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Factorization of all-pass matrix functions

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We consider the factorization of an all-pass matrix function $E(s)$ into proper, stable, minimum-phase factors. This is achieved by applying the state-space Wiener–Hopf factorization theory of Bart *et al.* (1983) to the state-space characterization of all-pass matrix functions given by Glover (1984). Important connections are made between the properties of $E(s)$ as a Nehari extension, its partial Wiener–Hopf factorization indices, and its Hankel singular values.

1. Introduction

The problem of factorizing an all-pass matrix function $E(s)$ into proper, stable, minimum-phase factors $V(s)$, $W(s)$ such that

$$E(s) = V(-s)^{-1} W(\bar{s})^* \tag{1.1}$$

was raised in the envisioned multivariable extension (Opdenacker and Jonckheere 1985) of the phase matching approach to stochastic model reduction developed in that paper and in Jonckheere and Helton (1985) and Harshavadhana and Jonckheere (1987). A number of problems with this envisioned multivariable algorithm were noted in Opdenacker and Jonckheere (1985), and have been addressed in Green and Anderson (1986). The resolution of these problems depends on the factorization theory of all-pass matrix functions. This development is considered to be of sufficient independent interest to warrant its exposition as an independent paper. Indeed, the connections made between all-pass Nehari extensions and their factorization properties, in addition to being interesting in themselves, allow some existing results to be extended.

Some of the results derived in this paper are contained, in somewhat different form and with different emphasis, in the work of Dym and Gohberg (1983 a, b), which has only recently come to our attention. Their results are derived from the operator-theoretic characterization of Nehari (and related) extensions (Nehari 1957) given by Adamjan *et al.* (1971, 1978) using the operator factorization theory of Gohberg and Krein (1960) and Clancy and Gohberg (1981). We obtain results similar to those of Dym and Gohberg, but by different methods. Our results are derived from the state-space factorization theory of Bart *et al.* (1983). Thus the contribution of this paper can be seen as a state-space parallel of Dym and Gohberg's results, in the same way as the Hankel norm approximation theory of Glover (1984) can be seen as a state-space parallel of Adamjan *et al.* (1971, 1978). In addition, our results are specifically tailored for application to the multivariable phase matching algorithm for stochastic model reduction (Green and Anderson 1986).

The organization of the paper is as follows. Section 2 consists of notation and

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definitions, § 3 introduces the topic with a consideration of the scalar case, and § 4 contains the multivariable results.

2. Definitions, notation and preliminaries

Let L_∞ denote the space of complex, measurable, $p \times p$ matrix functions which are bounded on the imaginary axis. Then

$$L_\infty = H_\infty^+ \oplus H_\infty^-$$

where

$$H_\infty^+ = \{H \in L_\infty : H(s) \text{ is analytic in } \operatorname{Re}(s) \geqslant 0 \text{ and such that } H(\infty) = 0\}$$

$$H_\infty^- = \{H \in L_\infty : H(s) \text{ is analytic in } \operatorname{Re}(s) \leqslant 0\}$$

Note that H_∞^+ contains all asymptotically stable, strictly proper, rational, $p \times p$ matrix functions.

Definition

If $H(s) \in L_\infty$, then $H(s)$ is uniquely decomposable as $H(s) = H_+(s) + H(s)$

$$H_\pm(s) \in H_\infty^\pm$$

and $H_+(s)$ is called the *stable part of* $H(s)$.

Remark

From now on, we will be dealing only with rational matrix functions. Thus for convenience, $H(s) \in L_\infty$ *will mean* $H(s)$ *is rational and in* L_∞.

Definition

Let $H(s) \in L_\infty$ and $H_+(s) = C(sI - A)^{-1}B$, with A $n \times n$, be a realization of $H_+(s)$. Let P, Q be the Hermitian solutions of the Lyapunov equations

$$AP + PA^* + BB^* = 0 \tag{2.1 a}$$

$$A^*Q + QA + C^*C = 0 \tag{2.1 b}$$

Denote by $\lambda_i(PQ)$ the eigenvalues of PQ, which are invariant under state-space transformation. The quantities $\sigma_i(H)$ given by

$$\sigma_i(H) \triangleq [\lambda_i(PQ)]^{1/2}, \quad i = 1, \ldots, n \tag{2.2 a}$$

are called the *Hankel singular values* of H, and by convention are ordered so that

$$\sigma_i(H) \geqslant \sigma_{i+1}(H), \quad i = 1, \ldots, n-1 \tag{2.2 b}$$

Remark

The number of non-zero Hankel singular values of $H(s)$ is equal to the McMillan degree of $H_+(s)$. Note also that $H(s)$ and $H_+(s)$ have the same Hankel singular values.

Definition

$H(s) \in L_\infty$ is called *all-pass* if it satisfies

$$H(s)H(-\bar{s})^* = I \text{ for all } s \tag{2.3}$$

Definition

$H(s) \in H_\infty^+ \oplus I$ is called *minimum-phase* if $H(s)$ is non-singular for all $s \in \{\mathrm{Re}(s) \geq 0\}$, but not necessarily at $s = \infty$.

Theorem 2.1 (Nehari 1957)

Let $H(s) \in L_\infty$. Then

$$\sigma_1(H) = \inf_{\tilde{F}_- \in H_\infty^-} \|H_+(j\omega) - \tilde{F}_-(j\omega)\|_\infty$$

Furthermore, there exists $F_-(s) \in H_\infty^-$ such that

$$\sigma_1(H) = \|H_+(j\omega) - F_-(j\omega)\|_\infty \tag{2.4}$$

A function $E(s) \triangleq H_+(s) - F_-(s)$ satisfying (2.4) is called a *Nehari extension* of $H_+(s)$.

Remarks

(1) $E(s)$ is *not unique*. Necessary and sufficient conditions for uniqueness are known (Glover 1984, Adamjan *et al.* 1978). A sufficient condition is $p = 1$. That is, scalar functions have unique Nehari extensions.

(2) For any $H_+(s) \in H_\infty^+$, there is always a Nehari extension $E(s)$ such that $1/\sigma_1 E(s)$ is all-pass (Glover 1984, Adamjan *et al.* 1978). Obviously if $E(s)$ is unique then $1/\sigma_1 E(s)$ is all-pass. However, not all Nehari extensions are of this type.

(3) Clearly from (2.4) if $E(s)$ is all-pass, then it is a Nehari extension of $E_+(s)$ if and only if $\sigma_1(E) = 1$.

The following result, the scalar version of which is in Latham (1984), relates the number of Hankel singular values of an all-pass matrix function which equal 1 to the number of stable and unstable poles.

