INT. J. CONTROL, 1987, VOL. 45, NO. 5, 1575–1602

State-space formulae for the factorization of all-pass matrix functions

MICHAEL GREEN† and BRIAN D. O. ANDERSON†

We consider the factorization of an all-pass matrix function $E(s)$ using proper stable minimum-phase factors. State-space formulae for the stable minimum-phase factors of $E(s)$ are derived (when they exist), as well as state-space formulae for the Wiener–Hopf factors. This is achieved using the state-space characterization of all-pass matrix functions given by Glover (1984) in conjunction with the results of Green and Anderson (1987).

1. Introduction

The recent trend toward the analysis of multivariable time series via the theory of canonical correlations (Akaike 1975, 1976, Jewel and Bloomfield 1983, Jewel *et al.* 1983) has motivated the study of the structure of all-pass matrices, since the canonical correlation operator associated with a stationary time series is the Hankel operator of an all-pass matrix. In particular, the analysis of time series via canonical correlations has led to the formulation of a stochastic model reduction technique, called phase matching (because of other connections), based on the approximation of the canonical correlation operator (Jonckheere and Helton 1985, Desai and Pal 1984, Opdenacker and Jonckheere 1985). In the latter paper the multivariable version of this model-reduction technique was envisioned and further developed by Green and Anderson (1986) exposing the necessity for a more detailed knowledge of all-pass matrices. In particular, the technique depends on the availability of a factorization theory for all-pass matrices. The phase matching, or canonical-correlations, approach to stochastic model reduction requires the factorization of an all-pass matrix (the phase matrix, the Hankel operator of which is the canonical correlation operator) $E(s)$ using proper stable minimum-phase factors $V(s)$, $W(s)$ such that

$$E(s) = V(-s)^{-1} W(\bar{s})^* \tag{1.1}$$

A complete solution requires that state-space realizations of the factors $V(s)$, $W(s)$ satisfying (1.1) be given. Our earlier paper (Green and Anderson 1987) was concerned with a number of fundamental issues. Does an arbitrary all-pass matrix have a factorization as in (1.1)? (It does not.) How can one characterize the class of all-pass matrices that do have such a factorization? Are the factors $V(s)$, $W(s)$ in (1.1) unique? (They are not.) What are the key properties of the factors $V(s)$ and $W(s)$? These questions were approached by relating the stable minimum-phase factorization problem to the Wiener–Hopf factorization problem (see e.g. Clancey and Gohberg 1981). The above questions were then tackled by applying the state-space Wiener–Hopf factorization theory of Bart *et al.* (1983) to the state-space characterization of all-pass matrices developed in Glover (1984), where the problem of optimal Hankel norm approximation was studied. The only formulae given in Green and

Received 7 July 1986.

† Department of Systems Engineering, Research School of Physical Sciences, Australian National University, G.P.O. Box 4, Canberra, A.C.T. 2601, Australia.

Anderson (1987) for the factors $V(s)$, $W(s)$ were, however, transfer-function formulae relating $V(s)$, $W(s)$ to the Wiener–Hopf factors of $E(s)$. This paper develops state-space formulae for $V(s)$, $W(s)$ satisfying (1.1) directly from the state-space characterization of all-pass matrices in Glover (1984). This enables a closed-form solution for the phase-matching/canonical-correlations stochastic model-reduction method of Green and Anderson (1986) to be given. As an extension (for completeness), state-space formulae for the Wiener–Hopf factors are also derived.

The organization of the paper is pedagogical rather than deductive, and is as follows. Section 2 consists of notation, definitions, the state-space characterization of all-pass matrices and a summary of the major results of Green and Anderson (1987). Section 3 shows that the factorization problem is related to the positive-real lemma, a link that was suggested by the stochastic model-reduction application. This provides proper, stable but not necessarily minimum-phase $V(s)$, $W(s)$ satisfying (1.1). Section 4 develops conditions that ensure that the solutions provided by § 3 are minimum-phase. This is done by treating two extreme cases and then showing how the general case can be solved by suitably combining the results of the two extreme cases. Section 5 extends the results to the Wiener–Hopf factorization.

2. Definitions, notation and preliminaries

2.1. *Definitions*

Let L_∞ denote the space of complex measurable $p \times p$ matrix functions that are bounded on the imaginary axis. Then

$$L_\infty = H_\infty^+ \oplus H_\infty^-$$

where

$$H_\infty^+ = \{H \in L_\infty : H(s) \text{ is analytic in } \operatorname{Re}(s) \geqslant 0 \text{ and such that } H(\infty) = 0\}$$

$$H_\infty^- = \{H \in L_\infty : H(s) \text{ is analytic in } \operatorname{Re}(s) \leqslant 0\}$$

Note that H_∞^+ contains all asymptotically stable, strictly proper, rational $p \times p$ matrix functions.

Definition

If $H(s) \in L_\infty$ then $H(s)$ is uniquely decomposable as $H(s) = H_+(s) + H_-(s)$,

$$H_\pm(s) \in H_\infty^\pm$$

and $H_+(s)$ is called the *stable part of* $H(s)$.

From now on we will be dealing only with rational matrix functions. Thus for convenience $H(s) \in L_\infty$ *will mean that* $H(s)$ *is rational and in* L_∞.

Definition

Let $H(s) \in L_\infty$ and $H_+(s) = C(sI - A)^{-1}B$, with A $n \times n$, be a realization of $H_+(s)$. Let P, Q be the hermitian solutions of the Lyapunov equations

$$AP + PA^* + BB^* = 0 \tag{2.1 a}$$

$$A^*Q + QA + C^*C = 0 \tag{2.1 b}$$

Denote by $\lambda_i(PQ)$ the eigenvalues of PQ, which are invariant under state-space transformation. The quantities $\sigma_i(H)$ given by

$$\sigma_i(H) \triangleq [\lambda_i(PQ)]^{1/2}, \quad i=1,\dots,n \tag{2.2 a}$$

are called the *Hankel singular values* of H, and by convention are ordered so that

$$\sigma_i(H) \geqslant \sigma_{i+1}(H), \quad i=1,\dots,n-1 \tag{2.2 b}$$

The number of non-zero Hankel singular values of $H(s)$ is equal to the McMillan degree of $H_+(s)$. Note also that $H(s)$ and $H_+(s)$ have the same Hankel singular values.

Definition

$H(s) \in H_\infty^+ \oplus I$ is called *minimum-phase* if $H(s)$ is non-singular for all $s \in \{\mathrm{Re}\,(s) \geqslant 0\}$, but not necessarily at $s=\infty$.

Definition

$H(s) \in L_\infty$ is called *all-pass* if it satisfies

$$H(s)H(-\bar{s})^* = I \quad \text{for all } s \tag{2.3}$$

Remark

Recall from Green and Anderson (1987), or deduce from Lemma 5.1 and Theorem 6.1 of Glover 1984, that all-pass matrices have Hankel singular values less than or equal to 1.

Definition

Let $E(s) \in L_\infty$ be all-pass with m_1 stable and m_2 unstable poles (counting multiplicities), *let r be the number of Hankel singular values of $E(s)$ that equal* 1. That is, r is defined by

$$\sigma_1 = \dots = \sigma_r > \sigma_{r+1} \geqslant \dots \geqslant \sigma_{m_1} \quad \text{if } \sigma_1 = 1 \tag{2.4 a}$$

$$r = 0 \qquad\qquad \text{if } \sigma_1 < 1 \tag{2.4 b}$$

By Lemma 2.1 of Green and Anderson (1987)

$$m_1 \geqslant r \geqslant \max(0, m_1 - m_2) \tag{2.5}$$

or

$$m_2 \geqslant m_1 - r \geqslant 0, \quad r \geqslant 0 \tag{2.6}$$

2.2. *State-space characterization of all-pass matrices*

Theorem 2.1

Let $E(s) \in L_\infty$ be all-pass with minimal realization

$$A_e = \begin{bmatrix} A & 0 \\ 0 & \tilde{A} \end{bmatrix} \tag{2.7 a}$$

$$B_e = \begin{bmatrix} B \\ \tilde{B} \end{bmatrix}, \quad C_e = [C \quad -\tilde{C}], \quad D = E(\infty) \tag{2.7 b}$$

where A is $m_1 \times m_1$ and stable, $\tilde{A}$ is $m_2 \times m_2$ with $-\tilde{A}$ stable, B and C^* are $m_1 \times p$, and $\tilde{B}$ and C^* are $m_2 \times p$. Further suppose (A, B, C) is balanced, with controllability/observability grammian

$$\Sigma = \operatorname{diag}(\sigma_{r+1}, \ldots, \sigma_{m_1}, I_r) \tag{2.8 a}$$

$$= \operatorname{diag}(\Sigma_1, I_r) \tag{2.8 b}$$

satisfying

$$A\Sigma + \Sigma A^* + BB^* = 0 \tag{2.8 c}$$

$$A^*\Sigma + \Sigma A + C^*C = 0 \tag{2.8 d}$$

where r is given by (2.4). Then we have the following.

(i) There exist unique $P_e = P_e^*$, $Q_e = Q_e^*$ such that

$$A_e P_e + P_e A_e^* + B_e B_e^* = 0 \tag{2.9 a}$$

$$A_e^* Q_e + Q_e A_e + C_e^* C_e = 0 \tag{2.9 b}$$

$$P_e Q_e = I \tag{2.9 c}$$

$$D^*D = I \tag{2.10 a}$$

$$D^*C_e + B_e^* Q_e = 0, \quad DB_e^* + C_e P_e = 0 \tag{2.10 b, c}$$

(ii) Partition

$$P_e = \begin{bmatrix} \Sigma & M \\ M^* & R \end{bmatrix}, \quad Q_e = \begin{bmatrix} \Sigma & N \\ N^* & S \end{bmatrix} \tag{2.11 a}$$

where R, S are $m_2 \times m_2$ and M, N are $m_1 \times m_2$. Then with $l = m_2 - (m_1 - r)$, non-negative by (2.6),

$$R = T \begin{bmatrix} \Sigma_1 \Gamma^{-1} & 0 \\ 0 & -I_l \end{bmatrix} T^*, \quad S = T^{-*} \begin{bmatrix} \Sigma_1 \Gamma & 0 \\ 0 & -I_l \end{bmatrix} T^{-1} \tag{2.11 b, c}$$

$$M = \begin{bmatrix} I_{m_1 - r} & 0 \\ 0 & 0_{r \times l} \end{bmatrix} T^*, \quad N = \begin{bmatrix} -\Gamma & 0 \\ 0 & 0_{r \times l} \end{bmatrix} T^{-1} \tag{2.11 d, e}$$

where

$$\Gamma = \Sigma_1^2 - I \tag{2.12}$$

and T is an $m_2 \times m_2$ non-singular matrix. (T is a similarity transformation on the realization $(\tilde{A}, \tilde{B}, \tilde{C})$ of $E_-(s)$.)

