plots (gain $-2$, $T_2 = 0.31$)*
——— full order
×– – reduced order
*a* 6th order to 5th
*b* 6th order to 4th
*c* 6th order to 3rd
*d* 6th order to 2nd

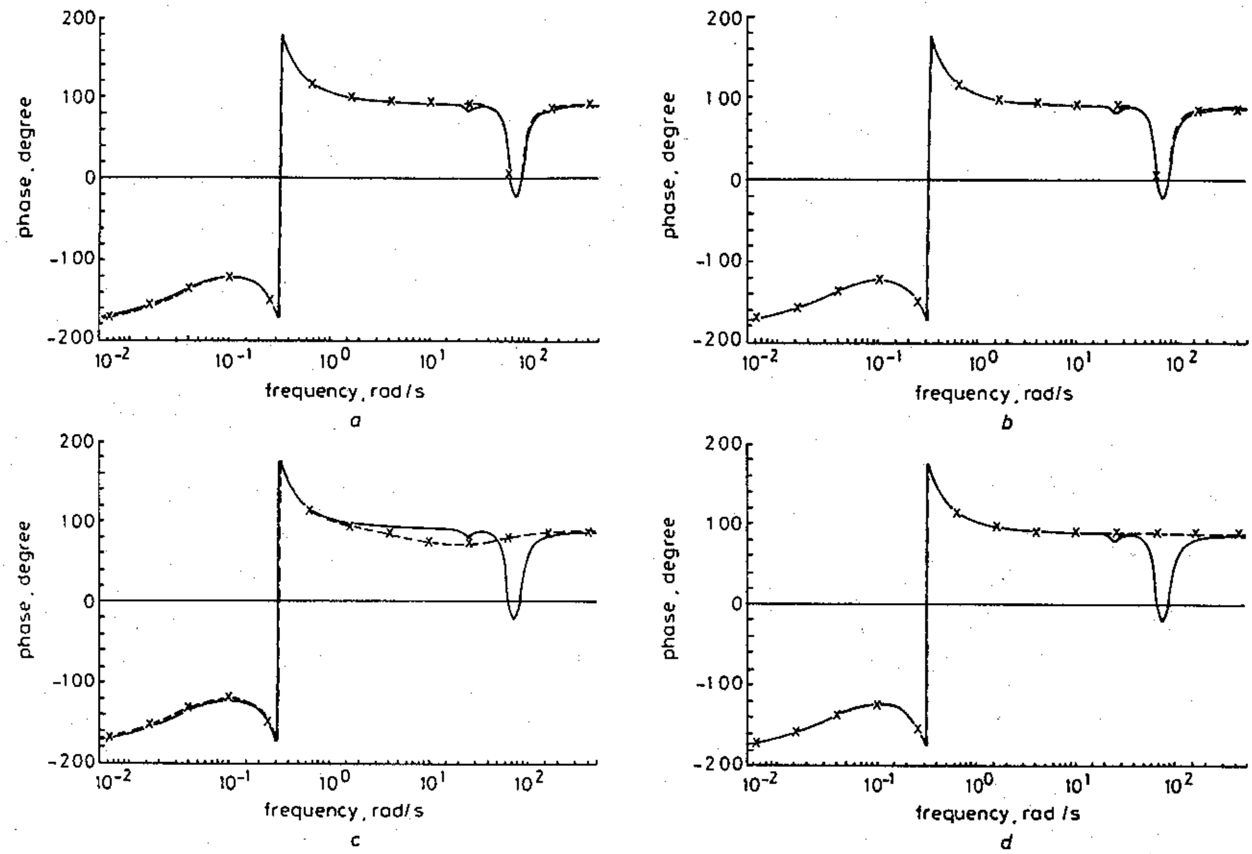


**Fig. 11** *Example 2 Bode plots (phase −1, $T_1 = 0$)*
——— full order
– –×– reduced order
*a* 6th order to 5th
*b* 6th order to 4th
*c* 6th order to 3rd
*d* 6th order to 2nd

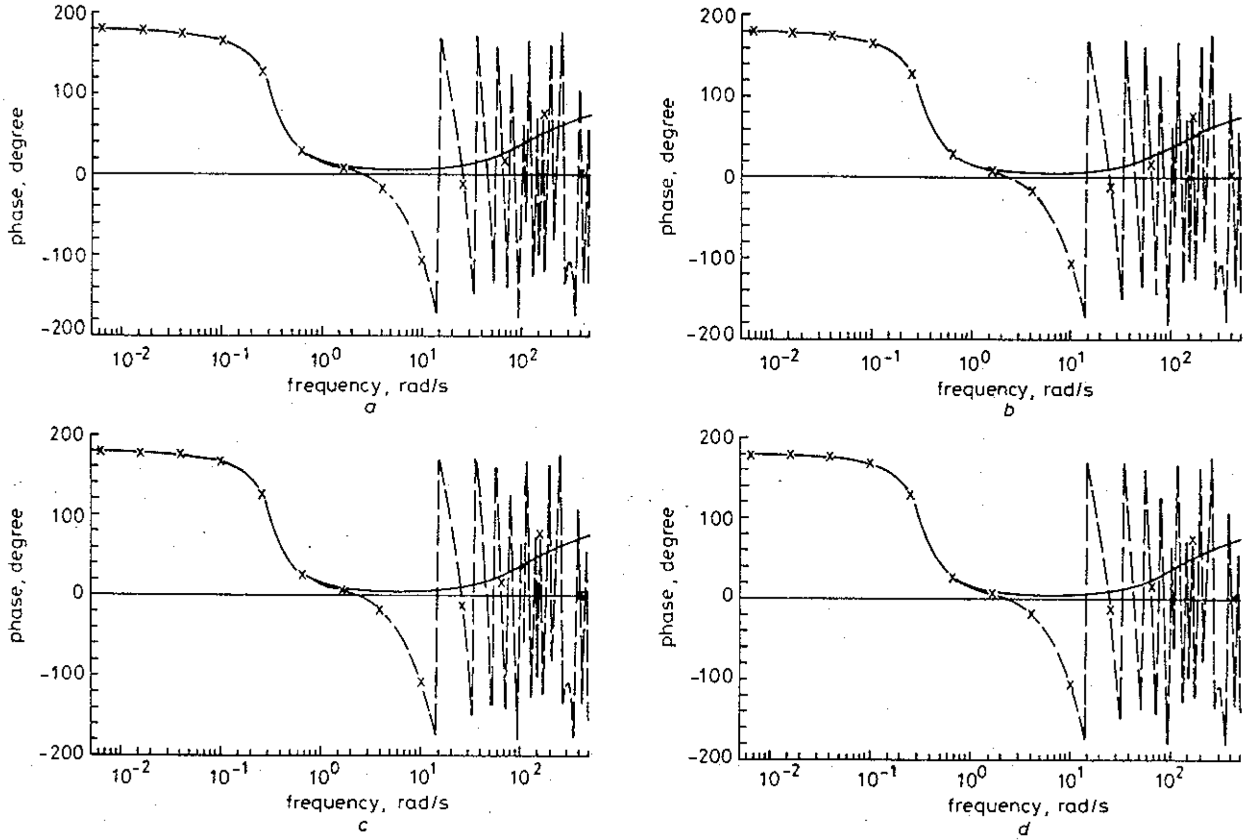


**Fig. 12** *Example 2 Bode plots (phase −2, $T_2 = 0.31$)*
· full order
× reduced order
*a* 6th order to 5th
*b* 6th order to 4th
*c* 6th order to 3rd
*d* 6th order to 2nd