Lemma 2.1

Let $E(s) \in L_\infty$ be all-pass with m_1 stable and m_2 unstable poles. Then

$$\sigma_i(E) \leq 1, \quad i = 1, \ldots, m_1 \tag{2.5 a}$$

and if $m_1 > m_2$,

$$\sigma_i(E) = 1, \quad i = 1, \ldots, m_1 - m_2 \tag{2.5 b}$$

$$\sigma_i(E) \leq 1 \quad i = m_1 - m_2 + 1, \ldots, m_1 \tag{2.5 c}$$

Proof

Let

$$L_\infty^k \triangleq \{H \in L_\infty : H_+ \text{ has McMillan degree} \leq k - 1\} \tag{2.6}$$

By Glover (1984), Adamjan *et al.* (1978 b)

$$\sigma_i(E) = \inf_{F \in L_\infty^i} \|E_+ - F\|_\infty \leq \|E_+ + E_-\|_\infty = \|E\|_\infty = 1$$

proving (2.5 *a*, *c*).

Now let $E(s) = E_1(s)E_2(s)$, where $E_1(s)$ is stable of degree m_1 and $E_2(-s)$ is stable of degree m_2, with $E_i(s)$ all-pass, $i = 1, 2$ (Newcomb 1966).

Then

$$\sigma_i(E) = \inf_{F \in L^i_\infty} \|E - F\|_\infty = \inf_{F \in L^i_\infty} \|E_1 E_2 - F\|_\infty = \inf_{F \in L^i_\infty} \|E_1 - F E_2^\dagger\|_\infty$$

since E_2 is all-pass, and where $E_2^\dagger(s) \triangleq E_2(-\bar{s})^*$.

Observe that $FE_2^\dagger$ has at most $i-1+m_2$ stable poles. That is if $F \in L^i_\infty$, then $FE_2^\dagger \in L^{i+m_2}_\infty$. Hence

$$\sigma_i(E) \geqslant \inf_{F \in L^{i+m_2}_\infty} \|E_1 - F\|_\infty = \sigma_{i+m_2}(E_1)$$

by (2.6), provided $i + m_2 \leqslant m_1$

Since $E_1(s)$ is stable and all-pass, $\sigma_j(E_1) = 1, j = 1, \ldots, m_1$ by Remark 5.2 of Glover (1984). The result follows. Note that this proof generalizes an idea of Latham (1984) to the multivariable case. □

Remark

For any all-pass matrix function $E(s) \in L_\infty$ as in Lemma 2.1, *let r be the number of Hankel singular values of E which equal* 1. That is, r is defined by

$$\sigma_1 = \ldots = \sigma_r > \sigma_{r+1} \geqslant \ldots \geqslant \sigma_{m_1}, \quad \text{if } \sigma_1 = 1 \tag{2.7 a}$$

$$r = 0, \quad \text{if } \sigma_1 < 1 \tag{2.7 b}$$

Lemma 2.1 is equivalent to

$$m_1 \geqslant r \geqslant \max(0, m_1 - m_2) \tag{2.8}$$

or

$$m_2 \geqslant m_1 - r \geqslant 0, \quad r \geqslant 0 \tag{2.9}$$

The result (2.9) also follows from the state-space results of Glover (1984), which are summarized in § 4. As a by-product of our development of factorization theory, this result will be strengthened.

3. Factorization of scalar all-pass functions

To facilitate our intuition and understanding of the new multivariable results of § 4, let us first consider the familiar scalar case. The basic factorization result (Lemma 3.1) is more or less common knowledge, and extends nicely to the multivariable case. The second result (Lemma 3.2) is a connection pointed out in Harshavadhana and Jonckheere (1987).

Lemma 3.1

Let $e(s)$ be a scalar, rational, all-pass function with m_1 stable and m_2 unstable poles. Then there exists a proper, stable, minimum-phase function $v(s)$ such that

$$e(s) = \frac{v(s)}{v(-s)} \tag{3.1}$$

if and only if $m_1 \geqslant m_2$.

Furthermore, $v(s)$ is strictly proper if and only if $m_1 \geqslant m_2 + 1$.

The function $v(s)$ is unique up to a constant scaling factor.

Proof

Since $e(s)$ is all-pass, it can be written

$$e(s)=\frac{n_+(s)d_+(-s)}{d_+(s)n_+(-s)} \tag{3.2}$$

where $n_+(s)$, $d_+(s)$ are coprime Hurwitz polynomials.

Thus

$$m_1=\deg d_+(s) \tag{3.3 a}$$

$$m_2=\deg n_+(s) \tag{3.3 b}$$

With $m_1 \geqslant m_2$ ($\geqslant m_2+1$) we define

$$v(s)=\frac{n_+(s)}{d_+(s)} \tag{3.4}$$

and (3.3) implies $v(s)$ is proper (strictly proper), stable and minimum phase. It follows from (3.2) that $v(s)$ given in (3.4) satisfies (3.1).

Conversely, if $v(s)$ is proper (strictly proper), stable and minimum phase, it can be written as in (3.4). Formulas (3.1), (3.4) then imply (3.2) holds and m_1, m_2 are given as in (3.3) with $m_1 \geqslant m_2$ ($m_1 \geqslant m_2+1$).

For the uniqueness result, observe that if $v(s)$ satisfies (3.1) then $\lambda v(s)$, $\lambda \in \mathbb{C}$, $\lambda \neq 0$ does also. Conversely, suppose $v_1(s)$ and $v_2(s)$ satisfy (3.1) and are proper, stable and minimum phase.

Define

$$g(s) \triangleq v_2(s)^{-1}v_1(s) \tag{3.5}$$

It follows from (3.1) and (3.5) that

$$g(s)=g(-s) \tag{3.6}$$

Since v_1, v_2 are stable and minimum phase, $g(s)$ and $g(s)^{-1}$ are analytic and non-zero in $\{\mathrm{Re}\,(s) \geqslant 0\}$. Furthermore, (3.6) implies that g(s) and $g(s)^{-1}$ are entire (analytic in $\mathbb{C}$). Moreover, either $g(s)$ or $g(s)^{-1}$ is bounded. Suppose it is $g(s)$. Thus $g(s)$ is a constant, λ say, and we have $v_2(s)=\lambda v_1(s)$. Obviously λ is non-zero. Consequently $g(s)^{-1}$ is also bounded. □

The next result is a connection, the sufficiency part of which is reasonably well known (Adamjan *et al.* 1978 a). The necessity part was pointed out in Harshavadhana and Jonckheere (1987). We prove it here differently using Lemma 2.1 and Lemma 3.1. This is the approach we will adopt in the proof of the multivariable version.