(iii) Partition A, B, C conformally with Σ as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = [C_1 \quad C_2] \tag{2.13}$$

where A_{11} is $(m_1 - r) \times (m_1 - r)$ and B_1, C_1^* are $(m_1 - r) \times p$. Note that B_2 is the matrix formed from the last r rows of B. Define

$$\tilde{A}_{11} = \Gamma^{-1}(A_{11}^* + \Sigma_1 A_{11} \Sigma_1 - C_1^* D B_1^*) \tag{2.14 a}$$

$$\tilde{B}_1 = \Gamma^{-1}(\Sigma_1 B_1 + C_1^* D), \quad \tilde{C}_1 = C_1 \Sigma_1 + D B_1^* \tag{2.14 b, c}$$

Then (with T as in (ii))

$$\tilde{A} = T\begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} T^{-1}, \quad \tilde{B} = T\begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}, \quad \tilde{C} = [\tilde{C}_1 \quad \tilde{C}_2] T^{-1} \tag{2.14 d}$$

where

$$B_2 \tilde{B}_2^* = 0, \quad \tilde{C}_2 = -D\tilde{B}_2^*, \quad \tilde{A}_{12} = -\Gamma^{-1} C_1^* \tilde{C}_2 \tag{2.15 a, b, c}$$

$$\tilde{A}_{21} = -\tilde{B}_2 B_1^*, \quad \tilde{A}_{22} + \tilde{A}_{22}^* = \tilde{B}_2 \tilde{B}_2^* \tag{2.15 d, c}$$

Proof

See Glover (1984). (i) is Glover's Lemma 5.1. (ii) For $\Gamma > 0$ (i.e. $\sigma_1(E) = 1$) is Glover's Lemma 8.2 with $k = 0$. For the case $r = 0$ the result follows via the same reasoning as in the proof of Glover's Lemma 8.2. (iii) For $r > 0$ is Glover's Lemma 8.5, and with $r = 0$ the result follows similarly. Note also that the $r = 0$ case can be proved simply from the $r > 0$ case using the device of Glover's Remark 8.4.

Remarks

(i) The non-singular matrix T is just a similarity transformation on the realization $(\tilde{C}, \tilde{A}, \tilde{B})$ of $E_-(s)$. From now on, we shall *assume that* $E(s)$ *is realized so that* $T = I$.

(ii) The zero columns of M, N in (2.11) and the subscript-2 blocks of $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ are present if and only if $l = m_2 - (m_1 - r) > 0$. The zero rows of M, N and the subscript-2 blocks of A, B, C are present if and only if $r > 0$, i.e. $\sigma_1(E) = 1$.

(iii) Observe from (2.7) and (2.8) that

$$E_+(s) = C(sI - A)^{-1} B \tag{2.16}$$

and (A, B, C) is a balanced realization of $E_+(s)$.

(iv) In addition to developing the characterization of all-pass matrices, Glover (1984) applied this characterization to the problem of optimal Hankel norm approximation of linear systems. This application was concerned with the *additive* decomposition of all-pass matrices. Here we apply Glover's all-pass characterization theorem to develop a *product* decomposition of all-pass matrices.

Definition

An all-pass matrix $E(s) \in L_\infty$ with m_1 stable and m_2 unstable poles and r defined by (2.4) will be called a *minimal all-pass matrix* if

$$m_2 = m_1 - r \tag{2.17}$$

The term 'minimal' derives from (2.6), and it follows from Theorem 2.1 that for a minimal all-pass matrix, the subscript-2 blocks of $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ do not exist. That is, $l = 0$ in Theorem 2.1. Thus all-pass matrices constructed by Glover's Theorem 6.3 are minimal. The following theorem derives from Glover's Theorem 6.3.

Theorem 2.2 (Minimal all-pass extension theorem)

Let $G(s) \in H_\infty^+$ be a matrix function of McMillan degree m with balanced realization (A, B, C) satisfying

$$G(s) = C(sI - A)^{-1} B \tag{2.18}$$

and controllability/observability gramian Σ satisfying (2.8). Partition A, B, C conformally with Σ as in (2.13) and let D be any unitary matrix satisfying

$$C_2 + DB_2^* = 0 \tag{2.19}$$

(Note that if $r = 0$, i.e. $\Sigma < I$, D may be an arbitrary unitary matrix, since B_2, C_2 do not exist.)

Define $\tilde{A}_{11}$ $(m-r) \times (m-r)$, $\tilde{B}_1$ $(m-r) \times p$, $\tilde{C}_1$ $p \times (m-r)$ by (2.14 *a*, *b*, *c*), and let $E(s)$ be given by

$$E(s) = D + C(sI - A)^{-1} B - \tilde{C}_1 (sI - \tilde{A}_{11})^{-1} \tilde{B}_1 \tag{2.20}$$

Then $E(s)$ is the unique minimal all-pass matrix satisfying

$$E_+(s) = G(s) \quad \text{and} \quad E(\infty) = D \tag{2.21}$$

Proof

This follows directly from Theorem 2.1. □

The message of Theorem 2.2 is that minimal all-pass matrices are completely determined by their stable part and their direct feedthrough term ($E(\infty)$).

Notice that minimality in the context of Theorem 2.2 has the interpretation that an all-pass matrix $E(s)$ satisfying (2.21), with $G(s)$ prescribed, is minimal just when it has least possible McMillan degree among the class of all all-pass matrices satisfying (2.21).

2.3. *Factorization*

In this subsection we review the basis of Wiener–Hopf factorizations, and their relationship to stable minimum phase factorizations, the material being a summary of the results developed in Green and Anderson 1987.

Theorem 2.3 (Clancey and Gohberg 1981: (generalized) Wiener–Hopf factorization)

Let $H(s) \in L_\infty$ have no zeros on $s = j\omega$, $\omega \in \mathbb{R} \cup \{\infty\}$. Then $H(s)$ has a Wiener–Hopf factorization

$$H(s) = H_-(s) D(s) H_+(s), \quad s = j\omega, \ \omega \in \mathbb{R} \tag{2.22 a}$$

where

(i) $H_+(s)$ is proper, stable and non-singular in $\{\text{Re}\,(s) \geqslant 0\} \cup \{\infty\}$;

(ii) $H_-(s)$ is proper, completely unstable ($\in H_\infty^-$) and non-singular in $\{\text{Re}\,(s) \leqslant 0\} \cup \{\infty\}$; and

(iii) $$D(s) = \text{diag}\left\{\left(\frac{s-1}{s+1}\right)^{k_i}, i = 1, \ldots, p\right\}, \quad k_i \geqslant k_{i+1} \tag{2.22 b}$$

The numbers k_i are integers, called the *partial indices* of H, and they are uniquely determined by $H(s)$.

Theorem 2.4

Let $H(s) \in L_\infty$ have no zeros on $s = j\omega$, $\omega \in \mathbb{R} \cup \{\infty\}$. Then

$$H(s) = V(-s)^{-1} W(\bar{s})^*, \quad s = j\omega \tag{2.23}$$

with $V(s)$, $W(s)$ proper, stable and minimum-phase if and only if $H(s)$ has no (strictly) negative partial indices. When this is the case

$$W(\bar{s})^* = C_+(s) H_+(s) \tag{2.24 a}$$

$$V(s) = C_+(-s) D(s) H_-(-s)^{-1} \tag{2.24 b}$$

where $H_\pm(s)$, $D(s)$ satisfy (2.22) and $C_+(s)$ is an arbitrary matrix function whose elements satisfy

$$[C_+(s)]_{ij} \text{ is constant if } k_j = 0 \tag{2.25 a}$$

$$[C_+(s)]_{ij} \text{ is a polynomial in } (1+s)^{-1} \text{ of degree} \leqslant k_j \text{ if } k_j \geqslant 0 \tag{2.25 b}$$

and

$$\det C_+(s) = \alpha (s+1)^{-k} \tag{2.26 a}$$

with

$$k = \sum_{j=0}^{p} k_j = m_1 - m_2 \tag{2.26 b}$$

and α an arbitrary non-zero constant.

Furthermore, let

$$\rho = \operatorname{rank} V(\infty) = \operatorname{rank} W(\infty) \tag{2.27}$$

then

$$\text{number of zero partial indices} \leqslant \rho \leqslant \begin{cases} p-1 & \text{if } k > 0 \\ p & \text{if } k = 0 \end{cases} \tag{2.28}$$

and for every ρ satisfying (2.28) there exists a $V(s)$, $W(s)$ pair satisfying (2.23), (2.27).

Theorem 2.4 connects proper stable mininum-phase factorization of matrix functions with Wiener–Hopf factorization. Note that from (2.28) it follows that the only case when the rank of $V(\infty)$, $W(\infty)$ is uniquely determined is when all the partial indices are zero, in which case $V(\infty)$, $W(\infty)$ are non-singular and $V(s)$, $W(s)$ are unique up to a constant non-singular matrix (by (2.25)). Also, $V(\infty)$, $W(\infty)$ can be zero (i.e. $V(s)$, $W(s)$ strictly proper) if and only if $k_j > 0$ for all j; of course they do not have to be zero in this case, but they do have to be singular.

Theorem 2.5

Let $E(s) \in L_\infty$ be all-pass with realization A_e, B_e, C_e, D as in Theorem 2.1. Then

(i) (*a*)
$$r = \sum_{k_j \geqslant 0} k_j \tag{2.29}$$

(*b*) the number of strictly positive partial indices = rank B_2;

(*c*) the strictly positive k_j are the (non-zero) controllability indices of (A_{22}, B_2);

(ii) (a)

$$m_2 - (m_1 - r) = -\sum_{k_j \leqslant 0} k_j \tag{2.30}$$

(b) the number of strictly negative partial indices is equal to rank $\tilde{B}_2$ (= rank $\tilde{C}_2$);

(c) *the strictly negative partial indices are a permutation of the negative of the* observability indices of $(\tilde{C}_2, \tilde{A}_{22})$—that is, $k_j < 0$ if and only if $-k_j$ is an observability index of $(\tilde{C}_2, \tilde{A}_{22})$ (we do not count zero observability indices here).

We now put together Theorems 2.5 and 2.4 to obtain a characterization of all-pass matrices that can be factored as in (1.1).

Theorem 2.6

Let $E(s) \in L_\infty$ be all-pass with m_1 stable and m_2 unstable poles. Let r be defined by (2.4) and k_j, $j = 1, \ldots, p$ be the partial indices of $E(s)$. Then

$$E(s) = V(-s)^{-1} W(\bar{s})^* \tag{2.31}$$

with $V(s)$, $W(s)$ proper, stable and minimum-phase if and only if $E(s)$ is a minimal all-pass matrix, i.e. $m_2 = m_1 - r$. Furthermore,

(a) $V(\infty)$, $W(\infty)$ are non-singular if and only if $r = 0$ (so $m_1 = m_2$);

(b) $V(\infty)$, $W(\infty)$ are singular if and only if $E(s)$ is a minimal-degree Nehari extension of $E_+(s)$, i.e. $r > 0$;

(c) $E(s)$ is the unique Nehari extension of $E_+(s)$ (among all possible Nehari extensions, whether or not all-pass) if and only if $k_j > 0$ for all j.

Proof

(a) follows from Theorem 2.4 and (2.30) of Theorem 2.5.

(b) follows from (a)—note that since $E(s)$ is all-pass $\|E(s)\|_\infty = 1$, so $E(s)$ is a Nehari extension of $E_+(s)$ iff $\sigma_1(E_+) = 1$, since a Nehari extension $N(s)$ must satisfy $N_+(s) = E_+(s)$ and $\|N(s)\|_\infty = \sigma_1(E_+)$.