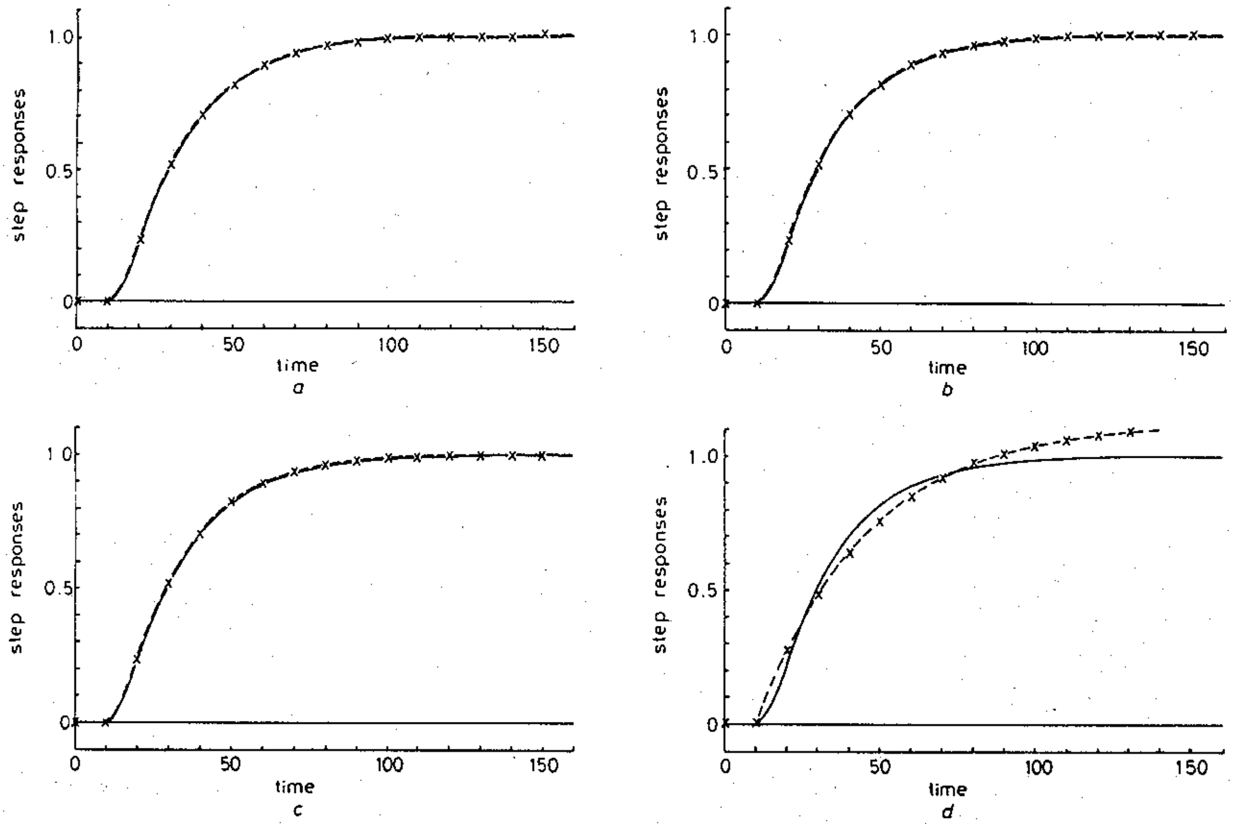


**Fig. 13** *Example 3 step responses (sampling time = 0.1, k = 10)*
—— full order
- –×– reduced order
*a* 6th order to 4th
*b* 6th order to 3rd
*c* 6th order to 2nd
*d* 6th order to 1st