Lemma 3.2

If $e(s)$ is a scalar, rational, all-pass function then $e(s)$ can be factored as

$$e(s)=\frac{v(s)}{v(-s)}$$

with $v(s)$ strictly proper, stable and minimum phase if and only if $e(s)$ is the unique Nehari extension of its stable part $e_+(s)$.

Proof

That a scalar Nehari extension, which is always unique, can be factored as desired follows from Adamjan *et al.* (1971).

For the necessity, observe that if $e(s)$ can be factored, we must have $m_1 \geqslant m_2 + 1$ by Lemma 3.1. Hence $\sigma_1(e) = \sigma_1(e_+) = 1$ by Lemma 2.1. Thus $e(s)$ is the Nehari extension of $e_+(s)$ (see definition), which is unique because $e_+(s)$ is scalar. □

Remark

In the scalar case, the Nehari extension is always unique. In the multivariable case, this is not true. The significance of putting the word unique (explicitly) in the statement of Lemma 3.2 is that Lemma 3.2, as it stands, generalizes completely to the multivariable case.

Examples

(1) Let

$$e(s) = \frac{(s+2)(s-1)}{(s-2)(s+1)} \quad \text{so } m_1 = m_2 = 1$$

$$e_+(s) = \frac{2/3}{s+1} \quad \text{and } \sigma_1 = \frac{1}{3}$$

$e(s)$ cannot be factored using strictly proper factors, and is not the Nehari extension of $e_+(s)$ (since $\sigma_1 < 1$).

(2)
$$e(s) = \frac{(s+2)(s-1)(s-3)}{(s-2)(s+1)(s+3)}$$

Take

$$v(s) = \frac{s+2}{(s+1)(s+3)}$$

and we have $e(s) = v(-s)^{-1} v(s)$.

Also

$$e_+(s) = \frac{-4/3}{s+1} + \frac{-12/5}{s+3}$$

and

$$\sigma_1 = 1, \quad \sigma_2 = \tfrac{2}{30}$$

So, $e(s)$ is the Nehari extension of $e_+(s)$.

4. Factorization of all-pass matrix functions

For the purposes of the phase-matching model-reduction algorithm, we would like to know whether Lemma 3.2 remains true for matrix all-pass functions. As already indicated, this is in fact the case, and will come as a by-product of the matrix generalization of Lemma 3.1. This will also enable us to strengthen Lemma 1.10 of Latham (1984), which is Lemma 2.1 for the scalar case.

To a large extent, this section is the result of applying the state-space factorization theory of Bart *et al.* (1983) to the state-space characterization of all-pass matrix functions in Glover (1984).

4.1. *State-space characterization of all-pass matrices*

Theorem 4.1

Let $E(s) \in L_\infty$ be all-pass with minimal realization

$$A_e = \begin{bmatrix} A & 0 \\ 0 & \tilde{A} \end{bmatrix} \tag{4.1 a}$$

$$B_e = \begin{bmatrix} B \\ \tilde{B} \end{bmatrix}, \quad C_e = [C \quad -\tilde{C}], \quad D_e = E(\infty) \tag{4.1 b}$$

where A is $m_1 \times m_1$ and stable; $\tilde{A}$ is $m_2 \times m_2$ with $-\tilde{A}$ stable; B and C^* are $m_1 \times p$; and $\tilde{B}$ and $\tilde{C}^*$ are $m_2 \times p$.

Further suppose A, B, C, is balanced, with controllability/observability gramian

$$\Sigma = \text{diag}\,(\sigma_{r+1}, \ldots, \sigma_{m_1}, I_r) \tag{4.2 a}$$

$$= \text{diag}\,(\Sigma_1, I_r) \tag{4.2 b}$$

satisfying

$$A\Sigma + \Sigma A^* + BB^* = 0 \tag{4.2 c}$$

$$A^*\Sigma + \Sigma A + C^*C = 0 \tag{4.2 d}$$

where r is given by (2.7).

Then:

(1) There exist unique $P_e = P_e^*$, $Q_e = Q_e^*$ such that

$$A_e P_e + P_e A_e^* + B_e B_e^* = 0 \tag{4.3 a}$$

$$A_e^* Q_e + Q_e A_e + C_e^* C_e = 0 \tag{4.3 b}$$

$$P_e Q_e = I \tag{4.3 c}$$

$$D_e^* D_e = I \tag{4.4 a}$$

$$D_e^* C_e + B_e^* Q_e = 0 \tag{4.4 b}$$

$$D_e B_e^* + C_e P_e = 0 \tag{4.4 c}$$

(2) Partition

$$P_e = \begin{bmatrix} \Sigma & M \\ M^* & R \end{bmatrix}, \quad Q_e = \begin{bmatrix} \Sigma & N \\ N^* & S \end{bmatrix} \tag{4.5 a}$$

where R, S are $m_2 \times m_2$ and M, N are $m_1 \times m_2$.

Then with $l = m_2 - (m_1 - r)$, non-negative by (2.9),

$$R = T \begin{bmatrix} \Sigma_1 \Gamma^{-1} & 0 \\ 0 & -I_l \end{bmatrix} T^* \tag{4.5 b}$$

$$S = T^{-*} \begin{bmatrix} \Sigma_1 \Gamma & 0 \\ 0 & -I_l \end{bmatrix} T^{-1} \tag{4.5 c}$$

$$M = \begin{bmatrix} I_{m_1-r} & 0 \\ 0 & 0_{r\times l} \end{bmatrix} T^* \tag{4.5 d}$$

$$N = \begin{bmatrix} -\Gamma & 0 \\ 0 & 0_{r\times l} \end{bmatrix} T^{-1} \tag{4.5 e}$$

where

$$\Gamma = \Sigma_1^2 - I \tag{4.6}$$

and T is an $m_2 \times m_2$ non-singular matrix. (T is a similarity transformation on the realization $(\tilde{A}, \tilde{B}, \tilde{C})$ of $E_-(s)$.)

(3) Partition A, B, C conformally with Σ as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = [C_1 \quad C_2] \tag{4.7}$$

where A_{11} is $(m_1 - r) \times (m - r)$, B_1, C_1^* are $(m_1 - r) \times p$. Note B_2 is the matrix formed from the last r rows of B.