(c) follows from Theorem 2.5 (i) (b) and Theorem 8.7 of Glover 1984. □

Theorem 2.2 provides state-space formulae for an additive decomposition of a minimal all-pass matrix, while Theorem 2.6 asserts that minimal all-pass matrices have a certain multiplicative decomposition. The subject of this paper will be to use the state-space characterization of Theorem 2.2 to develop state-space formulae for the multiplicative decomposition of Theorem 2.6. More generally, we will use Theorem 2.1 to develop state-space formulae for the Wiener–Hopf factorization of all-pass matrices.

3. Factorization and the positive-real lemma

The role that stable minimum-phase factorizations of (minimal) all-pass matrices play in the phase-matching stochastic model-reduction algorithm (Green and Anderson 1986) provides a clue as to how state-space formulae for the factors can be obtained. There is a natural relationship between $V(s)$, $W(s)$ satisfying (1.1) with $E(s)$ all-pass and a positive-definite hermitian matrix $P(s)$: since $E(s)$ is a minimal all-pass matrix, it follows that

$$I = E(s)E(-\bar{s})^* = V(-s)^{-1}W(\bar{s})^*W(-s)V(\bar{s})^{-*}$$

giving

$$P(s) \triangleq V(s)V(-\bar{s})^* = W(-\bar{s})^*W(s) \tag{3.1}$$

Thus $V(s)$, $W(s)$ are left and right stable, minimum-phase spectral factors of the 'power spectrum' $P(s)$. Given a power spectrum $P(s)$, it is known how state-space formulae for stable $V(s)$, $W(s)$ satisfying (3.1) can be obtained and how to ensure they are minimum phase (see e.g. Anderson and Vongpanitlerd 1973). An important role is played by the positive-real lemma in this construction. However, since we are dealing here with complex matrix functions, we shall generalize this concept to positive complex matrices and take the positive-real lemma, modified to allow complex matrices, as our definition.

Definition

A $p \times p$ complex proper rational matrix function

$$Z(s) = J + H(sI - F)^{-1}G, \quad \text{with } (H, F, G) \text{ minimal} \tag{3.2}$$

will be called a positive complex matrix if there exist complex matrices P, L, W with $P = P^* > 0$ such that

$$PF + F^*P + L^*L = 0 \tag{3.3 a}$$

$$PG = H^* - L^*W \tag{3.3 b}$$

$$W^*W = J + J^* \tag{3.3 c}$$

The number of rows of W and L is unspecified, while the number of columns of W and L, as well as the dimension of P, is automatically fixed.

If $P(s)$ is a proper rational matrix, non-negative Hermitian for all $s = j\omega$, then there exists a positive complex matrix $Z(s)$ such that

$$P(s) = Z(s) + Z(-\bar{s})^* \tag{3.4}$$

This decomposition gives the following spectral-factorization result.

Theorem 3.1 (Spectral factorization—see Anderson and Vongpanitlerd 1973)

Let $Z(s)$ be a positive complex matrix with realization as in (3.2). Let P, L, W be any solutions of (3.3). Define $W(s)$ by

$$W(s) = W + L(sI - F)^{-1}G \tag{3.5}$$

Then

$$P(s) \triangleq Z(s) + Z(-\bar{s})^* = W(-\bar{s})^*W(s) \tag{3.6}$$

Furthermore, $W(s)$ is minimum-phase if (P, L, W) is a minimal solution of (3.2), i.e. $P \leqslant \bar{P}$, where $(\bar{P}, \bar{L}, \bar{W})$ is any other solution of (3.3).

A left-spectral factor of $P(s)$ can be obtained using Theorem 3.1 on $Z(\bar{s})^*$ instead of $Z(s)$. This gives

$$P(s) = V(s)V(-\bar{s})^* \tag{3.7}$$

where

$$V(s) = V + H(sI - F)^{-1}K \tag{3.8}$$

and (Q, K, V) satisfy

$$FQ + QF^* + KK^* = 0 \tag{3.9 a}$$

$$QH^* = G - KV^* \tag{3.9 b}$$

$$VV^* = J + J^* \tag{3.9 c}$$

The following result is the major clue as to how the stable minimum-phase factorization of minimal all-pass matrices can be achieved.

Lemma 3.1

Let $Z(s)$ be a $p \times p$ positive complex matrix with realization as in (3.2). Let $V(s)$, $W(s)$ be $p \times p$ left- and right-stable, minimum-phase spectral factors given by (3.5), (3.8) and let $E(s) \triangleq V(-s)^{-1}W(\bar{s})^*$. Then

$$E_+(s) = K^*(sI - F^*)^{-1}L^* \tag{3.10}$$

Proof

See Green and Anderson (1986).

It follows from (3.6) and (3.7) that $E(s)$ is all-pass, and since $V(s)$, $W(s)$ are minimum-phase, $E(s)$ is minimal by Theorem 2.6.

The way that Theorem 3.1 is usually used (i.e. in the spectral-factorization problem) involves starting with (J, H, F, G) and finding (P, L, W). Lemma 3.1 suggests that to factorize an all-pass matrix $E(s)$ we might use Theorem 3.1 backwards. That is, we start with K, F, L (and therefore P, Q satisfying (3.3 *a*) and (3.9 *a*)) and we look for G, H, V, W satisfying (3.3 *b*, *c*) and (3.9 *b*, *c*). The result is the following.

Theorem 3.2

Let $E(s) \in L_\infty$ be a minimal all-pass matrix with realization $E(s) = D + C(sI - A)^{-1}B - \tilde{C}_1(sI - \tilde{A}_{11})^{-1}\tilde{B}_1$ as in Theorem 2.2. Let G, H^* $m \times p$, V, W $p \times p$ be *any* solutions of

$$\Sigma G = H^* - BW \tag{3.11 a}$$

$$\Sigma H^* = G - C^*V^* \tag{3.11 b}$$

$$V = (DW)^* \tag{3.11 c}$$

(a solution procedure is discussed below). Define

$$W(s) = W + B^*(sI - A^*)^{-1}G \tag{3.12 a}$$

$$V(s) = V + H(sI - A^*)^{-1}C^* \tag{3.12 b}$$

Then

$$V(-s)E(s) = W(\bar{s})^* \tag{3.13}$$

Before proceeding to the proof, we shall establish a connection between G, H and

$\tilde{B}_1, \tilde{C}_1$ that facilitates the proof, and solves (3.11) for part of G and H. We also observe how (3.11) can be solved for the rest of G, H not covered in the lemma.

Lemma 3.2

Let A, B, C, Σ, D and G, H, W, V be as in Theorem 3.2. Partition G, H conformally with B, C. Then

$$G_1 = -\tilde{B}_1 W \tag{3.14 a}$$

$$H_1 = -V\tilde{C}_1\Gamma^{-1} \tag{3.14 b}$$

where $\tilde{B}_1$, $\tilde{C}_1$, Γ are given by (2.14), (2.12).

Proof

Consider the 1-blocks of (3.11 *a*, *b*):

$$\Sigma_1 G_1 = H_1^* - B_1 W \tag{3.15 a}$$

$$\Sigma_1 H_1^* = G_1 - C_1^* DW \tag{3.15 b}$$

Premultiply (3.15 *a*) by Σ_1 and add to (3.15 *b*) to obtain

$$(\Sigma_1^2 - I)G_1 = -(\Sigma_1 B_1 + C_1^* D)W$$

and (3.14 *a*) follows from (2.12), (2.14). The procedure is similar for (3.14 *b*). □

The above lemma shows how to find G_1, H_1. To see how G_2, H_2, W, V are determined, consider now the 2-blocks of (3.11 *a*, *b*):

$$G_2 = H_2^* - B_2 W$$

$$H_2^* = G_2 - C_2^* DW$$

Obviously W must be chosen to satisfy

$$(C_2^* D + B_2)W = 0 \tag{3.16}$$

in which case G_2 can be chosen arbitrarily and H_2 defined by $H_2^* = G_2 + B_2 W$. Equation (2.19) guarantees that (3.16) is satisfied for all W. Thus G_2 and W can be chosen arbitrarily.

Proof of Theorem 3.2

By Theorem 2.2, $E(s)$ is given by (2.20). Thus

$$\begin{aligned} V(-s)E(s) &= [V + H(-sI - A^*)^{-1}C^*][D + C(sI-A)^{-1}B - \tilde{C}_1(sI - \tilde{A}_{11})^{-1}\tilde{B}_1] \\ &= VD + VC(sI-A)^{-1}B - V\tilde{C}_1(sI-\tilde{A}_{11})^{-1}\tilde{B}_1 + H(-sI-A^*)^{-1}C^*D \\ &\quad + H(-sI-A^*)^{-1}C^*C(sI-A)^{-1}B \\ &\quad - H(-sI-A^*)^{-1}C^*\tilde{C}_1(sI-\tilde{A}_{11})^{-1}\tilde{B}_1 \end{aligned} \tag{3.17}$$

Now observe from (2.8 *d*) that

$$\begin{aligned} H(-sI-A^*)^{-1}C^*C(sI-A)^{-1}B &= H(-sI-A^*)^{-1}[(-sI-A^*)\Sigma \\ &\quad + \Sigma(sI-A)](sI-A)^{-1}B \\ &= H\Sigma(sI-A)^{-1}B + H(-sI-A^*)^{-1}\Sigma B \end{aligned} \tag{3.18}$$

Also, it follows from the (1, 3)- and (2, 3)-blocks of (2.9 b), together with (2.11), that

$$C^*\tilde{C}_1 = -\begin{bmatrix} A_{11}^*\Gamma + \Gamma\tilde{A}_{11} \\ A_{12}^*\Gamma \end{bmatrix} \tag{3.19}$$

Thus

$$H(-sI - A^*)^{-1}C^*\tilde{C}_1(sI - \tilde{A}_{11})^{-1}\tilde{B}_1 = H_1\Gamma(sI - \tilde{A}_{11})^{-1}\tilde{B}_1 + H(-sI - A^*)^{-1}\begin{bmatrix} \Gamma\tilde{B}_1 \\ 0 \end{bmatrix} \tag{3.20}$$

Substituting (3.18) and (3.20) into (3.17), we obtain

$$V(-s)E(s) = VD + (VC + H\Sigma)(sI - A)^{-1}B + H(-sI - A^*)^{-1}\left(C^*D + \Sigma B - \begin{bmatrix} \Gamma\tilde{B}_1 \\ 0 \end{bmatrix}\right) - (H_1\Gamma + V\tilde{C}_1)(sI - \tilde{A}_{11})^{-1}\tilde{B}_1 \tag{3.21}$$

$$= W^* + G^*(sI - A)^{-1}B = W(\bar{s})^* \tag{3.22}$$

where the following have been used:

$$\left.\begin{aligned} VC + H\Sigma &= G^* && \text{by (3.11 } b\text{)} \\ H_1\Gamma + V\tilde{C}_1 &= 0 && \text{by (3.14 } b\text{)} \\ C^*D + \Sigma B - \begin{bmatrix} \Gamma\tilde{B}_1 \\ 0 \end{bmatrix} &= 0 && \text{by (2.19) and (2.14 } b\text{)} \end{aligned}\right\} \tag{3.23}$$

□

Theorem 3.2 essentially provides a class of solutions to the factorization part of our problem. There is no guarantee, however, that all $V(s)$, $W(s)$ pairs constructed via Theorem 3.2 are minimum-phase, although they are obviously proper and stable. In the next section we shall show how to choose the free parameters G_2 and W so that $V(s)$, $W(s)$ in (3.12) will be minimum-phase.