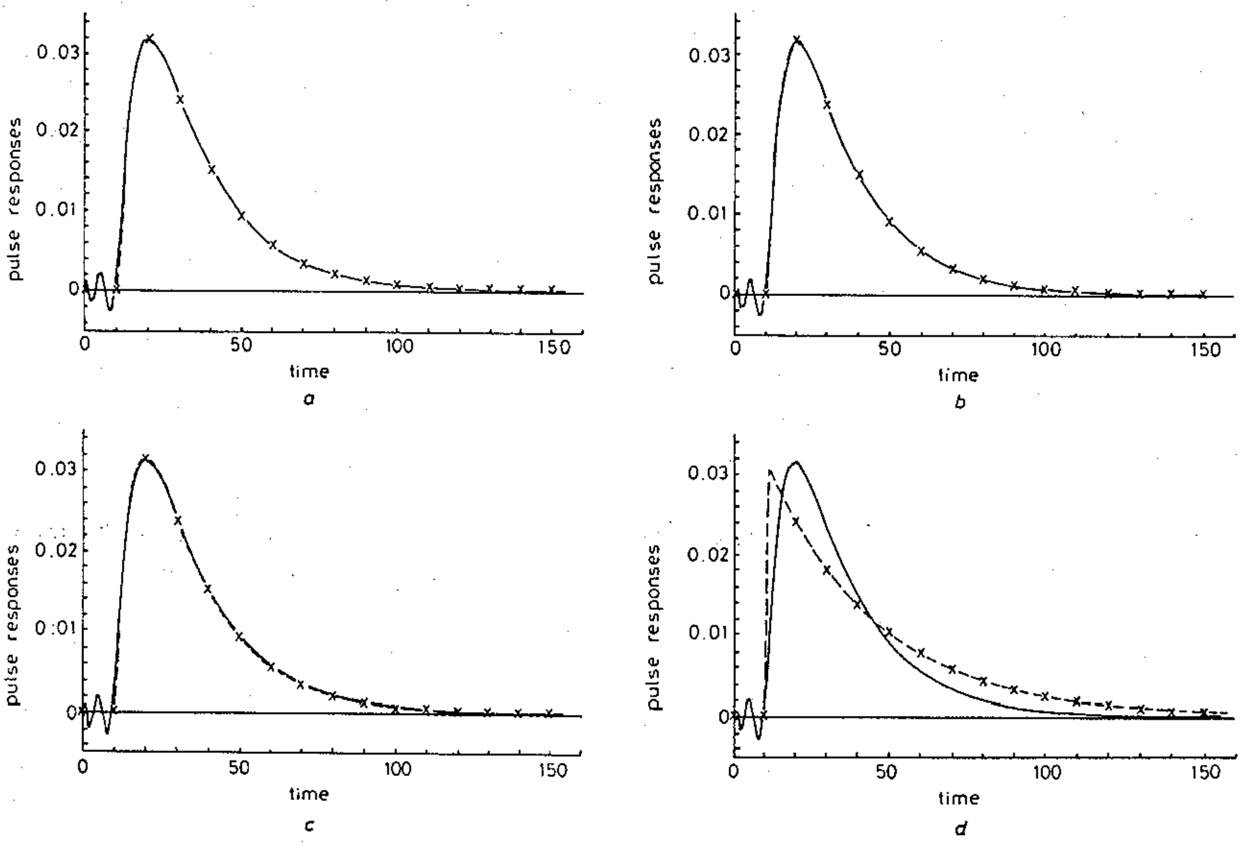


**Fig. 14** *Example 3 pulse responses (sampling time = 0.1, k = 10)*
—— full order
- –×– reduced order
*a* 6th order to 4th
*b* 6th order to 3rd
*c* 6th order to 2nd
*d* 6th order to 1st

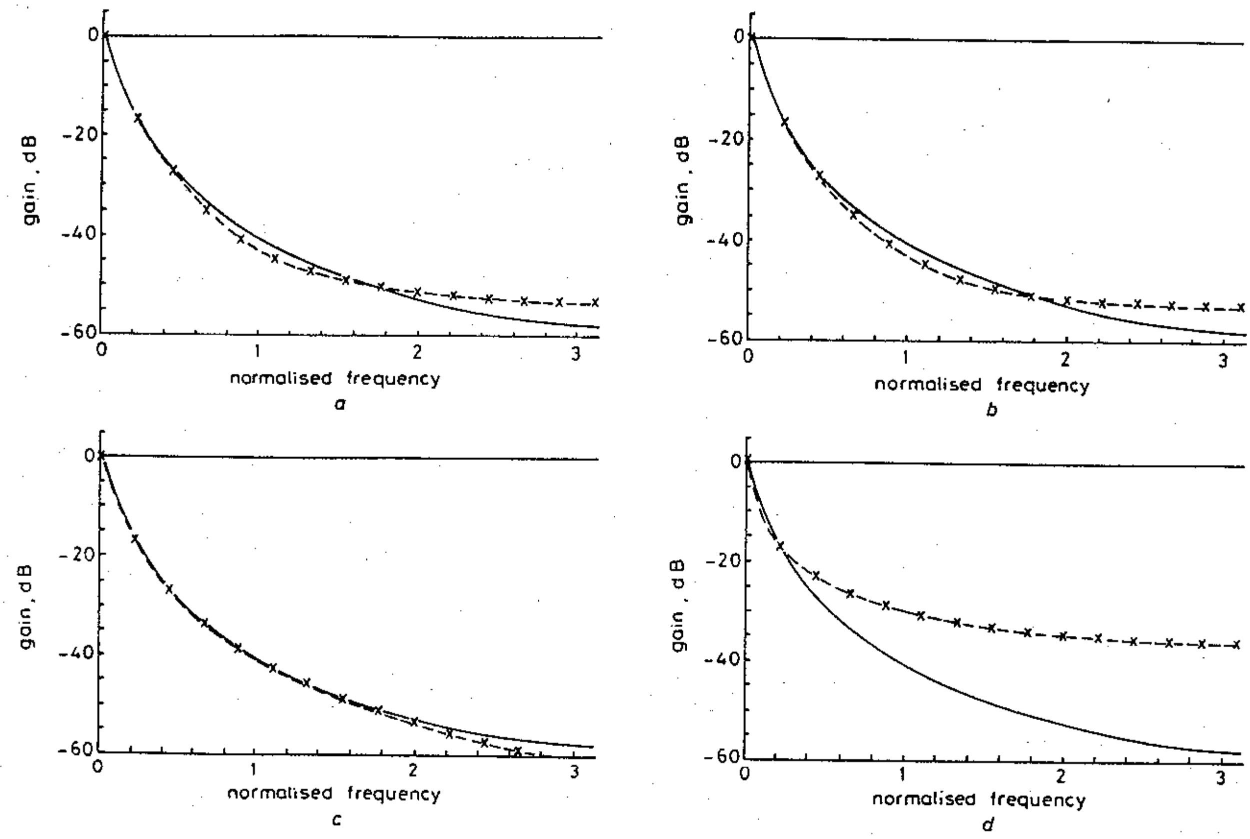


**Fig. 15** *Example 3 gains (sampling time = 0.1, k = 10)*
——— full order
- —×— reduced order
*a* 6th order to 4th
*b* 6th order to 3rd
*c* 6th order to 2nd
*d* 6th order to 1st