Define

$$\tilde{A}_{11} = \Gamma^{-1}(A_{11}^* + \Sigma_1 A_{11} \Sigma_1 - C_1^* D_e B_1^*) \tag{4.8 a}$$

$$\tilde{B}_1 = \Gamma^{-1}(\Sigma_1 B_1 + C_1^* D_e) \tag{4.8 b}$$

$$\tilde{C}_1 = C_1 \Sigma_1 + D_e B_1^* \tag{4.8 c}$$

Then

$$\tilde{A} = T\begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} T^{-1}, \quad \tilde{B} = T\begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}, \quad \tilde{C} = [\tilde{C}_1 \quad \tilde{C}_2] T^{-1} \tag{4.8 d}$$

where

$$B_2 \tilde{B}_2^* = 0 \tag{4.9 a}$$

$$\tilde{C}_2 = -D_e \tilde{B}_2^* \tag{4.9 b}$$

$$\tilde{A}_{12} = -\Gamma^{-1} C_1^* \tilde{C}_2 \tag{4.9 c}$$

$$\tilde{A}_{21} = -\tilde{B}_2 B_1^* \tag{4.9 d}$$

$$\tilde{A}_{22} + \tilde{A}_{22}^* = \tilde{B}_2 \tilde{B}_2^* \tag{4.9 e}$$

Proof

See Glover (1984). Part 1 is Theorem 5.1 of Glover (1984). Part 2 for $r > 0$ (i.e. $\sigma_1(E) = 1$) is Lemma 8.2 of Glover (1984) with $k = 0$. For the case $r = 0$, the result follows via the same reasoning as in the proof of Glover's Lemma 8.2. Part 3 for the case $r > 0$ is Glover's Lemma 8.5, and with $r = 0$, the result follows similarly. Note that the $r = 0$ case can be proved simply from the $r > 0$ using the device of Remark 8.4 of Glover (1984). □

Remarks

(1) It is shown in the proof of Lemma 8.2 of Glover (1984) that (2.9) holds.

(2) The zero columns of M, N in (4.5) and the subscript 2 blocks of $T^{-1}\tilde{A}T$, $T^{-1}\tilde{B}$,

$\tilde{C}T$ occur if and only if $l = m_2 - (m_1 - r) > 0$. The zero rows of M, N and the subscript 2 blocks of A, B, C occur if and only if $r > 0$, i.e. $\sigma_1(E) = 1$.

(3) The fact that (A_e, B_e, C_e) is minimal implies (A, B, C) and $(\tilde{A}, \tilde{B}, \tilde{C})$ are minimal. This implies A and $\tilde{A}$ have no eigenvalue on the imaginary axis, since if

$$x^*A = \lambda x^* \text{ with } \lambda + \bar{\lambda} = 0$$

then

$$(\lambda + \bar{\lambda})x^*\Sigma x + x^*BB^*x = 0$$

by (4.2 c), and hence $x^*B = 0$, which implies (A, B) is not controllable by the Popov–Belevitch–Hautus test. Similarly for $\tilde{A}$. Note that this result is part 4 of Theorem 3.3 of Glover (1984). We will use this fact later.

(4) Observe from (4.1) and (4.2) that

$$E_+(s) = C(sI - A)^{-1}B \tag{4.10}$$

and (A, B, C) is a balanced realization of $E_+(s)$.

(5) Given (A, B, C) and Σ satisfying (4.2) with $\sigma_i < 1$, $i = r+1, \ldots, m_1$ Theorem 4.1 allows us to construct an all-pass matrix $E(s)$ with m_2 unstable poles, $m_2 \geqslant m_1 - r$, such that $E_+(s) = C(sI - A)^{-1}B$: This is done, along the lines of Theorem 6.3 of Glover (1984), as follows. Partition Σ, A, B, C as in (4.2), (4.7). By Theorem 5.1 of Glover (1984), there exists a D_e satisfying (4.4 a) and

$$D_e^* C_2 + B_2^* = 0$$

This does not uniquely define D_e in general (only when rank $B_2 = p$). Choose $l \geqslant 0$ and let $m_2 = l + m_1 - r$ (if rank $B_2 = p$, we must take $l = 0$, otherwise l can be an arbitrary non-negative integer). Now pick an $l \times p$ matrix $\tilde{B}_2$ to satisfy (4.9 a) and an $l \times l$ matrix $\tilde{A}_{22}$ to satisfy (4.9 e). (Note that (4.9 a, e) do not in general uniquely define $\tilde{B}_2$ or $\tilde{A}_{22}$. Later we will see how the choice of $\tilde{B}_2, \tilde{A}_{22}$ can effect the factorization properties of $E(s)$.) Now determine $\tilde{C}_2, \tilde{A}_{12}, \tilde{A}_{21}$ by (4.9 b, c, d) and $\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1$ by (4.8 a, b, c). Define $\tilde{A}, \tilde{B}, \tilde{C}$ by (4.8 d) with T arbitrary non-singular and finally A_e, B_e, C_e, D_e by (4.1). Then $E(s) = D_e + C_e(sI - A_e)^{-1}B_e$ is all-pass with m_1 stable and m_2 unstable poles and $E_+(s) = C(sI - A)^{-1}B$. Note that $E(s)$ is independent of the choice of T.

4.2. *All-pass matrices as Nehari extensions*

Lemma 4.1

Let $E(s) \in L_\infty$ be all-pass and r be given by (2.7). Then $E(s)$ is a Nehari extension of $E_+(s)$ if and only if $r > 0$ (i.e. $\sigma_1(E) = 1$). It is a minimal degree, all-pass, Nehari extension if and only if, in addition, $m_2 = m_1 - r$. It is the unique Nehari extension of $E_+(s)$ if and only if in addition rank $B_2 = p$, $B_2 =$ last r rows of B, with B given as in Theorem 4.1.

Proof

A Nehari extension N of E_+ satisfies

$$\sigma_1(E_+) = \|N\|_\infty \text{ and } E_+ = \text{stable part of } N$$

Since $E(s)$ is all-pass, $\|E\|_\infty = 1$. Thus $E(s)$ is a Nehari extension of E_+ if and only if $\sigma_1(E_+) = \sigma_1(E) = 1$.

By minimal degree extension, we mean minimal number of unstable poles, i.e. m_2 as small as possible. By Lemma 2.1, $m_2 \geqslant m_1 - r$. Hence for $E(s)$ to be a minimal degree extension, $m_2 = m_1 - r$. These minimal extensions are characterized by Theorem 6.3 of Glover (1984) or (equivalently) by Theorem 4.1 with $m_2 = m_1 - r$ ($l = 0$).