4. Minimum-phase conditions and the product decomposition of $V(s)$, $W(s)$

In this section we seek to define a subset of solutions to (3.11) such that the associated factors $V(s)$, $W(s)$ of $E(s)$ are minimum-phase. This task will be divided into three subsections. Initially, we consider the case where $\Sigma < I$ ($r = 0$), implying that the G_2 block does not exist, so we only have to decide how to choose W. Next we deal with the other extreme case $\Sigma = I$, which implies that $E(s)$ is a stable all-pass matrix. The final subsection shows how for the general case $0 < \Sigma \leqslant I$ the matrices $V(s)$, $W(s)$ can be obtained by combining the solutions for these two extremes.

4.1. *The case* $0 < \Sigma < I$ ($r = 0$)

For this case the subscript-2 blocks of § 3 are non-existent, so that G, H are completely determined by (3.11) as in Lemma 3.2, and the only free (matrix) parameter is W.

Lemma 4.1

Let A be $m \times m$ stable B, C^* $m \times p$ and satisfying (2.8 c, d) with $0 < \Sigma < I$. Let G, H^* be $m \times p$, V, W $p \times p$ and satisfying (3.11) with D an arbitrary unitary matrix. Then $V(s)$, $W(s)$ given by (3.12) are minimum-phase if and only if W is non-singular.

Proof

We prove the result by showing $W(s)^{-1}$ and $V(s)^{-1}$ are stable. Observe from Lemma 3.2 that $W(s)$ is given by

$$W(s) = [I - B_1^*(sI - A_{11}^*)^{-1}\tilde{B}_1]W$$

(Since $r = 0$, we have $B = B_1$ and $A = A_{11}$. We use A_{11}, B_1 for later reference.) Thus if W is singular then $W(s)$ is singular for all s and so cannot be minimum-phase.

Now suppose W is non-singular. By the matrix-inversion lemma,

$$W(s)^{-1} = W^{-1}[I + B_1^*(sI - (A_{11}^* + \tilde{B}_1 B_1^*)^{-1}\tilde{B}_1] \tag{4.1}$$

so we need to show that $A_{11}^* + \tilde{B}_1 B_1^*$ is a stable matrix. From the (1, 3)-block of (2.9 a), together with (2.11), it follows that

$$A_{11}^* + \tilde{B}_1 B_1^* = -\tilde{A}_{11} \tag{4.2}$$

Thus $W(s)^{-1}$ is stable, since $\tilde{A}_{11}$ is completely unstable by Theorem 2.1, so $W(s)$ is minimum-phase.

Similarly by Lemma 3.2,

$$V(s) = V[I - \tilde{C}_1\Gamma^{-1}(sI - A_{11}^*)^{-1}C_1^*]$$

Thus singular W implies singular V by (3.11 c), so $V(s)$ is not minimum-phase.

Conversely, with W non-singular V is non-singular, and it follows as for $W(s)$ that $V(s)$ is minimum-phase from the observation that by the (1, 3)-block of (2.9 b):

$$A_{11}^* + C_1^*\tilde{C}_1\Gamma^{-1} = -\Gamma\tilde{A}_{11}\Gamma^{-1} \tag{4.3}$$

□

Corollary 4.1

Let $E(s) \in L_\infty$ be a minimal all-pass matrix with $\sigma_1(E) < 1$ and realization $E(s) = D + C_1(sI - A_{11})^{-1}B_1 - \tilde{C}_1(sI - \tilde{A}_{11})^{-1}\tilde{B}_1$ as in Theorem 2.2. Then $V(s)$, $W(s)$ satisfy (1.1) and are proper, stable and minimum-phase if and only if

$$W(s) = [I - B_1^*(sI - A_{11})^{-1}\tilde{B}_1]W \tag{4.4 a}$$

$$V(s) = W^*D^*[I - \tilde{C}_1\Gamma^{-1}(sI - A_{11})^{-1}C^*] \tag{4.4 b}$$

where W is an arbitrary $p \times p$ non-singular matrix.

Proof

$E(s) = V(-s)^{-1}W(\bar{s})^*$ follows from Theorem 3.2 and Lemma 3.2. $V(s)$, $W(s)$ are stable by the stability of A and are minimum-phase by Lemma 4.1. That (4.4) defines all solutions follows from Theorem 2.4—see the remarks following Theorem 2.4. □

4.2. *Factorization of stable all-pass matrices—the case $\Sigma = I$*

Subsection 4.1 dealt with the case when the subscript-2 blocks of § 3 were non-existent. This subsection deals with the opposite extreme, where the subscript-1 blocks are non-existent. We shall continue to use a subscript 2 so that we can later refer to the results without confusion. Note that when $\Sigma = I$ we have $r = m$ in the minimal all-pass extension theorem, so the minimal all-pass extension of $C_2(sI - A_{22})^{-1}B_2$ is just $D + C_2(sI - A_{22})^{-1}B_2$, where D satisfies (2.19), and this is a stable all-pass matrix. Note that the Lyapunov equations (2.8 c, d) are

$$A_{22} + A_{22}^* = -B_2B_2^* = -C_2^*C_2 \tag{4.5}$$

Lemma 4.2

Let A be $r \times r$ stable, B, C^* $r \times p$ and satisfying (2.8 c, d) with $\Sigma = I$. Let G, H^* be $r \times p$, V, W $p \times p$ and satisfying (3.11) with D satisfying (2.19). Let $V(s)$, $W(s)$ be given by (3.12). Then s_0 is a finite zero of $W(s)$ if and only if $-s_0$ is a finite zero of $V(s)$.

Proof

s_0 is a finite zero of $W(s)$ if and only if there exist vectors x, y not both zero such that

$$\begin{bmatrix} s_0I - A_{22}^* & G \\ -B_2^* & W \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\Leftrightarrow (s_0I - A_{22}^*)x + Gy = 0 \quad \text{and} \quad -B_2^*x + Wy = 0$$

$$\Leftrightarrow (s_0I - A_{22} + B_2B_2^*)x + (H_2^* - B_2W)y = 0 \quad \text{and} \quad -B_2^*x + Wy = 0 \quad \text{by (4.5) and (3.11 } a)$$

$$\Leftrightarrow (s_0I - A_{22})x + H_2^*y - B_2(-B_2^*x + Wy) = 0 \quad \text{and} \quad -B_2^*x + Wy = 0$$

$$\Leftrightarrow (s_0I - A_{22})x + H_2^*y = 0 \quad \text{and} \quad C_2x + V^*y = 0 \quad \text{by (2.19) and (3.11 } c)$$

$$\Leftrightarrow (-x^* \; -y^*)\begin{bmatrix} -s_0I - A_{22}^* & C_2^* \\ -H_2 & V \end{bmatrix} = 0$$

$\Leftrightarrow -s_0$ is a finite zero of $V(s)$. □

Corollary 4.2

Let $E(s) \in L_\infty$ be a stable all-pass matrix. Let $V(s)$, $W(s)$ be proper, stable and satisfy (1.1). Then $V(s)$, $W(s)$ are minimum-phase if and only if they are non-singular for all finite s.

Proof

This follows from Lemma 4.2. Alternatively, consider the following argument. Suppose that $V(s)$, $W(s)$ are minimum-phase. Then $V(-s)$, $W(\bar{s})^*$ have no common zeros. It follows from (1.1) that the poles of $E(s)$ are the zeros of $V(-s)$ and the poles of $W(\bar{s})^*$. Clearly then, since $E(s)$ is stable and any zeros of $V(-s)$ must be unstable, $V(s)$ has no finite zeros. That $W(s)$ has no finite zeros follows by applying the above argument to $E(s)^{-1} = E(-\bar{s})^*$. Conversely if $V(s)$, $W(s)$ have no finite zeros then they are minimum-phase. □

As a consequence of Lemma 4.2, we must show how to choose G_2 and W in Theorem 3.2 such that $V(s)$, $W(s)$ have no finite zeros. First, consider $W(s) =$

$D(s)^{-1}N(s)$, where $N(s)$, $D(s)$ are coprime polynomial matrices with $D(s)$ row-reduced. The finite zeros of $W(s)$ are given by the solutions of $\det N(s)=0$. Thus for $W(s)$ to have no finite zeros, we must have $\det N(s)=$ non-zero constant, i.e. $N(s)$ is a unimodular polynomial matrix. Observe from (3.12 *b*) that $D(s)$ is determined and the choice available for W, G_2 affects only $N(s)$. The simplest way to make $N(s)$ unimodular is to make it lower-triangular with constants on the diagonal. This can be done as follows.

Lemma 4.3

Let A_{22} be $r\times r$ stable, B_2 $r\times p$ with (A_{22}, B_2) controllable. Let $q \triangleq \operatorname{rank} B_2$ and T_1 be a $p\times p$ non-singular matrix such that $B_2=[\hat{B}_2 \quad 0]T_1$, with $\hat{B}_2$ $r\times q$ and of rank q. Let T_2 be the $r\times r$ non-singular similarity transformation transforming $(A_{22}, \hat{B}_2)$ to controllable canonical form (see equation (3.6.9) of Wolovich 1974) and ν_i, $i=1,\ldots,q$, the controllability indices of $(A_{22}, \hat{B}_2)$. Let A_m, B_m be defined as in Wolovich's structure theorem (Wolovich 1974, p. 105). Define

$$W=T_1^*\begin{bmatrix}\hat{W}_1 & 0\\ 0 & \hat{W}_2\end{bmatrix} \tag{4.6 a}$$

$$\hat{W}_2 \text{ is an arbitrary } (p-q)\times(p-q) \text{ non-singular matrix} \tag{4.6 b}$$

$$\hat{W}_1^* B_m^{-1} \text{ is an arbitrary } q\times q \text{ lower-triangular matrix with } \textit{zeros on the diagonal} \tag{4.6 c}$$

$$G_2=[T_2^*\hat{G}_2 \quad Y] \quad (r\times p) \tag{4.7 a}$$

$$Y \text{ is an arbitrary } r\times(p-q) \text{ matrix} \tag{4.7 b}$$

$$\hat{G}_2^*=\hat{W}_1^* B_m^{-1}A_m+N \tag{4.7 c}$$

$$N_i = i\text{th row of } N=[N_{i1} \quad \ldots \quad N_{ii} \quad 0 \quad \ldots \quad 0] \tag{4.8 a}$$

$$N_{ij} \text{ is a } 1\times\nu_j \text{ matrix} \tag{4.8 b}$$

$$=\begin{cases}0 & \text{if } i<j\\ [n_i \quad 0 \quad \ldots \quad 0], \text{ where } n_i \text{ is any non-zero number} & \text{if } i=j\\ \text{arbitrary} & \text{if } i>j\end{cases} \tag{4.8 c}$$

Then

$$W(s)=W+B_2^*(sI-A_{22}^*)^{-1}G_2 \tag{4.9}$$

has no finite zeros.