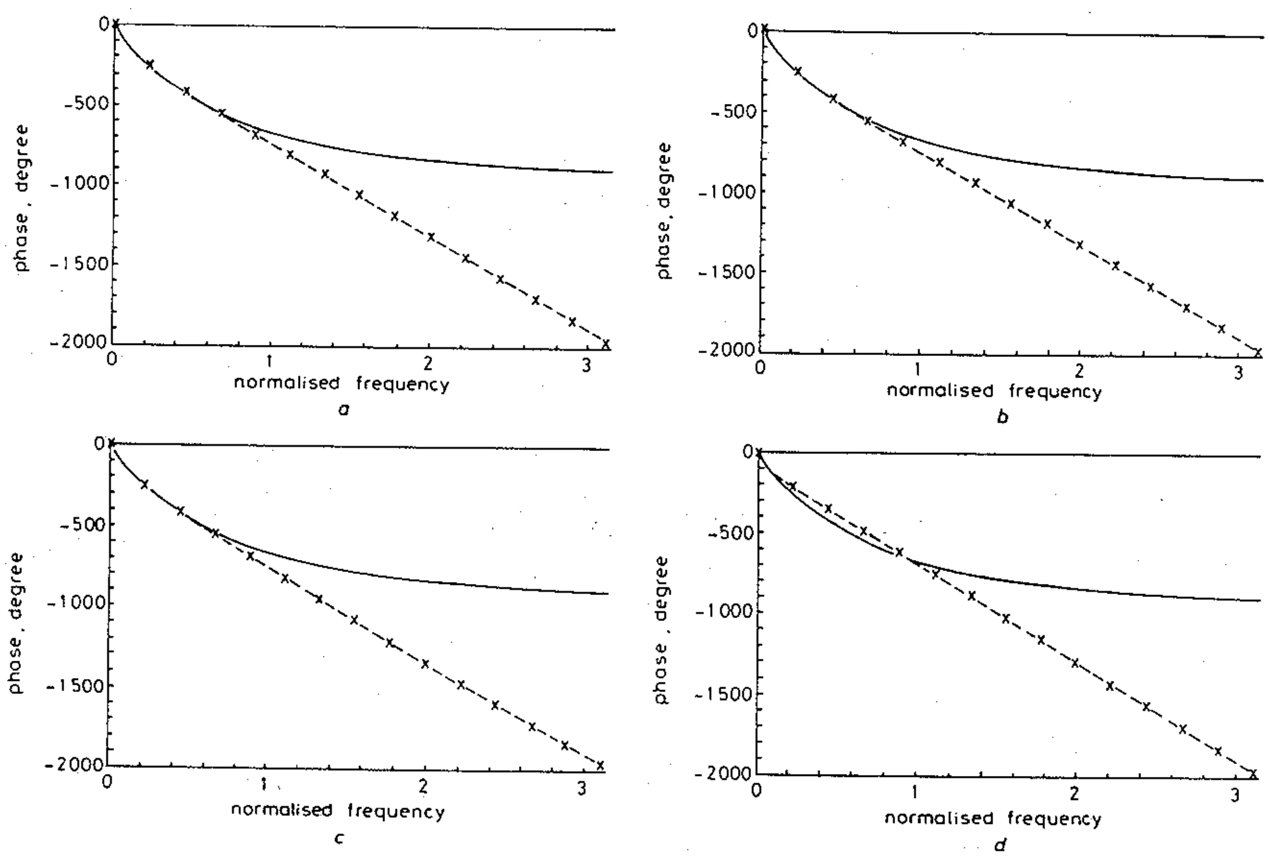


**Fig. 16** *Example 3 phases (sampling time = 0.1, k = 10)*
——— full order
—×— reduced order
*a* 6th order to 4th
*b* 6th order to 3rd
*c* 6th order to 2nd
*d* 6th order to 1st

We introduce a time delay $k = 2$, and we have the first 3 Markov parameters $M_0 = 0$, $M_1 = 0.397242 \times 10^{-2}$, $M_2 = 0.110629 \times 10^{-1}$ and $\sum[\bar{G}(z)] = \text{diag}\{0.723728, 0.304016, 0.52995 \times 10^{-2}, 0.489425 \times 10^{-2}, 0.150281 \times 10^{-2}\}$. The first term of the error bound using condition 29*a* in theorem 4.3 is 0.0203593, and using condition 29*b* is 0.0331886. By using our procedure, we reduced the system to 4th order, 3rd order, 2nd order and 1st order (together with the time delay). The performance comparisons are shown in Figs. 17 to 20. The error bounds are shown in Table 4.

**Table 4: Whole error bounds for example 4**

| Reduced order | Exact errors of the reduction with time delay | Error bounds using theorem 4.3 condition 29*a*, for the first term | Error bounds using theorem 4.3, condition 29*b*, for the first term term |
|---|---|---|---|
| 4th order | 0.0174061 | 0.023365 | 0.0361942 |
| 3rd order | 0.0224762 | 0.0331535 | 0.0459827 |
| 2nd order | 0.0228013 | 0.0437525 | 0.0565818 |
| 1st order | 0.585287 | 0.651785 | 0.664614 |

## 6 Remarks and conclusions

(i) In Section 2, we made the assumption that the continuous-time transfer function to be approximated had only distinct poles and was strictly proper. This assumption is not necessary. Suppose $G(s)$ has $v$ different poles $(\beta_1, \beta_2, \ldots, \beta_v)$ and $\beta_i$ has multiplicity $r_i$. We have $\sum_{i=1}^{v} r_i = n$, the order of the transfer function $G(s)$. Hence,

$$G(s) = \sum_{k=1}^{v} \sum_{i=1}^{r_k} \frac{A_{ki}}{(s-\beta_k)^{r_k-i+1}}$$

where $A_{ki}$ is $m \times l$ constant matrix. Let

$$\bar{G}(s) = \sum_{k=1}^{v} \sum_{i=1}^{r_k} \frac{B_{ki}(s)A_{ki}}{(s-\beta_k)^{r_k-i+1}} \tag{32}$$

where

$$B_{ki}(s) = \text{diag}\left\{e^{\beta_k T_1} \sum_{d=0}^{r_k-i} \frac{T_1^d}{d!}(s-\beta_k)^d, \ldots, \times e^{\beta_k T_m} \sum_{d=0}^{r_k-i} \frac{T_m^d}{d!}(s-\beta_k)^d\right\}$$

It is easy to see that $\bar{G}(s)$ is rational, strictly proper and stable. And it is a rational, strictly proper and stable approximation of diag $\{e^{sT_1}, \ldots, e^{sT_m}\}G(s)$. So we can use a truncation of a balanced realisation to find the reduced-order system $\tilde{\bar{G}}(s)$ with $r$th order $(r < n)$, and form the approximation of $G(s)$ with time delays as in eqn. 7. And finally, it can be proved that all previous results about the error bound are still valid in this case, because it is not very difficult to verify that eqn. 20 still holds when $\bar{G}(s)$ is chosen as in eqn. 32. On the other hand, if $G(s)$ is proper, but not strictly proper, say $G(s) = D + C(sI - A)^{-1}B$ with $D \neq 0$, then we can directly apply our procedure to the strictly proper part of $G(s)$ and modify the result as diag $\{e^{-sT_1}, \ldots, e^{-sT_m}\}\tilde{\bar{G}}(s) + D$. Then we can see all error bound results are still valid, provided that we replace the impulse response of $G(s)$ by the impulse response of the strictly proper part of