The uniqueness result is part of Theorem 8.7 of Glover (1984). Note tht the alternative uniqueness condition rank $C_2 = p$ follows from rank $B_2 = p$, (4.4 *c*), the structure of P_e given by (4.5) and the fact that $E(s)$ is square (which implies D_e is non-singular). □

4.3. *Factorization*

We now apply the Wiener–Hopf factorization results of Bart *et al.* (1983) to the state-space characterization of all-pass matrices given in Glover (1984) and summarized in Theorem 4.1. We will review the basis of Wiener–Hopf factorizations, and relate them to our factorization problem.

Theorem 4.2 Bart *et al.* (1983), Gohberg and Krein (1960), Clancey and Gohberg (1981)

Let $H(s) \in L_\infty$ have no zeros or poles on $s = j\omega$, $\omega \in \mathbb{R} \cup \{\infty\}$. Then $H(s)$ has a Wiener–Hopf factorization

$$H(s) = H_-(s)D(s)H_+(s), \quad s = j\omega, \quad \omega \in \mathbb{R} \tag{4.11 a}$$

where $H_+(s)$ is proper, stable and non-singular in $\{\mathrm{Re}\,(s) \geqslant 0\} \cup \{\infty\}$; $H_-(s)$ is proper, completely unstable ($\in H_\infty^-$) and non-singular in $\{\mathrm{Re}\,(s) \leqslant 0\} \cup \{\infty\}$; and

$$D(s) = \mathrm{diag}\left\{\left(\frac{s-1}{s+1}\right)^{k_i}, \quad i = 1, \ldots, p\right\} \quad k_i \geqslant k_{i+1} \tag{4.11 b}$$

The numbers k_i are integers, called the *partial indices* of H, and they are unique.

Remarks

(1) The matrix functions $H_\pm(s)$ in (4.11 *a*) are not unique, but can be characterized (Gohberg and Krein 1960, Clancey and Gohberg 1981). State-space formulas for $H_\pm(s)$ (for proper rational $H(s)$) are given in Bart *et al.* (1983). It is also possible to replace the diagonal factor D with

$$\mathrm{diag}\left[\left(\frac{s-a}{s+b}\right)^{k_i}, \quad i = 1, \ldots, p\right]$$

where Re (a), Re (b) > 0, otherwise arbitrary.

(2) Let n_1 be the number of stable ($\mathrm{Re}\,(s) < 0$) zeros of $H(s)$. With m_1, m_2 the number of stable and unstable poles of $H(s)$, as before, then (Clancey and Gohberg 1981)

$$k \triangleq \sum_{i=1}^{p} k_i = m_1 - n_1 \tag{4.12 a}$$

Thus, when $H(s)$ is all-pass, implying $n_1 = m_2$, we have

$$k = \sum_{i=1}^{p} k_i = m_1 - m_2 \tag{4.12 b}$$

Theorem 4.3

Let $H(s) \in L_\infty$ have no poles or zeros on $s = j\omega$, $\omega \in \mathbb{R} \cup \{\infty\}$. Then

$$H(s) = V(-s)^{-1} W(\bar{s})^*, \quad s = j\omega \tag{4.13}$$

with $V(s)$, $W(s)$ proper, stable and minimum phase if and only if $H(s)$ has no (strictly) negative partial indices. When this is the case

$$W(\bar{s})^* = C_+(s)H_+(s) \tag{4.14 a}$$

$$V(s) = C_+(-s)D(s)H_-(-s)^{-1} \tag{4.14 b}$$

where $H_\pm(s)$, $D(s)$ satisfy (4.11) and $C_+(s)$ is an arbitrary matrix function whose elements satisfy

$$[C_+(s)]_{ij} \text{ is constant if } k_j = 0 \tag{4.15 a}$$

$[C_+(s)]_{ij}$ is a polynomial in $(1+s)^{-1}$

$$\text{of degree} \leqslant k_j \text{ if } k_j \geqslant 0 \tag{4.15 b}$$

and

$$\det C_+(s) = \alpha(s+1)^{-k} \tag{4.16 a}$$

with

$$k = \sum_{j=0}^{p} k_j \tag{4.16 b}$$

and α an arbitrary constant.

Furthermore, let

$$\rho = \text{rank } V(\infty) = \text{rank } W(\infty) \tag{4.17}$$

then

$$\text{number of zero partial indices} \leqslant \rho \leqslant \begin{cases} p-1 & \text{if } k > 0 \\ p & \text{if } k = 0 \end{cases} \tag{4.18}$$

and for every ρ satisfying (4.18) there exists a $V(s)$, $W(s)$ pair satisfying (4.17), (4.13).

Proof

By hypothesis, the conditions of Theorem 4.2 are satisfied and $H(s)$ has a factorization as in (4.11). Suppose (4.13) holds and let

$$C_+(s) = W(\bar{s})^* H_+(s)^{-1} \tag{4.19 a}$$

$$C_-(s) = V(-s)H_-(s) \tag{4.19 b}$$

Let

$$\pi_+ = \{\text{Re}(s) \geqslant 0\}$$

$$\pi_- = \{\text{Re}(s) \leqslant 0\}$$

From (4.11), (4.13) and (4.19) we have

$$C_+(s) = C_-(s)D(s) \tag{4.20 a}$$

and formula (4.14) follows since $D(-s) = D(s)^{-1}$.

Now we rewrite (4.20 *a*) entry by entry as

$$[C_+(s)]_{ij} = [C_-(s)]_{ij}\left(\frac{s-1}{s+1}\right)^{k_j} \tag{4.20 b}$$

The left-hand side of (4.20 *b*) is analytic in $\pi_+ \cup \{\infty\}$ by (4.19 *a*) and the properties of W, H_+.

Suppose (to obtain a contradiction) that $k_j < 0$. Then the right-hand side of (4.20 *b*) is analytic in $\pi_- \cup \{\infty\}$, by (4.19 *b*) and the properties of V, H_-. Hence $[C_+(s)]_{ij}$ is a constant for all i which must be zero by considering $s = -1$ in (4.20 *b*). Thus $C_+(s)$ has a zero column and from (4.19 *a*) we see $W(s)$ is singular for all s, contradicting the assumption that $W(s)$ is minimum phase. This proves the necessity of $k_j \geqslant 0\ \forall_j$.

Now suppose $k_j \geqslant 0$ in (4.20 *b*). Then the right-hand side of (4.20 *b*) is analytic in $\pi_- \cup \{\infty\}$ except for a pole of order $\leqslant k_j$ at $s = -1$. Thus $[C_+(s)]_{ij}$ is a polynomial in $(s+1)^{-1}$ of degree $\leqslant k_j$, proving (4.15).