Proof

By (4.9) and the definition of T_1,

$$\begin{bmatrix}I & 0\\ 0 & T_1^{-*}\end{bmatrix}\begin{bmatrix}sI-A_{22}^* & G_2\\ -B_2^* & W\end{bmatrix}=\begin{bmatrix}sI-A_{22}^* & T_2^*\hat{G}_2 & Y\\ -\hat{B}_2^* & \hat{W}_1 & 0\\ 0 & 0 & \hat{W}_2\end{bmatrix}$$

Hence $W(s)$ has no finite zeros if and only if $\hat{W}_2$ is non-singular and

$$\hat{W}(s) \triangleq \hat{W}_1+\hat{B}_2^*(sI-A_{22}^*)^{-1}T_2^*\hat{G}_2 \tag{4.10}$$

has no finite zeros. $\hat{W}_2$ is non-singular by construction, and we prove $\hat{W}(s)$ has no finite zeros by applying Wolovich's structure theorem (Wolovich 1974, p. 106, Theorem 4.3.3) to $\hat{W}(\bar{s})^*$. This gives

$$\hat{W}(\bar{s})^* = [(\hat{G}_2^* - \hat{W}_1^* B_m^{-1} A_m)S(s) + \hat{W}_1^* B_m^{-1} \operatorname{diag}(s^{v_i})]D(s)^{-1} \tag{4.11}$$

where

$$S(\bar{s})^* = \text{black diag}([1, s, \ldots, s^{v_i - 1}]), \quad i = 1, \ldots, q \tag{4.12 a}$$

$D(s)$ is a polynomial matrix, non-singular almost everywhere,

with column degrees v_i, $i = 1, \ldots, q$ (4.12 *b*)

Now from (4.6 *c*)

$$\hat{W}_1^* B_m^{-1} \operatorname{diag}(s^{v_i}) = 0 \tag{4.13 a}$$

and from (4.7 *c*)

$$\hat{G}_2^* - \hat{W}_1^* B_m^{-1} A_m = N \tag{4.13 b}$$

Hence, substituting (4.8), (4.13) into (4.11),

$$\hat{W}(\bar{s})^* = N(s)D(s)^{-1} \tag{4.14 a}$$

$N(s)$ is a lower-triangular polynomial matrix with column degrees $< v_i$
and with n_i on the diagonal (4.14 *b*)

Clearly $\det N(s) = \prod n_i =$ a non-zero constant by (4.8 *c*), so $\hat{W}(s)$ has no finite zeros. □

The precise formulae for $D(s)$, A_m, B_m can be found in Wolovich (1974).

The lemma allows us to construct $W(s)$ so that $W = W(\infty)$ has arbitrary rank between $p - q$ and $p - 1$ (by (4.6)). This connects nicely with Theorem 2.4, (2.27) and (2.28), since

$$\begin{aligned} p - q &= p - \operatorname{rank} B_2 = p - \#\{k_j > 0\} \quad \text{by Theorem 2.5} \\ &= \#\{k_j = 0\} \end{aligned}$$

where k_j are the partial indices of $E(s) = D + C_2(sI - A_{22})^{-1} B_2$.

Corollary 4.3

Let $E(s) \in L_\infty$ be a stable all-pass matrix of degree r with realization $D + C_2(sI - A_{22})^{-1} B_2$ satisfying (4.5). Let G_2, $r \times p$, and W, $p \times p$, be constructed in accordance with Lemma 4.3. Define

$$V = (DW)^*$$

$$H_2 = G_2^* - VC_2$$

Then

$$W_2(s) = W + B_2^*(sI - A_{22}^*)^{-1} G_2 \tag{4.15 a}$$

$$V_2(s) = V + H_2(sI - A_{22}^*)^{-1} G_2^* \tag{4.15 b}$$

are proper, stable, minimum-phase and satisfy

$$E(s) = V_2(-s)^{-1} W_2(\bar{s})^* \tag{4.16}$$

Proof

This follows from Lemmas 4.2, 4.3 and Theorem 3.2. □

4.3. *Product decomposition of $V(s)$, $W(s)$—The general case $0 < \Sigma \leqslant I$*

In this subsection, taking our lead from Theorem 2.4, we shall show how to decompose $V(s)$, $W(s)$ given by (3.12) as a product of two matrix functions:

$$W(s) = W_1(s)W_2(s) \tag{4.17 a}$$

$$V(s) = V_2(s)V_1(s) \tag{4.17 b}$$

where $V_1(s)$, $W_1(s)$ are stable, minimum-phase and non-singular at infinity (i.e. they are like the case treated in § 4.1) and $L(s) = V_2(-s)^{-1}W_2(\bar{s})^*$ is stable all-pass (i.e. $L(s)$ is like the case treated in § 4.2). The result is the following.

Theorem 4.1

Let $E(s) \in L_\infty$ be a minimal all-pass matrix with realization $E(s) = D + C(sI - A)^{-1}B - \tilde{C}_1(sI - \tilde{A}_{11})^{-1}\tilde{B}_1$ as in Theorem 2.2. Define

$$W_1(s) = I - B^*(sI - A^*)^{-1}[B_1^* \quad 0]^* \tag{4.18 a}$$

$$V_1(s) = I - [\tilde{C}_1\Gamma^{-1} \quad 0](sI - A^*)^{-1}C^* \tag{4.18 b}$$

$$L(s) = D + C_2(sI - A_{22})^{-1}B_2 \tag{4.19}$$

Then

(i) $V_1(s)$, $W_1(s)$ are proper, stable, minimum phase and non-singular at infinity;

(ii) $L(s)$ is a stable all-pass matrix;

(iii) $E(s) = V_1(-s)^{-1}L(s)W_1(\bar{s})^*$ (4.20)

Proof

(ii) That $L(s)$ is a stable all-pass matrix follows from the (2, 2)-blocks of (2.8 *c*, *d*) and Theorem 5.1 of Glover (1984).

(i) Since A is stable, $V_1(\infty) = W_1(\infty) = I$, and (4.18) holds, it follows that $V_1(s)$, $W_1(s)$ are proper, stable and non-singular at infinity. Thus we need to show that they are minimum-phase. As in the proof of Lemma 4.1, we do this by showing $W_1(s)^{-1}$ and $V_1(s)^{-1}$ are stable.

To show $W_1(s)^{-1}$ is stable, we need to show that $A^* + [\tilde{B}_1^* \quad 0]^*B$ is a stable matrix:

$$A^* + \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix} B^* = \begin{bmatrix} A_{11}^* + \tilde{B}_1 B_1^* & A_{21}^* + \tilde{B}_1 B_2^* \\ A_{12}^* & A_{22}^* \end{bmatrix} = \begin{bmatrix} -\tilde{A}_{11} & 0 \\ A_{12}^* & A_{22}^* \end{bmatrix} \tag{4.21}$$

where we have used (4.2) and the (2, 3)-block of (2.9 *a*), which gives

$$A_{21} + B_2\tilde{B}_1^* = 0 \tag{4.22}$$

Since $-\tilde{A}_{11}$ and A_{22} are stable, it follows from (4.21) that $W(s)$ is minimum-phase.

For $V(s)$ we need to show that $A^* + C^*[\tilde{C}_1\Gamma^{-1} \quad 0]$ is a stable matrix. This follows as above using (4.3) and the (2, 3)-block of (2.9 *b*), which gives

$$-A_{12}^*\Gamma - C_2^*\tilde{C}_1 = 0 \tag{4.23}$$

(iii) The proof of this is along the lines of the proof of Theorem 3.2:

$$V_1(-s)E(s) = [I + [\tilde{C}_1\Gamma^{-1} \quad 0](sI + A^*)^{-1}C^*] \times [D + C(sI - A)^{-1}B - \tilde{C}_1(sI - \tilde{A}_{11})^{-1}\tilde{B}_1]$$

Observe that

$$(sI + A^*)^{-1}C^*C(sI - A)^{-1}B = -\Sigma(sI - A)^{-1}B + (sI + A^*)^{-1}\Sigma B \quad \text{by (2.8 } d)$$

and

$$(sI + A^*)^{-1}C^*\tilde{C}_1(sI - \tilde{A}_{11})^{-1}\tilde{B}_1 = -[\Gamma \quad 0]^*(sI - \tilde{A}_{11})^{-1}\tilde{B}_1 + (sI + A^*)[\tilde{B}_1^*\Gamma \quad 0]^*$$

using (3.20). Hence

$$\begin{aligned} V_1(-s)E(s) &= D + [C_1 - \tilde{C}_1\Sigma_1\Gamma^{-1} \quad C_2](sI - A)^{-1}B \\ &\quad + [\tilde{C}_1\Gamma^{-1} \quad 0](sI + A^*)^{-1}(\Sigma B + C^*D - [\tilde{B}_1^*\Gamma \quad 0]^*) \\ &= D + [-D\tilde{B}_1^* \quad C_2](sI - A)^{-1}B \end{aligned} \tag{4.24}$$

where we have used the 3-block of (2.10 c), which gives

$$D\tilde{B}_1^* + C_1\Sigma_1\Gamma^{-1} = 0 \tag{4.25}$$

and (3.23).

By the matrix-inversion lemma,

$$W_1(\bar{s})^{-*} = I + [\tilde{B}_1^* \quad 0](sI - \hat{A})^{-1}B \tag{4.26}$$

where

$$\hat{A} = A + B[\tilde{B}_1^* \quad 0] \tag{4.27}$$

Observe from (4.27) that

$$(sI - A)^{-1}B[\tilde{B}_1^* \quad 0](sI - \hat{A})^{-1} = (sI - \hat{A})^{-1} - (sI - A)^{-1} \tag{4.28}$$

Hence by (4.24), using (4.28),

$$\begin{aligned} V_1(-s)E(s)W_1(\bar{s})^{-*} &= D + [0 \quad C_2](sI - \hat{A})^{-1}B \\ &= D + C_2(sI - A_{22})^{-1}B_2, \quad \text{using (4.27) and (4.21)} \\ &= L(s) \end{aligned}$$

□

Theorem 4.1 gives a multiplicative characterization of minimal all-pass matrices instead of the additive characterization provided by Theorem 2.2. Of course, with Theorem 4.1 and Corollary 4.3 we have our stable minimum-phase factorization theorem.

Corollary 4.4

Let $E(s) \in L_\infty$ be a minimal all-pass matrix with realization $E(s) = D + C(sI - A)^{-1}B - \tilde{C}_1(sI - \tilde{A}_{11})^{-1}\tilde{B}_1$ as in Theorem 2.2. Let G_2, H_2, W, V be constructed in accordance with Corollary 4.3 and define G_1, H_1 by (3.14). Then $V(s)$, $W(s)$ given by (3.12) are proper, stable, minimum-phase and satisfy $E(s) = V(-s)^{-1}W(\bar{s})^*$. Furthermore,

$$W(s) = W_1(s)W_2(s) \tag{4.29 a}$$

$$V(s) = V_2(s)V_1(s) \tag{4.29 b}$$

with $V_1(s)$, $W_1(s)$ as in Theorem 4.1, see (4.18), and $V_2(s)$, $W_2(s)$ as in Corollary 4.3, see (4.15).