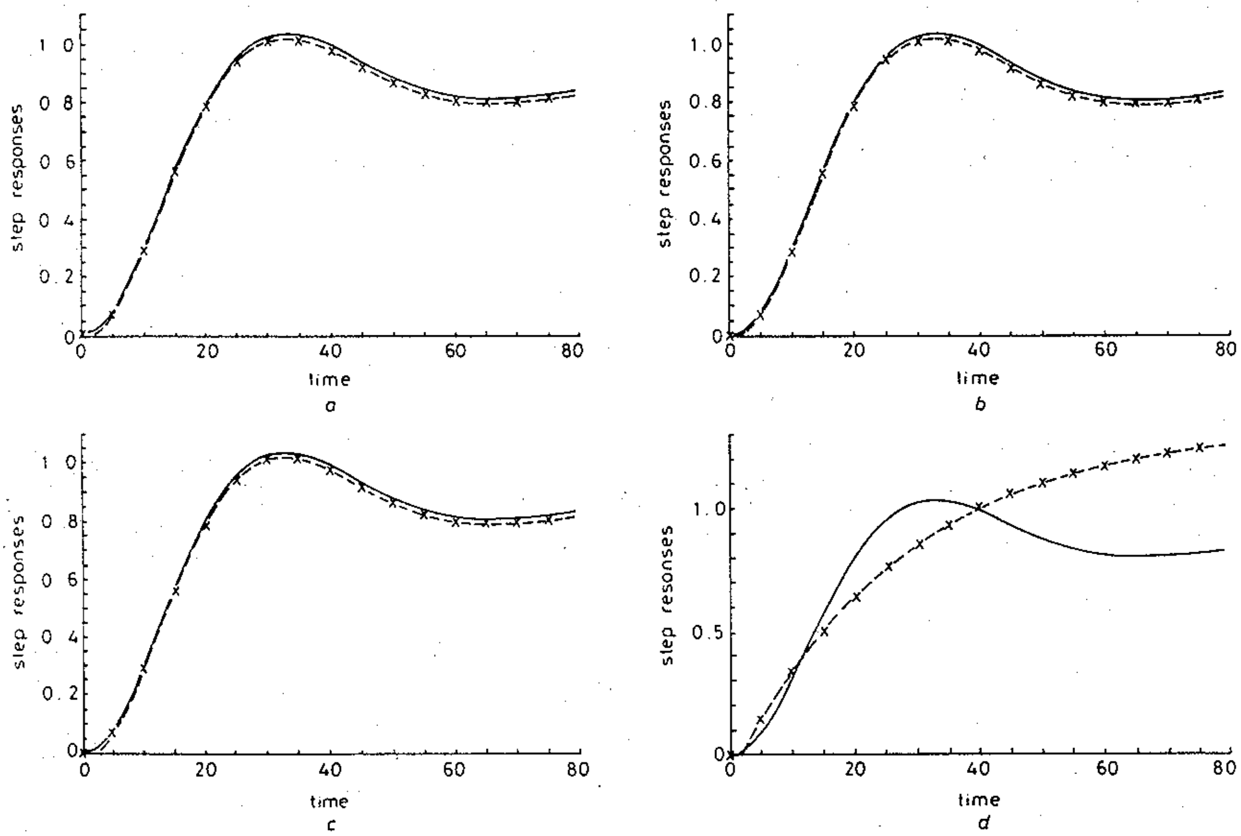


**Fig. 17** *Example 4 step responses (discrete time, k = 2)*
—— full order
×—× reduced order
*a* 5th order to 4th
*b* 5th order to 3rd
*c* 5th order to 2nd
*d* 5th order to 1st

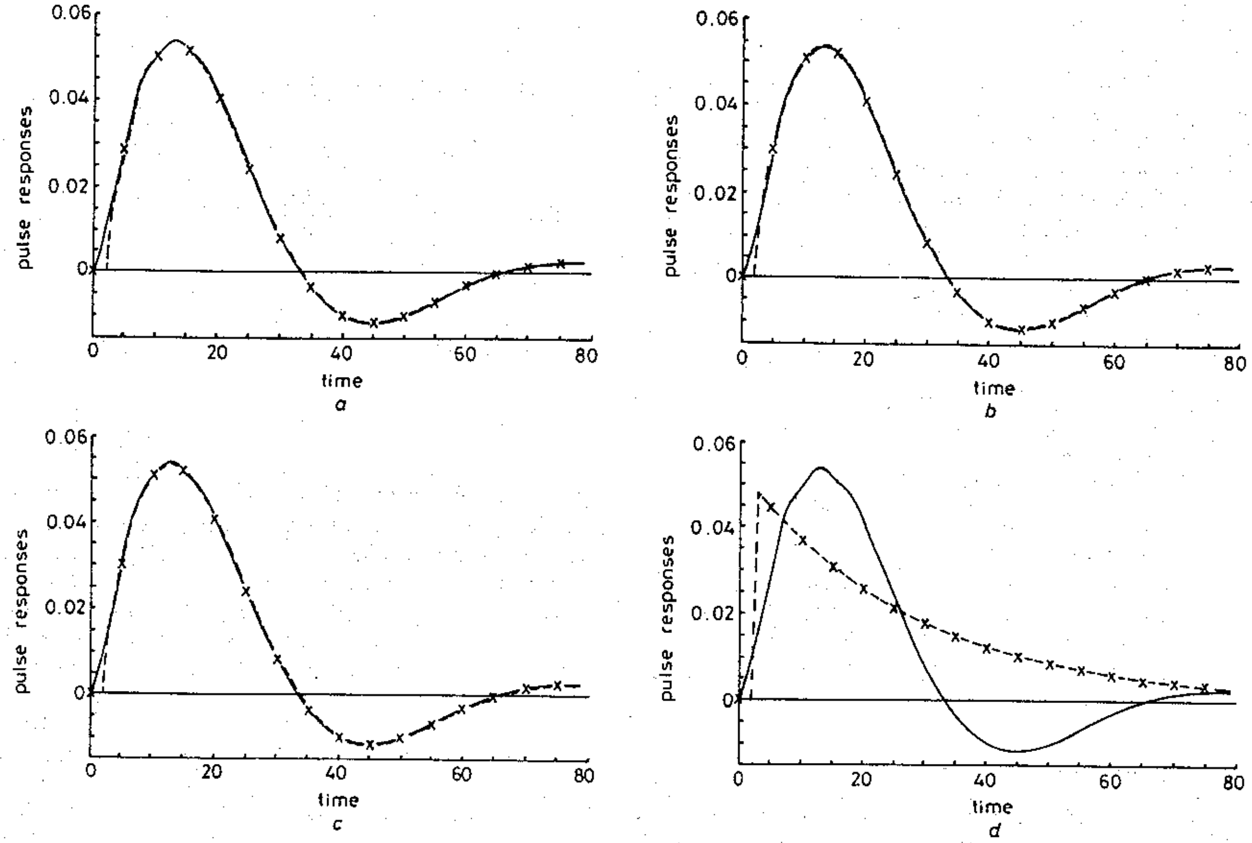


**Fig. 18** *Example 4 pulse responses (discrete time, k = 2)*
——— full order
- -×- reduced order
*a* 5th order to 4th
*b* 5th order to 3rd
*c* 5th order to 2nd
*d* 5th order to 1st