Now (4.15) implies det $C_+(s)$ is a polynomial in $(s+1)^{-1}$ of degree $\leqslant k$, with k as in (4.16 *b*). That is

$$\det C_+(s) = \frac{\alpha(s)}{(s+1)^k} \tag{4.21}$$

with $\alpha(\cdot)$ a polynomial of degree $\leqslant k$.

By (4.19 *a*) and the properties of W, H_+, $\alpha(s)$ has no zeros in π_+. By (4.20 *a*)

$$\det C_-(-s) = \det C_+(-s)\det D(s) = \frac{\alpha(-s)}{(-s+1)^k}\left(\frac{s-1}{s+1}\right)^k = \frac{\alpha(-s)(-1)^k}{(s+1)^k}$$

and it follows from the properties of V, H_- that $\alpha(-s)$ is non-zero in π_+. Thus $\alpha(s)$ is polynomial and non-zero through $\mathbb{C}$, and hence is a constant. Formula (4.14) now follows from (4.21).

For the converse, let $C_+(s)$ satisfy (4.15), (4.16) and V, W be given by (4.14). Obviously $W(s)$ is proper, stable and minimum phase. Also observe that $[C_+(-s)D(s)]_{ij}$ can be written as a polynomial in $(s+1)^{-1}$ of degree $\leqslant k_j$, so $V(s)$ is proper and stable. It follows from (4.16) that $C_+(-s)D(s)$ is non-singular in π_+, so $V(s)$ is also minimum phase. Finally,

$$V(-s)^{-1}W(\bar{s})^* = H_-(s)D(s)C_+(s)^{-1}C_+(s)H_+(s) = H_-(s)D(s)H_+(s) = H(s)$$

Now for the rank inequality. Note that rank $W(\infty) =$ rank $V(\infty)$ follows from (4.13) and the non-singularity of $H(s)$ at ∞. From (4.14 *a*)

$$\rho = \text{rank } C_+(\infty)$$

Now the number of columns of $C_+(s)$ which are constant is greater than or equal to the number of zero indices of $H(s)$ by (4.15). These constant columns must be independent since $C_+(s)$ is non-singular almost everywhere. This proves the left-hand inequality in (4.18). The second follows from (4.16).

To see that any ρ satisfying (4.17), (4.18) is possible, let $\mu_j, j = 2, \ldots, p$ be any non-negative integers with

$$\mu_j \leqslant k_j, \quad j = 1, \ldots, p$$

For $i, j = 1, \ldots, p$, let

$$[C_+(s)]_{ij} = \begin{cases} (s+1)^{-k_i} & j = i \\ (s+1)^{-\mu_j} & j = i+1, \quad i \leqslant p-1 \\ 0 & \text{otherwise} \end{cases}$$

Observe that $C_+(s)$ satisfies (4.15), (4.16) and rank $C_+(\infty)$ is equal to the number of zero μ_j (or is equal to p if $k_j=0, j=1, \ldots, p$). In particular if ρ satisfies (4.18) and $k>0$, let

$$\mu_{p-\rho+1}, \ldots, \mu_p = 0, \quad \mu_1, \ldots, \mu_{p-\sigma} > 0$$

while if $k=0$, let $\mu_i=0, i=2, \ldots, p$. Then rank $C_+(\infty)=\rho$. □

Remarks

(1) The condition that $k_j \geqslant 0 \, \forall j$ is equivalent to the equation

$$\sum_{k_j \geqslant 0} k_j = k$$

with k given by (4.12 *a*).

(2) From (4.18) it follows that the only case when the rank of $V(\infty)$, $W(\infty)$ is uniquely determined is when all the partial indices are zero, in which case $V(\infty)$, $W(\infty)$ are non-singular. Also, $V(\infty)$, $W(\infty)$ can be zero (i.e. $V(s)$, $W(s)$ strictly proper) if and only if $k_j>0, \forall_j$.

(3) $C_+(s)^{-1}$ is a polynomial matrix, and is in fact an interactor matrix for $W(\bar{s})^*$ (Wolovich and Falb 1976), since $H_+(\infty)$ is non-singular.

Corollary 4.1

Let $H(s) \in L_\infty$ satisfy the conditions of Theorem 4.3 and have non-negative partial indices. Let $V_1(s)$, $W_1(s)$ be proper, stable, minimum-phase and satisfy

$$H(s) = V_1(-s)^{-1} W_1(\bar{s})^* \tag{4.22 a}$$

Then $V_2(s)$, $W_2(s)$ are proper, stable, minimum-phase and satisfy

$$H(s) = V_2(-s)^{-1} W_2(\bar{s})^* \tag{4.22 b}$$

if and only if

$$W_2(s) = W_1(s) U(\bar{s})^* \tag{4.23 a}$$

$$V_2(s) = U(-s) V_1(s) \tag{4.23 b}$$

where $U(s)$ is a unimodular (i.e. constant determinant) polynomial matrix in s such that

$$U(-s)V_1(s) \text{ is proper} \tag{4.23 c}$$

Furthermore a unimodular matrix $U(s)$ is a solution of (4.23 *c*) if and only if it can be written as

$$U(s) = C_{2+}(s) C_{1+}(s)^{-1} \tag{4.23 d}$$

where $C_{1+}(s)$ is a particular (i.e. determined by $V_1(s)$) matrix function satisfying (4.15), (4.16) and $C_{2+}(s)$ is an arbitrary matrix function satisfying (4.15), (4.16).

If $k_j=0 \, \forall_j$, or $p=1$ (i.e. $H(s)$ scalar), then $U(s)=U$ is a constant non-singular matrix.

Proof

That (4.23) and (4.22 *a*) imply (4.22 *b*) is obvious.

Conversely, that (4.22) implies (4.23) follows easily from Theorem 4.3, in particular (4.15) and (4.16). This also gives the characterization (4.23 *d*) of the solutions of

(4.23 *c*). Note that a $U(s)$ satisfying (4.23 *d*) and the associated conditions on $C_{1+}(s)$, $C_{2+}(s)$ must be unimodular.

Alternatively, eqns (4.22) imply

$$V_2(-s)V_1(-s)^{-1} \triangleq U(s) = W_2(\bar{s})^* W_1(\bar{s})^{-*}$$

Since $V_i(s)$, $W_i(s)$, $i=1,2$ are stable and minimum phase, it follows that $U(s)$ has no poles or zeros, except possibly at infinity. Thus $U(s)$ is unimodular. Now condition (4.23 *c*) must hold since $V_2(s)$ is proper.