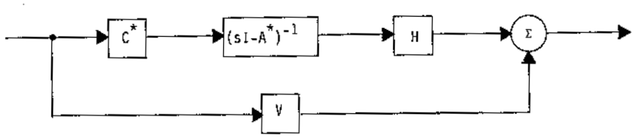
Proof

If (4.29) is true then it follows immediately from Corollary 4.3 and Theorem 4.1 that $V(s)$, $W(s)$ are proper, stable, minimum-phase and satisfy (1.1). Thus we need to prove (4.29). Consider the right-hand side of (4.29 b):

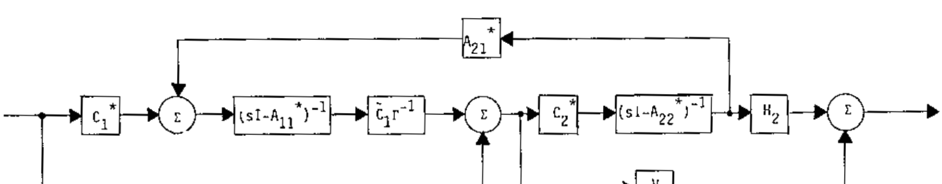
$$\begin{aligned}
V_2(s)V_1(s) &= V + H_2(sI - A_{22}^*)^{-1}C_2^* - [V\tilde{C}_1\Gamma^{-1} \quad 0](sI - A^*)^{-1}C^* \\
&\quad + H_2(sI - A_{22}^*)^{-1}[A_{12}^* \quad 0](sI - A^*)^{-1}C^*, \quad \text{using (4.23)} \\
&= V + [H_1 \quad 0](sI - A^*)^{-1}C^* \\
&\quad + H_2(sI - A_{22}^*)^{-1}[[0 \quad I](sI - A^*) + [A_{12}^* \quad 0]] \\
&\quad \times (sI - A^*)^{-1}C^*, \quad \text{using (3.14 } b) \\
&= V + [H_1 \quad 0](sI - A^*)^{-1}C^* \\
&\quad + H_2(sI - A_{22}^*)^{-1}[0 \quad sI - A_{22}^*](sI - A^*)^{-1}C^* \\
&= V + H(sI - A^*)^{-1}C^*
\end{aligned}$$

Thus (4.29 b) is equivalent to (3.12 b), with H_2, V defined as in Corollary 4.3.

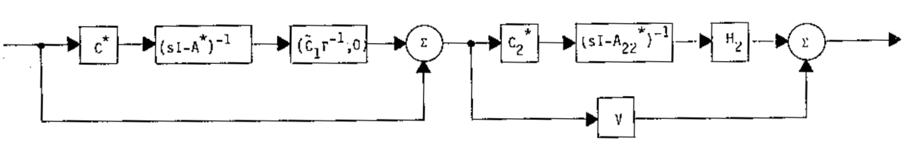
It follows similarly, using (4.22) and (3.14 a), that $W(s)$ defined by the right-hand side of (4.29 a) is equivalent to (3.12 a). □



(a)



(b)



(c)

Figure 1. Structure of $V(s)$: (a) Minimal realization; (b) product decomposition with feedback (minimal); (c) product decomposition (non-minimal).

Consider $V(s)$ given by (3.12 *b*). Now recall the product formula for transfer matrices, viz (in obvious notation)

$$T_2(s)T_1(s) = D_2D_1 + [D_2C_1 \quad C_2]\left(sI - \begin{bmatrix} A_1 & 0 \\ B_2C_1 & A_2 \end{bmatrix}\right)^{-1}\begin{bmatrix} B_1 \\ B_2D_1 \end{bmatrix} \tag{4.30}$$

Now observe the formula (3.12 *b*) for $V(s)$ and consider (3.14 *b*) and (4.23). $V(s)$ is easily seen to be decomposable as a product with feedback from the state of T_2 to the state of T_1, the feedback gain matrix being A_{21}^* (see Fig. 1). To completely decompose $V(s)$ as a product, we need to take the feedback from T_2 into account in our calculation of the state of T_1, which we do by augmenting the state of T_1 with the state of T_2 and assigning the output matrix from the T_2 state to be zero. This leads to (4.18 *b*) for $V_1(s)$. The observation that $V(s)$, $W(s)$ have the product described above plus feedback form is in fact how formulae (4.18) came about. Thus the product decomposition of $V(s)$ in (4.29 *b*) contains two copies of the state of $V_2(s)$, one of which is uncontrollable. This can be simply verified by multiplying V_2 and V_1 using the above product formula and performing an obvious state transformation.

5. Wiener–Hopf factorization of all-pass matrices

Until now, we have been dealing with the factorization of minimal all-pass matrices since we have been primarily concerned with the factorization (1.1). Now we turn to deriving formulae for the Wiener–Hopf factors of Theorem 2.3 in the all-pass case (for the general case see Bart *et al.* 1983). It is easily observed that Theorem 4.1 in fact provides a 'pre-Wiener–Hopf' factorization of a minimal all-pass matrix. The factorization of Theorem 4.1 is not quite a Wiener–Hopf factorization, because the centre term $L(s)$ is not diagonal, nor does it have centralized singularities—see (2.22 *b*). In this section we first extend Theorem 4.1 to the case when $E(s)$ is not minimal, thus providing a 'pre-Wiener–Hopf' factorization. We then show how to Wiener–Hopf-factorize the remaining central term ($L(s)$ in Theorem 4.1).

5.1. *Reduction to a simple all-pass matrix*

In this subsection we extend Theorem 4.1 to non-minimal all-pass matrix functions.

Definition

Let $E(s) \in L_\infty$ be all-pass and write

$$E(s) = E(\infty) + E_+(s) + E_-(s) \tag{5.1}$$

with $E_+(s)$, $E_-(-s) \in H_\infty^+$. $E(s)$ will be called a *simple all-pass matrix* when $E(\infty) + E_+(s)$ and $E(\infty) + E_-(s)$ are both all-pass.

Obviously, stable all-pass matrices are simple, as are completely unstable all-pass matrices.

Lemma 5.1

Let $E(s) \in L_\infty$ be all-pass. Then $E(s)$ is simple if and only if it can be written as

$$E(s) = D + C_2(sI - A_{22})^{-1}B_2 - \tilde{C}_2(sI - \bar{A}_{22})^{-1}B_2 \tag{5.2}$$

where (D, C_2, A_{22}, B_2) satisfy (4.5) and (2.19), and $(\tilde{C}_2, \tilde{A}_{22}, \tilde{B}_2)$ satisfy (2.15 *a, b, e*).

Proof

This follows from Theorem 2.1.

It also follows (from Lemma 5.1 or directly) that $E(s)$ is simple if and only if $E_+(s)E_-(-\bar{s})^* = 0$, where $E_\pm(s)$ are as in (5.1) (see (2.15 *a*)). Additionally, simple all-pass matrices can also be characterized by their factorization properties: by Lemma 5.1 and Theorem 2.5, an all-pass matrix $E(s)$ is simple if and only if the positive partial factorization indices of $E(s)$ are the controllability indices of $E(\infty) + E_+(s)$ and the negative partial indices are the negatives of the observability indices of $E(\infty) + E_-(s)$.

Theorem 5.1

Let $E(s) \in L_\infty$ be all-pass with realization

$$E(s) = D + C(sI - A)^{-1}B - \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} \tag{5.3}$$

as in Theorem 2.1. Define

$$W_1(s) = I - B^*(sI - A^*)^{-1}[[\tilde{B}_1^* \quad 0]^* + [C_1\Gamma^{-1} \quad 0]^*\tilde{C}_2(sI + \tilde{A}_{22})^{-1}\tilde{B}_2] \tag{5.4 a}$$

$$V_1(s) = I - [[\tilde{C}_1\Gamma^{-1} \quad 0] + \tilde{C}_2(sI + \tilde{A}_{22})^{-1}\tilde{B}_2[B_1^*\Gamma^{-1} \quad 0](sI - A^*)^{-1}C^* \tag{5.4 b}$$

$$E_S(s) = D + C_2(sI - A_{22})^{-1}B_2 - \tilde{C}_2(sI - \tilde{A}_{22})^{-1}\tilde{B}_2 \tag{5.5}$$

Then

(i) $V_1(s)$, $W_1(s)$ are proper, stable, minimum-phase and non-singular at infinity;

(ii) $E_S(s)$ is a simple all-pass matrix;

(iii)
$$E(s) = V_1(-s)^{-1}E_S(s)W_1(\bar{s})^* \tag{5.6}$$

Proof

(i) Obviously $V_1(s)$, $W_1(s)$ are proper and non-singular at infinity. Also, since A is stable and $-\tilde{A}_{22}$ is stable (by (2.15 *e*)), we see that $V_1(s)$, $W_1(s)$ are stable. Thus we need to show $V_1(s)$ and $W_1(s)$ are minimum-phase. Write $W_1(\bar{s})^*$ as

$$W_1(\bar{s})^* = I - [\tilde{B}_1^* \quad 0 \quad \tilde{B}_2^*]\left[sI - \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ -\tilde{A}_{12}^* & 0 & -\tilde{A}_{22}^* \end{bmatrix}\right]^{-1}\begin{bmatrix} B_1 \\ B_2 \\ 0 \end{bmatrix} \tag{5.7}$$

where we have used (2.15 *c*) and the formula for triangular-block matrix inversion. $W_1(s)$ will be minimum-phase if $W_1(\bar{s})^{-*}$ is stable. By the matrix-inversion lemma, $W_1(\bar{s})^*$ is stable if $\hat{A}$ is stable, where

$$\hat{A} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ -\tilde{A}_{12}^* & 0 & -\tilde{A}_{22}^* \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ 0 \end{bmatrix}[\tilde{B}_1^* \quad 0 \quad \tilde{B}_2] \tag{5.8 a}$$

Now using (4.2), (4.22) and (2.15 *d*), observe that

$$\hat{A} = \begin{bmatrix} -\tilde{A}_{11}^* & A_{12} & -\tilde{A}_{21}^* \\ 0 & A_{22} & 0 \\ -\tilde{A}_{12}^* & 0 & -\tilde{A}_{22}^* \end{bmatrix} \tag{5.8 b}$$

and clearly, for an obvious non-singular X,

$$\hat{A} = X \begin{bmatrix} -\tilde{A}_{11}^* & -\tilde{A}_{21}^* & A_{12} \\ -\tilde{A}_{12}^* & -\tilde{A}_{22}^* & 0 \\ 0 & 0 & A_{22}^* \end{bmatrix} X^{-1} \tag{5.8 c}$$

which is stable since $-\tilde{A}$ and A_{22} are stable. Thus $W_1(s)$ is minimum-phase. It can similarly be shown, using (4.3), (4.23) and (2.15 *c*) that $V_1(s)$ is minimum phase.

(ii) follows from Lemma 5.1.