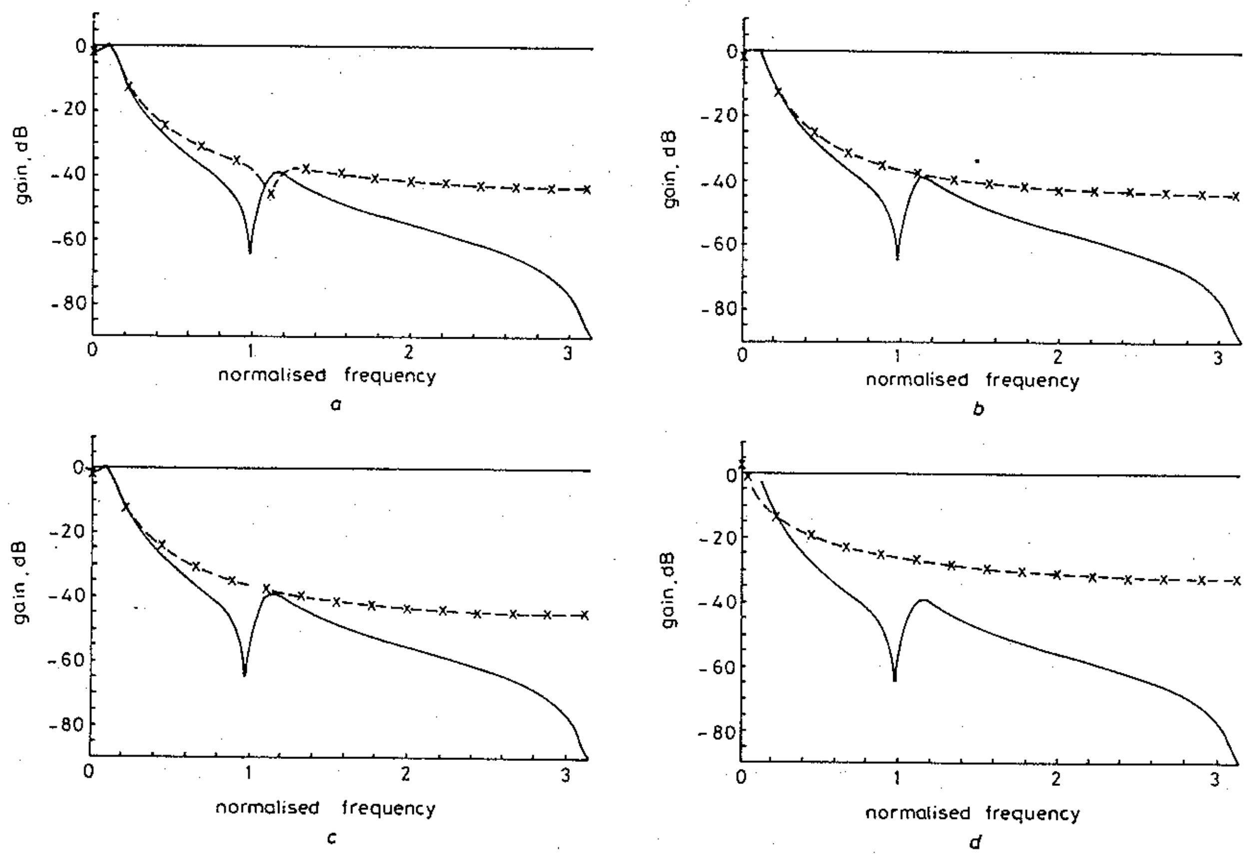


**Fig. 19** *Example 4 gains (discrete time, k = 2)*
——— full order
- -×- reduced order
*a* 5th order to 4th
*b* 5th order to 3rd
*c* 5th order to 2nd
*d* 5th order to 1st

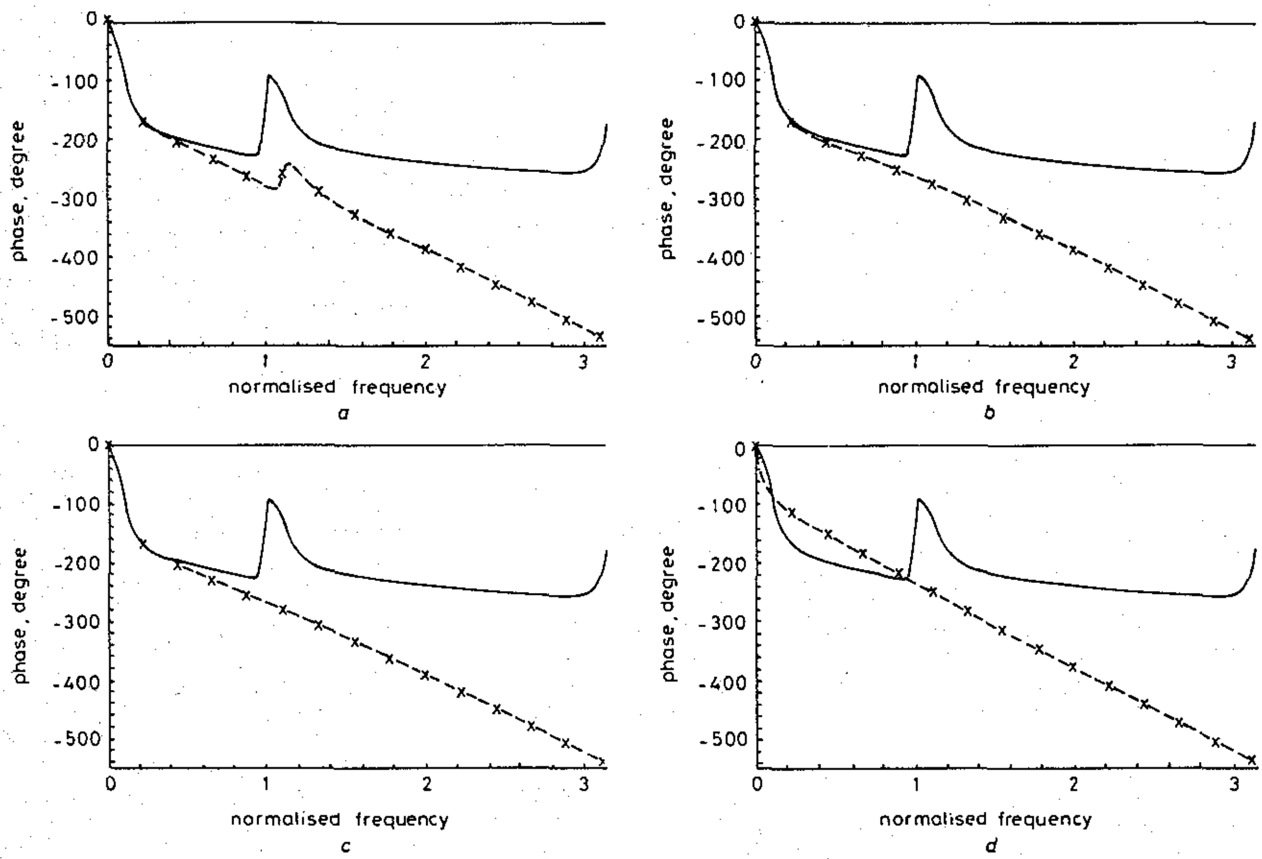


**Fig. 20** *Example 4 phases (discrete time, k = 2)*
------ full order
-×- reduced order
*a* 5th order to 4th
*b* 5th order to 3rd
*c* 5th order to 2nd
*d* 5th order to 1st

$G(s)$. It does, however, seem to us unlikely (though not impossible) that a physical system would be modelled by direct feed through plus a time delay in states with a system with rational transfer function. So the idea here may be of little importance.