It is trivial to observe that unimodular scalar polynomials are non-zero constants, and that the only solutions of (4.23 *c*) are non-singular constant matrices when $k_j = 0\ \forall_j$ follows from (4.15)—see also Remark 2 above. □

Remarks

(1) The particular $C_{1+}(s)^{-1}$ in (4.23 *d*) can be replaced by any fixed interactor matrix for $W(\bar{s})^*$ (Wolovich and Falb 1976). This is equivalent to taking a different $H_\pm(s)$ pair in the Wiener–Hopf factorization (4.11).

(2) Theorem 4.3 connects proper, stable, minimum-phase factorization of matrix functions with Wiener–Hopf factorization.

We now apply the state-space techniques of Bart *et al.* (1983) to the characterization of all-pass matrix functions in Theorem 4.1 to make statements about the partial indices.

Theorem 4.4

Let $E(s) \in L_\infty$ be all-pass with realization A_e, B_e, C_e, D_e as in Theorem 4.1 with $T = I$. Then

(1) (*a*) $$r = \sum_{k_j \geq 0} k_j \tag{4.24}$$

(*b*) The number of strictly positive partial indices = rank B_2.

(*c*) The strictly positive k_j are the (non-zero) reachability indices of (A_{22}, B_2).

(2) (*a*) $$m_2 - (m_1 - r) = -\sum_{k_j \leq 0} k_j \tag{4.25}$$

(*b*) The number of strictly negative partial indices is equal to rank $\tilde{B}_2$ (= rank $\tilde{C}_2$).

(*c*) The strictly negative partial indices are a permutation of the negative of the observability indices of $(\tilde{C}_2, \tilde{A}_{22})$. That is, $k_j < 0$ if and only if $-k_j$ is an observability index of $(\tilde{C}_2, \tilde{A}_{22})$ (we do not count zero observability indices here).

Proof

Recall that since $E(s)$ is all-pass, it has no poles or zeros on $s = j\omega$, $\omega \in \mathbb{R} \cup \{\infty\}$. Thus it has a factorization $H_- D H_+$ as in (4.11) in Theorem 4.2. Since A_e, B_e, C_e, D_e is minimal, A_e has no eigenvalues on $s = j\omega$.

Define the subspace $S \subset \mathbb{R}^n$, $n = m_1 + m_2$ by

S = largest A_e invariant subspace of $\mathbb{R}^n$ such that $A_e|_S$ is completely unstable

$$= \left\{ y \in \mathbb{R}^n : y = \begin{bmatrix} 0 \\ x \end{bmatrix} \quad x \in \mathbb{R}^{m_2}, \text{ arbitrary} \right\} \tag{4.26}$$

Let A_{ex} denote the state-matrix of $E(s)^{-1}$, viz.

$$A_{ex} = A_e - B_e D_e^* C_e$$

and define $S^x \subset \mathbb{R}^n$ by

S^x = largest A_{ex} invariant subspace of $\mathbb{R}^n$ such that $A_{ex}|_{S^x}$ is completely stable.

Since $E(s)E^*(-\bar{s})$, we have

$$A_{ex} = P_e(-A_e^*)Q_e$$

with P_e, Q_e as in (4.3), (4.5). See proof of Theorem 5.1 of Glover (1984) for this.

Hence

$$S^x = Q_e^{-1}S = P_eS \text{ by (4.3 } c) = \left\{ y \in \mathbb{R}^n : y = \begin{bmatrix} Mx \\ Rx \end{bmatrix}, \quad x \in \mathbb{R}^{m_2} \text{ arbitrary} \right\} \tag{4.27}$$

where M, R are given by (4.5).

Now, for any integer $j \geq 0$, define the subspaces

$$F_j = S + S^x + \text{Im } B_e + \text{Im } A_eB_e + \ldots + \text{Im } A_e^{j-1}B_e \tag{4.28 a}$$

$$G_j = S \cap S^x \cap \ker C_e \cap \ker C_eA_e \cap \ldots \cap \ker C_eA_e^{j-1} \tag{4.28 b}$$

where for $j = 0$, we mean $F_0 = S + S^x$, $G_0 = S \cap S^x$.

Consider the positive indices.

Let ω be the first integer such that $F_\omega = F_{\omega+1}$. Such an ω exists since $F_j \subset \mathbb{R}^n \, \forall_j$. In fact $F_\omega = \mathbb{R}^n$. This follows in this case from (4.28 *a*) and the minimality of (A_e, B_e, C_e). More generally, (i.e. if A_e, B_e, C_e were non-minimal), it follows as in Bart *et al.* (1983).

By (4.7) of Bart *et al.* (1983), for any $l > 0$

$$\begin{aligned} \text{number of } k_j \text{ equal to } l &= \dim(F_l - F_{l-1}) - \dim(F_{l+1} - F_l) \\ &= 2 \dim F_l - \dim F_{l-1} - \dim F_{l+1} \end{aligned}$$

Thus

$$\begin{aligned} \sum_{k_j \geq 0} k_j &= \sum_{l=1}^{\omega} (2 \dim F_l - \dim F_{l-1} - \dim F_{l+1})l \\ &= \dim F_\omega - \dim F_0 \quad \text{since } F_{\omega+1} = F_\omega \\ &= m_1 + m_2 - \dim(S + S^x) \end{aligned}$$

Now

$$S + S^x = \left\{ y \in \mathbb{R}^n : y = \begin{bmatrix} Mx \\ z \end{bmatrix}, \quad x, z \in \mathbb{R}^{m_2} \text{ arbitrary} \right\} \tag{4.29}$$

So

$$\dim(S + S^x) = m_2 + \text{rank } M = m_2 + m_1 - r \quad \text{by (4.5 } d)$$

Thus

$$\sum_{k_j \geq 0} k_j = r$$

proving (4.24).

Of course (4.25) now follows from (4.24) and (4.12 *b*), providing parts (1 *a*) and (2 *a*).

Now for the number of indices of each type. According to eqn (4.5) of Bart *et al.* (1983), the number of strictly positive partial indices equals $\dim(F_1 - F_0)$. From (4.29), (4.28 *a*) and (4.5 *d*) we have

$$F_1 - F_0 = \left\{ y \in \mathbb{R}^n : y = \begin{bmatrix} 0 \\ B_2 w \\ 0 \end{bmatrix}, \quad w \in \mathbb{R}^p \text{ arbitrary} \right\} \tag{4.30}$$

Thus part (1 *b*) is proven.