(iii) We show that

$$V_1(-s)E(s) = E_S(s)W_1(\bar{s})^* \tag{5.9}$$

First observe from (2.9) together with (2.11) that

$$(sI + A^*)^{-1}C^*C(sI - A)^{-1} = -\Sigma(sI - A)^{-1} + (sI + A^*)^{-1}\Sigma \tag{5.10 a}$$

$$(sI + A^*)^{-1}C^*\tilde{C}(sI - \tilde{A})^{-1} = N(sI - \tilde{A})^{-1} - (sI + A^*)^{-1}N \tag{5.10 b}$$

where N is as in (2.11 *e*). Using (5.3), (5.4 *b*) and (5.10),

$$\begin{aligned} V_1(-s)E(s) = E(s) &+ [[\tilde{C}_1\Gamma^{-1} \quad 0] - \tilde{C}_2(sI - \tilde{A}_{22})^{-1}\tilde{B}_2[B_1^*\Gamma^{-1} \quad 0]] \\ &\times [-\Sigma(sI - A)^{-1}B + (sI + A^*)^{-1}(\Sigma B + C^*D + N\tilde{B}) - N(sI - \tilde{A})^{-1}\tilde{B}] \end{aligned} \tag{5.11}$$

Recall from (2.19), (2.14 *b*) and (2.11 *e*)—see (3.23)—that $\Sigma B + C^*D + N\tilde{B} = 0$. Now consider the $(sI - \tilde{A})^{-1}\tilde{B}$ terms in (5.11):

$$\begin{aligned} &[-\tilde{C} - [\tilde{C}_1\Gamma^{-1} \quad 0]N + \tilde{C}_2(sI - \tilde{A}_{22})^{-1}\tilde{B}_2[B_1\Gamma^{-1} \quad 0](sI - \tilde{A})^{-1}\tilde{B} \\ &\quad = [[0 \quad -\tilde{C}_2] + \tilde{C}_2(sI - \tilde{A}_{22})^{-1}[\tilde{A}_{21} \quad 0]](sI - \tilde{A})\tilde{B} \quad \text{by (2.11 } e\text{) and (2.15 } d\text{)} \\ &\quad = -\tilde{C}_2(sI - \tilde{A}_{22})^{-1}[[0 \quad sI - \tilde{A}_{22}] - [\tilde{A}_{21} \quad 0]](sI - \tilde{A})^{-1}\tilde{B} \\ &\quad = -\tilde{C}_2(sI - \tilde{A}_{22})^{-1}[0 \quad I]\tilde{B} \\ &\quad = -\tilde{C}_2(sI - \tilde{A}_{22})^{-1}\tilde{B}_2 \end{aligned} \tag{5.12}$$

Thus

$$\begin{aligned} V(-s)E(s) = D &- \tilde{C}_2(sI - \tilde{A}_{22})^{-1}\tilde{B}_2 \\ &+ [[C_1 - \tilde{C}_1\Sigma_1\Gamma^{-1} \quad C_2] - \tilde{C}_2(sI - \tilde{A}_{22})^{-1}[\tilde{A}_{21}\Sigma_1\Gamma^{-1} \quad 0]] \\ &\times (sI - A)^{-1}B \end{aligned} \tag{5.13}$$

We now calculate $E_S(s)W_1(\bar{s})^*$. First observe (2.15 *a*) and that

$$(sI - \tilde{A}_{22})^{-1}\tilde{B}_2\tilde{B}_2^*(sI + \tilde{A}_{22}^*)^{-1} = (sI - \tilde{A}_{22})^{-1} - (sI + \tilde{A}_{22}^*)^{-1} \tag{5.14}$$

by (2.15 *e*). Now using (5.4 *a*) and (5.5),

$$\begin{aligned} E_S(s)W_1(\bar{s})^* &= E_S(s) - E_S(s)[[\tilde{B}_1^* \quad 0] + \tilde{B}_2^*(sI + \tilde{A}_{22}^*)^{-1}\tilde{C}_2^*[C_1\Gamma^{-1} \quad 0]](sI - A)^{-1}B \\ &= E_S(s) + [[-D\tilde{B}_1^* \quad 0] + C_2(sI - A_{22})^{-1}[A_{21}, 0] \\ &\quad + \tilde{C}_2(sI - \tilde{A}_{22})^{-1}(\tilde{B}_2[\tilde{B}_1^* \quad 0] + \tilde{C}_2[C_1\Gamma^{-1} \quad 0])](sI - A)^{-1}B \end{aligned} \tag{5.15}$$

using (2.15 *b*) and (4.22). Consider the $C_2(sI - A_{22})^{-1}$ terms:

$$
\begin{aligned}
C_2(sI-A_{22})^{-1}&[B_2+[A_{21}\quad 0](sI-A)^{-1}B] \\
&= C_2(sI-A_{22})^{-1}[[0\quad I](sI-A)+[A_{21}\quad 0]](sI-A)^{-1}B \\
&= C_2(sI-A_{22})^{-1}[0\quad sI-A_{22}](sI-A)^{-1}B \\
&= [0\quad C_2](sI-A)^{-1}B
\end{aligned} \tag{5.16}
$$

Also

$$
\begin{aligned}
\tilde{B}_2\tilde{B}_1^* + \tilde{C}_2C_1\Gamma^{-1} &= \tilde{B}_2(D^*C_1 + B_1^*\Sigma_1)\Gamma^{-1} - \tilde{B}_2D^*C_1\Gamma^{-1}, \quad \text{by (2.14 } b\text{) and (2.15 } b\text{)} \\
&= \tilde{B}_2B_1^*\Sigma_1\Gamma^{-1} = -\tilde{A}_{21}\Sigma_1\Gamma^{-1}, \quad \text{by (2.15 } d\text{)}
\end{aligned} \tag{5.17}
$$

Now substituting (5.17), (5.16) and (4.25) into (5.15),

$$
\begin{aligned}
E_S(s)W(\bar{s})^* = D &- \tilde{C}_2(sI-\tilde{A}_{22})^{-1}\tilde{B}_2 \\
&+ [[C_1 - \tilde{C}_1\Sigma_1\Gamma^{-1}\quad C_2] - \tilde{C}_2(sI-\tilde{A}_{22})^{-1}[\tilde{A}_{21}\Sigma_1\Gamma^{-1}\quad 0]] \\
&\times (sI-A)^{-1}B \\
= {} & V(-s)E(s), \quad \text{by (5.13)}
\end{aligned} \tag{5.18}
$$

□

Figure 2 gives a block diagram for Theorem 5.1, from which the formulae for, and the symmetry between, $W(\bar{s})^*$ and $V(-s)^{-1}$ can be seen.

It now remains to show how to convert a simple all-pass matrix to a diagonal all-pass matrix with centralized singularities (nomenclature from Bart *et al.* 1983), i.e. how to Wiener–Hopf-factorize a simple all-pass matrix.

5.2. *Wiener–Hopf factorization of simple all-pass matrices*

Definition

Let $H_i(s) \in L_\infty$, $i = 1, 2$, have no zeros on $s = j\omega$, $\omega \in \mathbb{R} \cup \{\infty\}$. $H_1(s)$ will be said to be *factorization-equivalent* to $H_2(s)$ if $H_1(s)$ and $H_2(s)$ have the same partial factorization indices.

Equivalently, $H_i(s)$, $i = 1, 2$, are factorization-equivalent if and only if there exist $H_+(-s)$, $H_-(-s)$ proper, stable, minimum phase and non-singular at infinity such that $H_2(s) = H_-(s)H_1(s)H_+(s)$. Thus in Theorem 5.1, $E(s)$ and $E_S(s)$ are factorization-equivalent. This also follows from the definition, since by Theorem 2.5 the partial indices of $E(s)$ are determined by A_{22}, B_2 and $\tilde{A}_{22}$, $\tilde{B}_2$, implying that $E(s)$ and $E_S(s)$ have the same partial indices.

This subsection shows how to construct $H_\pm(s)$, with the above properties, such that $E_2(s) = H_-(s)E_1(s)H_+(s)$, where $E_1(s)$ and $E_2(s)$ are factorization-equivalent simple all-pass matrices. The idea is quite simple: write $E_i(s) = X_i(s)Y_i(s)$ for some rational matrix functions $X_i(s)$, $Y_i(s)$ (not necessarily polynomial, or proper). Clearly

$$E_2(s) = X_2(s)X_1(s)^{-1}E_1(s)Y_1(s)^{-1}Y_2(s)$$

The problem is then to show how to choose $X_i(s)$, $Y_i(s)$ such that $Y_1(s)^{-1}Y_2(s)$ and $X_2(-s)X_1(-s)^{-1}$ are proper, stable, minimum-phase and non-singular at infinity.

First consider the simplest case—that of stable all-pass matrices with all partial indices strictly positive (i.e. B full column rank by Theorem 2.5). For this we drop our previous subscript conventions on realization, because we now need to differentiate between realizations of different all-pass matrices.

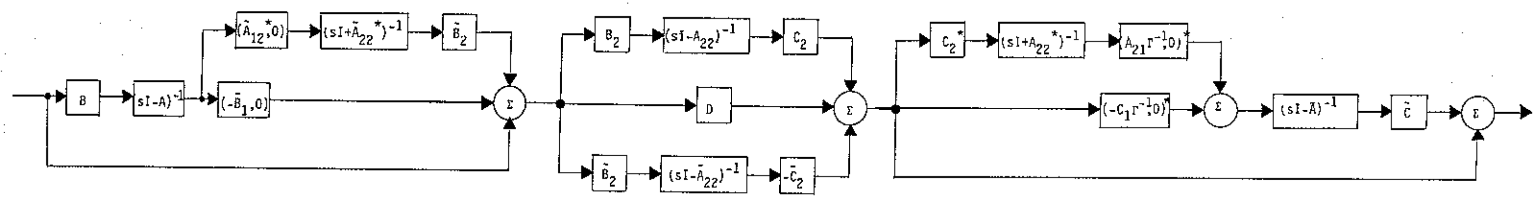


Figure 2. Factorization structure of an all-pass matrix (Theorem 5.1). $E(s) = D + C(sI - A)^{-1}B - \tilde{C}(sI - \tilde{A})^{-1}\tilde{B}$.

Lemma 5.2

Let $E_1(s) \in L_\infty$ be a stable all-pass matrix with partial indices $k_j > 0$, $j = 1, \ldots, p$. Let $E_2(s) \in L_\infty$ be stable, all-pass and factorization-equivalent to $E_1(s)$. Let $E_i(s) = N_i(-s)M_i(s)^{-1}$, $i = 1, 2$, with $N_i(s)$, $M_i(s)$ coprime polynomial matrices and $M_i(s)$ column-proper (Wolovich 1974). Define

$$H_-(s) = N_2(-s)N_1(-s)^{-1} \tag{5.19 a}$$

$$H_+(s) = M_1(s)M_2(s)^{-1} \tag{5.19 b}$$

Then $H_+(s)$, $H_-(s)$ are proper, stable, minimum-phase, non-singular at infinity and

$$E_2(s) = H_-(s)E_1(s)H_+(s) \tag{5.20}$$

Proof

It is trivial to verify (5.20). What need to be proved are the properties of $H_\pm(s)$. Consider first $H_+(s)$. Since $E_i(s)$, $i = 1, 2$, are simple with no negative indices, their partial indices are their controllability indices. Since $E_i(s)$ are factorization-equivalent, they have the same partial indices. Hence $E_i(s)$ have the same controllability indices k_j. Consequently, $M_i(s)$ have the same column degrees k_j, and have only LHP zeros since $E_i(s)$ are stable. It follows that $H_+(s)$ is proper (by Lemma 6.3-10 of Kailath 1980), stable (since $M_2(s)$ has only LHP zeros), minimum-phase (since $M_1(s)$ has only LHP zeros) and non-singular at infinity (since $M_i(s)$ have the same column degrees and are column-proper).