(ii) Consider the SISO case and recall that, in our procedure, we approximate $e^{sT}G(s)$ by the ratio·al strictly proper transfer function $\tilde{G}(s)$, the main point being that we approximate each $e^{sT}/(s-\beta_i)$ by $e^{\beta_i T}/(s-\beta_i)$. If we expand $e^{sT}$ around the point $s=\beta_i$ as a Taylor series, we have

$$e^{sT} = e^{\beta_i T} + e^{\beta_i T}\cdot T(s-\beta_i) + e^{\beta_i T}\cdot T^2(s-\beta_i)^2/2! + \cdots + e^{\beta_i T}\cdot T^n(s-\beta_i)^n/n! + \cdots$$

One natural consideration is to use the first *two* terms of this expansion instead of only the first one, i.e. we approximate each $e^{sT}/(s-\beta_i)$ by

$$e^{\beta_i T}[1 + T(s-\beta_i)]/(s-\beta_i) = e^{\beta_i T}/(s-\beta_i) + e^{\beta_i T}\cdot T.$$

Hence, we have a nonstrictly proper approximation of $e^{sT}G(s)$ as

$$\tilde{G}_{new}(s) = \sum_{i=1}^{n} \frac{\alpha_i e^{\beta_i T}}{s-\beta_i} + T\sum_{i=1}^{n}\alpha_i e^{\beta_i T} = \tilde{G}_{old}(s) + Tg(T)$$

where $g(T)$ is the value of the impulse response of $G(s)$ at time $t = T$.

It is possible that $\tilde{G}_{new}(s)$ is a better rational approximation of $e^{sT}G(s)$, and we can go on to obtain the low-order system $\hat{\tilde{G}}_{new}(s)$. It is not guaranteed, however, that we have a better error bound. We know that the Hankel singular values are independent of the direct feed-through term of a system. So we have

$$\|\tilde{G}_{new}(j\omega) - \hat{\tilde{G}}_{new}(j\omega)\|_\infty \leqslant 2\ \mathrm{tr}\ \{\Sigma_2[\tilde{G}_{new}]\} \equiv 2\ \mathrm{tr}\ \{\Sigma_2[\tilde{G}_{old}]\}$$

On the other hand, the first term of the error bound now becomes

$$\begin{aligned}\|G(j\omega) - e^{-j\omega T}\tilde{G}_{new}(j\omega)\|_\infty &= \|G(j\omega) - e^{-j\omega T}\tilde{G}_{old}(j\omega) - e^{-j\omega T}g(T)\cdot T\|_\infty \\ &= \left\| \int_0^T g(T-t)e^{j\omega t}\,dt - g(T)\cdot T \right\|_\infty \end{aligned} \quad (33)$$

It is easy to see that the value of eqn. 33 depends on the characteristic of the impulse response $g(t)$. The bound can be bigger or smaller than the value of $\|\int_0^T g(T-t)e^{j\omega t}\,dt\|_\infty$, the error bound we obtained before.

For the discrete-time approach, we also can consider a nonstrictly causal approximation of $G(z)$, say $\tilde{G}_{new}(z)$, by simply changing $F(z)$ in eqn. 8 to

$$F_{new}(z) = \sum_{i=0}^{k-1} M_i z^{k-i},$$

where $M_i$ are as before the Markov parameters of the transfer function $G(z)$. Now the first term of the error bound becomes $\|F_{new}(z)\|_\infty$, and, as in the continuous-time case, the second term of the error bound does not change. So we still cannot be definitive about the improvement of the error bound in this case. Certainly, all these arguments are valid also for MIMO systems.

(iii) One disadvanage of the method presented is that we have to know the time delay of the system before reduction. It is not straightforward to work out the exact optimum value of the delay starting with the original rational model, although there are several ways to estimate it. We note that the first term of the error bound obtained is independent of the reduced-order transfer function, and only depends on the impulse response of the original system and the time delay. That means we can calculate it before doing any reduction. So, in some sense, we can use this value to aid us to obtain the time delay. Let $\xi = \|G(j\omega) - e^{-j\omega T}\bar{G}(j\omega)\|_\infty$, the first term of the error bound. Then, from corollary 4.2, condition 16, we have that $\xi$ is overbounded by $\sigma$ where

$$\sigma^2 \triangleq \int_0^T g^2(t)\,dt \cdot T$$

Hence,

$$\frac{d}{dT}\sigma^2 = g^2(T) \cdot T + \int_0^T g^2(t)\,dt \tag{34}$$

This means that the sensitivity of the first term of the error bound is heavily dependent on the characteristic of the impulse response of the original system with transfer function $G(s)$. This could give us another way to estimate the value of the time delay. However, because $\sigma^2$ is monotone in $T$, while $\xi$ generally is not, eqn. 34 may be of limited utility. Consequently, one may be thrown back into one-dimensional searching.

(iv) In our discussion of multioutput problems, we postulated the inclusion of delays at each output in the reduced-order model. We can alternatively conceive of including time delays in the system inputs (or even in both inputs and outputs). It is easy to see that the error bound analysis should have no substantial difference between these new schemes and the previous one.

(v) The above examples illustrate that the error in the step and impulse responses between the full-order system and the reduced-order systems with time delay are rather small. The larger difference in the frequency responses only occurs at high frequency where there is a very small gain. The large difference in phase is not surprising in view of the introduction of time delay into the reduced-order systems. Another interesting observation about these examples is that the values of the first term error bound almost always dominate in the whole error bound for continuous-time systems. So, from this point of view, there would seem to be little advantage in the continuous-time case in using the Hankel-norm optimal approximation method in approximating the transfer function $\bar{G}(s)$ by $\tilde{G}(s)$, instead of the balanced realisation truncating method we used. Hankel-norm optimal approximation can only improve the second term of the error bound, and certainly it will introduce more complicated calculations. Generally speaking, the scheme is a quite acceptable model reduction method, in that it performs adequately on some examples, and the error bounds we obtained are also quite accurate. Obviously, however, one could find examples where approximations would be poor (and the same is probably true of any approximation scheme).