Now since $E_i(\infty)$ are non-singular, it follows that the column degrees of $N_i(s)$ are equal to those of $M_i(s)$, and $N_i(s)$ are column-proper. Thus $H_-(s)$ is proper and non-singular at infinity. Since $E_i(-s)^{-1}$ are stable (equivalently the zeros of $E_i(s)$ are in the right half plane) it follows that $H_-(-s)$ is stable and minimum-phase. □

If state-space formulae for $H_\pm(s)$ are desired, they can easily be obtained from (5.19) and Wolovich's structure theorem (Wolovich 1974). Also note that $W(\bar{s})^* = M(s)^{-1}$ and $V(s) = N(s)^{-1}$ provides a stable minimum-phase factorization of $E(s)$ with $V(s)$, $W(s)$ strictly proper. We now use Lemma 5.2 to handle the case of simple all-pass matrices.

Firstly consider (2.15 *a*): $B\tilde{B}^* = 0$. It follows that there exists a unitary $p \times p$ matrix U such that

$$B = [\beta \quad 0 \quad 0]U \quad \text{and} \quad \tilde{B} = [0 \quad 0 \quad \tilde{\beta}]U$$

where β, $\tilde{\beta}$ have full column rank. Also observe from Lemma 5.1 and Theorem 2.5 that for factorization-equivalent simple all-pass matrices $E_i(s)$, $i = 1, 2$, rank B_1 = rank B_2 and rank $\tilde{B}_1$ = rank $\tilde{B}_2$.

Lemma 5.3

Let $E_i(s) \in L_\infty$, $i = 1, 2$, be factorization-equivalent simple all-pass matrices with partial indices k_j, $j = 1, \ldots, p$, and balanced realizations

$$E_i(s) = D_i + C_i(sI - A_i)^{-1}B_i - \tilde{C}_i(sI - \tilde{A}_i)^{-1}\tilde{B}_i, \quad i = 1, 2 \tag{5.21}$$

Let $q \triangleq$ rank B_i and $t \triangleq$ rank $\tilde{B}_i$, $i = 1, 2$. Let U_i be $p \times p$ unitary such that

$$B_i = [\beta_i \quad 0 \quad 0]U_i, \quad i = 1, 2 \tag{5.22 a}$$

$$\tilde{B}_i = [0 \quad 0 \quad \tilde{\beta}_i]U_i, \quad i = 1, 2 \tag{5.22 b}$$

with β_i, $\tilde{\beta}_i$ full column rank. Let

$$I-\beta_1^*(sI-A_1)^{-1}\beta_1=N_1(-s)M_1(s)^{-1} \tag{5.23 a}$$

$$I-\beta_2^*(sI-A_2)^{-1}\beta_2=N_2(-s)M_2(s)^{-1} \tag{5.23 b}$$

$$I+\tilde{\beta}_1^*(sI-\tilde{A}_1)^{-1}\tilde{\beta}_1=\tilde{M}_1(-s)^{-1}\tilde{N}_1(s) \tag{5.23 c}$$

$$I+\tilde{\beta}_2^*(sI-\tilde{A}_2)^{-1}\tilde{\beta}_2=\tilde{M}_2(-s)^{-1}\tilde{N}_2(s) \tag{5.23 d}$$

be irreducible polynomial matrix-fraction descriptions. Define

$$H_-(s)=D_2U_2^{-1}\begin{bmatrix} N_2(-s)N_1(-s)^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \tilde{M}_2(-s)^{-1}\tilde{M}_1(-s)\end{bmatrix}\begin{array}{l} q \text{ rows} \\ U_1D_1^* \\ t \text{ rows}\end{array} \tag{5.24 a}$$

$$H_+(s)=U_1^{-1}\begin{bmatrix} M_1(s)M_2(s)^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \tilde{N}_1(s)^{-1}\tilde{N}_2(s)\end{bmatrix}\begin{array}{l} q \text{ rows} \\ U_2 \\ t \text{ rows}\end{array} \tag{5.24 b}$$

Then $H_+(s)$, $H_-(-s)$ are proper, stable, minimum-phase and satisfy

$$E_2(s)=H_-(s)E_1(s)H_+(s) \tag{5.25}$$

Proof

First note that A_1, A_2 are the same dimension $r\times r$, where r is given by (2.4) and $\tilde{A}_1$, $\tilde{A}_2$ are the same dimension $l\times l$ (by factorization equivalence and Theorem 2.5, since $r=$ sum of positive partial indices, and $l=-$sum of negative partial indices). We also have (by factorization equivalence and Theorem 2.5) rank $B_1=$ rank $B_2\triangleq q$ and rank $\tilde{B}_1=$ rank $\tilde{B}_2\triangleq t$. Also note that $q+t\leqslant p$ by (2.15 a) and Lemma 5.1. By (2.19), for $i=1,2$,

$$\begin{aligned} E_i(s) &= D_i[I-B_i^*(sI-A_i)^{-1}B_i+\tilde{B}_i^*(sI-\tilde{A}_i)^{-1}B_i] \\ &= D_iT_i^*[I-[\beta_i\ \ 0\ \ 0]^*(sI-A_i)^{-1}[\beta_i\ \ 0\ \ 0] \\ &\qquad +[0\ \ 0\ \ \tilde{\beta}_i]^*(sI-\tilde{A}_i)^{-1}[0\ \ 0\ \ \tilde{\beta}_i]]T_i \\ &= D_iT_i^*\begin{bmatrix} I-\beta_i^*(sI-A_i)^{-1}\beta_i & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I+\tilde{\beta}_i^*(sI-\tilde{A}_i)^{-1}\tilde{\beta}_i\end{bmatrix}T_i \end{aligned} \tag{5.26}$$

Thus (5.25) follows from (5.26), (5.24) and (5.23).

By Theorem 5.1 of Glover (1984), $I-\beta_i^*(sI-A_i)^{-1}\beta_i$ is stable and all-pass, and by Theorem 2.5 it has strictly positive partial indices $k_1,\ldots,k_q$, since β_i has full column rank. Also, $I-\beta_1^*(sI-A_1)^{-1}\beta_1$ is factorization-equivalent to $I-\beta_2^*(sI-A_2)^{-1}\beta_2$. Thus the hypotheses of Lemma 5.2 are satisfied.

Similarly, $[I+\tilde{\beta}_i^*(sI-\tilde{A}_i)^{-1}\tilde{\beta}_i]^{-1}$, $i=1,2$, are factorization-equivalent stable all-pass matrices with strictly positive partial indices $-k_{p-t+1},\ldots,-k_p$.

It now follows from Lemma 5.2 that $H_+(s)$ and $H_-(s)$ have the desired properties. □

With Lemma 5.3 in hand, we can now Wiener–Hopf-factorize a simple all-pass matrix, and therefore, in conjunction with Theorem 5.1, we can Wiener–Hopf-factorize any all-pass matrix $E(s) \in L_\infty$.

Corollary 5.1

Let $E(s) \in L_\infty$ be all-pass with realization $E(s) = D + C(sI - A)^{-1}B - \tilde{C}(sI - \tilde{A})^{-1}\tilde{B}$ and partial indices $k_j, j = 1, \ldots, p$. Define $V_1(s)$, $W_1(s)$ and $E_S(s)$ as in Theorem 5.1 and $E_2(s) = E_S(s)$. Define

$$E_1(s) = D(s) = \operatorname{diag}\,[(s-1)^{k_j}(s+1)^{-k_j}], \quad j = 1, \ldots, p \tag{5.27}$$

Define $H_\pm(s)$ relating $E_1(s)$ and $E_2(s)$ as in Lemma 5.3. Then

$$E(s) = [V_1(-s)^{-1}H_-(s)]D(s)[H_+(s)W_1(\bar{s})^*] \tag{5.28}$$

is a Wiener–Hopf factorization of $E(s)$.

Note that state-space formulae for $D(s)$ in (5.27) are easily derived, and have been given by Bart *et al.* (1983).

The Wiener–Hopf factorization of general rational matrices (in L_∞) is considered in Bart *et al.* (1983), which is quite opaque. This theory could have been specialized to the case of all-pass matrices, but it is almost certainly easier to develop the factorization theory of all-pass matrices directly from the results of Glover (1984) using linear algebra and linear system theory, as we have done. The results of Bart *et al.* were used only to prove Theorem 2.5, and even this could now, in retrospect, be dispensed with.

6. Conclusions

State-space formulae for the stable minimum-phase factors of a minimal all-pass matrix and the Wiener–Hopf factors of an arbitrary all-pass matrix have been derived from the state-space characterization of all-pass matrices developed by Glover (1984), furthering the understanding of the structure of all-pass matrices. For example, we have related the mysterious importance of B_2 in Glover's work to the factorization properties of all-pass matrices. The results of this paper are, however, to be seen as of prime use in the analysis of the canonical correlation structure of stationary multiple time series. Applications in this area include the selection of canonical variables, the simplification of proofs, and most importantly the closed-form solution of a stochastic model-reduction algorithm, based on canonical correlation analysis, using practically implementable state-space formulae. This model-reduction algorithm in turn can be seen as providing a technique for the design of reduced-order Kalman filters for stochastic processes.

References

Akaike, H., 1975, *SIAM J. Control*, **13**, 162; 1975, Canonical correlation analysis of time series and the use of an information criterion. *Systems Identification: Advances and Case Studies*, edited by R. K. Mehra and D. G. Lainiotis (New York: Academic Press).

Anderson, B. D. O., and Vongpanitlerd, S., 1973, *Network Analysis and Synthesis* (Englewood Cliffs, NJ: Prentice-Hall).

Bart, H., Gohberg, I., and Kaashoek, M. A., 1983, Wiener–Hopf factorization of analytic operator functions and realization, Rep. 231, Vrije University of Amsterdam.

Clancey, K., and Gohberg, I. C., 1981, *Factorization of Matrix Functions and Singular Integral Operators* (Basel: Birkhauser).

Desai, U. B., and Pal, D., 1984, *I.E.E.E. Trans. autom. Control*, **29**, 1097.
Glover, K., 1984, *Int. J. Control*, **39**, 1115.
Green, M., and Anderson, B. D. O., 1986, The approximation of power spectra by phase matching, 25th Conf. on Decision and Control, Athens, Greece, pp. 1085–1090; 1987, *Int. J. Control*, **45**, 215.
Jewel, N. P., and Bloomfield, P., 1983, *Ann. Statist.*, **11**, 837.
Jewel, N. P., Bloomfield, P., and Bartmann, F. C., 1983, *Ann. Statist.*, **11**, 848.
Jonckheere, E. A., and Helton, J. W., 1985, *I.E.E.E. Trans. autom. Control*, **30**, 1192.
Kailath, T., 1980, *Linear Systems* (Englewood Cliffs, NJ: Prentice-Hall).
Opdenacker, P., and Jonckheere, E. A., 1985, A state space approach to approximation by phase matching. *MTNS '85*, Stockholm, June 1985.
Wolovich, W. A., 1974, *Linear Multivariable Systems* (Berlin: Springer-Verlag).