(vi) We have given virtually no attention to the problem of approximating unstable systems. Let $G(s)$ be an unstable rational transfer function, which can be written as $G(s) = \nu(s)/\delta(s)$, where $\nu(s) = n(s)/d(s)$, $\delta(s) = p(s)/d(s)$ are both stable proper transfer functions. From some points of view, approximation of $G(s)$ can then be regarded as a problem of approximating $\nu(s)$, $\delta(s)$ by stable proper transfer functions $\bar{\nu}(s) = [e^{-sT}\bar{n}(s)/\bar{d}(s)]$, $\bar{\delta}(s) = [\bar{p}(s)/\bar{d}(s)]$ with $\bar{d}(s)$ a Hurwitz polynomial of degree less than $d(s)$. The SIMO analysis (with time delay in only one output) encompasses this case. Some work by using this idea to approximate an unstable rational proper transfer function by a lower-order transfer function (without time delay) has been done [22].

(vii) There may be a desire to have a more accurate approximation at certain frequencies, perhaps because input signals are known to be concentrated there, or because such frequencies may encompass the unity gain crossover point in a loop including the system under consideration. This means we are interested in minimising

$$\|[G(j\omega) - e^{-j\omega T}\tilde{G}(j\omega)]H(j\omega)\|_\infty = \|[e^{j\omega T}G(j\omega) - \tilde{G}(j\omega)]H(j\omega)\|_\infty \tag{35}$$

where $H(j\omega)$ is some weighting function. Certainly, it is rather difficult to find a $\tilde{G}(s)$ which directly minimises eqn. 35. One way of solving this problem is first to find a rational, proper and stable approximation $\bar{G}(s)$ of $e^{sT}G(s)$ as before, defined as in eqn. 4, then for the rational, proper and stable weighting function $H(s)$, find a rational, proper and stable low-order truncating approximation $\tilde{G}(s)$ of the frequency weighted balanced realisation [6] of $\bar{G}(s)$. Alternatively, one could find a rational, proper and stable low-order frequency weighted Hankel-norm optimal approximation $\tilde{G}(s)$ of $\bar{G}(s)$ for the proper, strictly stable and minimum-phase frequency weighting function $H(s)$ [10, 11]. Finally, form the frequency weighted approximation with time delay of $G(s)$ as $e^{-sT}\tilde{G}(s)$ which makes eqn. 35 small. All results about the error bounds obtained before can easily be modified for the scheme using frequency weighted Hankel-norm optimal approximation, as the error bound is available for the frequency weighted Hankel-norm optimal approximation method [11]. But there is no error bound available for the frequency weighted balanced realisation truncation method. The suggested scheme also fails to reflect the weighting from $H(j\omega)$ in the first step, where $\bar{G}(s)$ is determined from $G(s)$.

## 7 References


1 DOUGLAS, J.M.: 'Process dynamics and control — Vol. 1' (Prentice-Hall Inc., 1972), pp. 262–264

2 GRUCA, A., and BERTRAND, P.: 'Approximation of high-order systems by low-order models with delays', *Int. J. Control*, 1978, **28**, (6), pp. 953–965

3 MARSHALL, S.A.: 'Comments on Approximation of high-order system by low-order models with delays', *ibid.*, 1979, **30**, (4), pp. 721–723

4 MOORE, B.C.: 'Principal component analysis in linear system: Controllability, observability, and model reduction', *IEEE Trans.*, 1981, **AC-26**, (1), pp. 17–32

5 PERNEBO, L., and SILVERMAN, L.M.: 'Model reduction via balanced state space representations', *ibid.*, 1982, **AC-27**, (2), pp. 382–387

6 ENNS, D.: Ph. D. thesis, Department of Electrical Engineering, Standford University, 1984

7 ADAMJAN, V.M., AROV, D.Z., and KREIN, M.G.: 'Analytic properties of Schmidt pairs for a Hankel operator and the generalized Schur-Takagi problem', *Math. USSR Sbornik*, 1971, **15**, (1), pp. 31–73

8 KUNG, S.Y., and LIN, D.W.: 'Optimal Hankel-norm model reductions: multivariable systems', *IEEE Trans.*, 1981, **AC-26**, (4), pp. 832–852

9 GLOVER, K.: 'All optimal Hankel-norm approximations of linear multivariable systems and their $L^\infty$-error bounds', *Int. J. Control*, 1984, **39**, (6), pp. 1115–1193

10 LATHAM, G.A., and ANDERSON, B.D.O.: 'Frequency-weighted optimal Hankel-norm approximation of stable transfer functions', *Syst. & Control Lett.*, 1985, **5**, (4), pp. 229–236

11 ANDERSON, B.D.O.: 'Linear controller approximation: a method with bounds'. Proc. IFAC Workshop on Model Error Concepts and Compensation, 1985
12 YENGST, W.C.: 'Procedures of modern network synthesis' (Macmillan, 1964), pp. 446–458
13 LASTMAN, G.J., SINHA, N.K., and RÓZSA, P.: 'On the selection of states to be retained in a reduced-order model', *IEE Proc. D, Control Theory & Appl.*, 1984, **131**, (1), pp. 15–22
14 WARWICK, K.: 'A new approach to reduced-order modelling', *ibid.*, 1984, **131**, (2), pp. 74–78
15 THERAPOS, C.P.: 'Low-order modelling via discrete stability equations', *ibid.*, 1984, **131**, (6), pp. 248–252
16 LIAW, C.M., PAN, C.T., and OUYANG, M.: 'Model reduction of discrete systems using power decomposition method and the system identification method', *ibid.*, 1986, **133**, (1), pp. 30–34
17 CHEN, C.T., CHANG, C.Y., and HAN, K.W.: 'Reduction of transfer functions by the stability-equation method', *J. Franklin Inst.*, 1979, **308**, pp. 389–404
18 WITTMEYER, V.H.: 'Einfluß der Änderung einer Matrix', *Z. Angew. Math. & Mech.*, 1936, **16**, pp. 287–300
19 BODEWIG, E.: 'Matrix calculus' (North-Holland Publishing Co., 1956), pp. 66–67
20 FUKATA, S., MOHRI, A., and TAKATA, M.: 'Optimization of linear systems with integral control for time-weighted quadratic performance indices', *Int. J. Control*, 1983, **37**, (5), pp. 1057–1070
21 ROGERS, R.O., and SWORDER, D.D.: 'Suboptimal control of linear systems derived from models of lower dimension', *AIAA J.*, 1971, **9**, (8), pp. 1461–1467
22 PARKER, P.J., and ANDERSON, B.D.O.: 'Unstable rational function approximation', *Int. J. Control* (accepted for publication)