For part (1 *c*), note that the (2, 3) block of (4.3 *a*) implies $A_{21} = -B_2 \tilde{B}_1^*$, so $\operatorname{Im} A_{21} \subset \operatorname{Im} B_2$. It follows from (4.29), (4.28 *a*) and (4.5 *d*) that

$$F_j = S + S^\times \oplus \operatorname{Im} [B_2 \quad A_{22} B_2 \quad \ldots \quad A_{22}^{j-1} B_2]$$

By (4.6) of Bart *et al.* (1983), recalling the ordering $k_j \geqslant k_{j-1}$, the strictly positive partial indices are

$$k_j = \#\{i: \dim F_i - \dim F_{i-1} \geqslant i\} \quad j = 1, \ldots, \operatorname{rank} B_2$$

It follows (see, for example, (13)–(48) of Fuhrmann 1981) that the strictly positive partial indices are the reachability indices of (A_{22}, B_2).

The number of strictly negative indices equals $(G_0 - G_1)$ by (5.5) of Bart *et al.* (1983). It follows from (4.28 *b*), (4.26), (4.27) that

$$G_0 = S \cap S^\times = \left\{ y \in \mathbb{R}^n : y = \begin{bmatrix} 0 \\ z \end{bmatrix}, \quad z \in \mathbb{R}^{m_2 - (m_1 - r)}, \text{ arbitrary} \right\} \tag{4.31 a}$$

and

$$G_1 = \left\{ y \in \mathbb{R}^n : y = \begin{bmatrix} 0 \\ z \end{bmatrix}, \quad z \in \mathbb{R}^{m_2 - (m_1 - r)}, \tilde{C}_2 z = 0 \right\}$$

Thus

$$\dim(G_0 - G_1) = \operatorname{rank} \tilde{C}_2$$

and we have proved part (2 *b*), observing that $\operatorname{rank} \tilde{B}_2 = \operatorname{rank} \tilde{C}_2$ by (4.9 *b*). Now for part (2 *c*). Note that by (4.9 *c*) $\tilde{C}_2 z = 0$ implies $\tilde{A}_{12} z = 0$. It follows that

$$G_j = \left\{ y \in \mathbb{R}^n : y = \begin{bmatrix} 0 \\ z \end{bmatrix}, \quad z \in \mathbb{R}^{m_2 - (m_1 - r)}, \tilde{0}_2^j z = 0 \right\} \quad j \geqslant 1 \tag{4.31 b}$$

where

$$\tilde{0}_2^j = [\tilde{C}_2^*, \tilde{A}_{22}^* \tilde{C}_2^*, \ldots, (\tilde{A}_{22}^{j-1})^* \tilde{C}_2^*]^* \tag{4.31 c}$$

Thus

$$\dim(G_j - G_{j-1}) = \dim \ker \tilde{0}_2^j - \dim \ker \tilde{0}_2^{j-1} = \operatorname{rank} \tilde{0}_2^j - \operatorname{rank} \tilde{0}_2^{j-1} \tag{4.32}$$

Now the non-positive indices exist only when $\operatorname{rank} B_2 < p$, in which case they are k_j, $j = \operatorname{rank} B_2 + 1, \ldots, p$, by part 1(*b*) and the assumed ordering of the k_j $(k_j \geqslant k_{j+1})$. By part (2 *b*) the strictly negative indices are k_j, $j = p - \operatorname{rank} \tilde{B}_2 + 1, \ldots, p$ (note that by (4.9 *a*) $\operatorname{rank} B_2 + \operatorname{rank} \tilde{B}_2 \leqslant p$) and by equation (5.6) of Bart *et al.* (1983) are given by

$$-k_{p-j} = \#\{i: \dim(G_i - G_{i-1}) \geqslant j + 1\} \quad j = 0, \ldots, \operatorname{rank} \tilde{B}_2 - 1 \tag{4.33}$$

Now using (4.32) and (4.31 *c*) it is easy to see that these are just the non-zero observability indices of $(\tilde{C}_2, \tilde{A}_{22})$, and part (2 *c*) is proved. □

Remarks

(1) There is a consistency question about the statement of Theorem 4.4, since there are two conditions for $E(s)$ to have no strictly positive indices. They are

$$r = \sum_{k_j \geqslant 0} k_j = 0 \quad \text{and} \quad \text{rank } B_2 = 0$$

Now B_2 is an $r \times p$ matrix, so $r = 0$ certainly implies rank $B_2 = 0$. What about the converse? If rank $B_2 = 0$ but $r > 0$, then B_2 is a matrix of zeros. It follows from (4.2 *c*) and (4.2 *a*) that $A_{22} + A_{22}^* = 0$ and hence A has imaginary axis eigenvalues. This contradicts Remark 3 following Theorem 4.1. It can similarly be shown that rank $\tilde{C}_2 = 0$ is equivalent to

$$m_2 - (m_1 - r) = \sum_{k_j \geqslant 0} k_j = 0$$

(2) Formula (4.24) makes an important connection between the Hankel singular values of an all-pass matrix function and its factorization partial indices. A number of consequences follow from this result. In fact, eqn (4.24) can be viewed as a strengthened, more precise statement of Lemma 2.1. Lemma 2.1 amounts to the statement that $m_2 \geqslant m_1 - r$, which is (2.9). To prove this from (4.24) is easy:

$$\begin{aligned} m_2 &= m_1 - \sum_{j=1}^{p} k_j && \text{by (4.12 } b\text{)} \\ &\geqslant m_1 - \sum_{k_j \geqslant 0} k_j && \text{since } \sum_{j=1}^{p} k_j \leqslant \sum_{k_j \geqslant 0} k_j \\ &= m_1 - r && \text{by (4.24)} \end{aligned}$$

This in particular is worth stating for the scalar case.

Corollary 4.2.

Let $e(s)$ be a scalar, rational, all-pass function with m_1 stable and m_2 unstable poles. Then

$$r = \max(0, m_1 - m_2) \tag{4.34}$$

That is, if $m_1 > m_2$, then

$$\sigma_i = 1, \quad i = 1, \ldots, m_1 - m_2 \tag{4.35 a}$$

$$\sigma_i < 1, \quad i = m_1 - m_2 + 1, \ldots, m_1 \tag{4.35 b}$$

and if $m_1 \leqslant m_2$, then

$$\sigma_i < 1 \quad \forall i \tag{4.35 c}$$

Proof

For the scalar case, there is only one factorization index $k_1 = k = m_1 - m_2$ by (4.12 *b*). Hence

$$\sum_{k_j \geqslant 0} k_j = \max(0, k)$$

from which (4.34) follows by (4.24). It is easily seen from the definition (2.7) that (4.34) is equivalent to (4.35). □

Remark

Corollary 4.2 differs from Lemma 1.10 of Latham (1984) (i.e. Lemma 2.1) in that we have strict inequality in (4.35 *b*), the possibility of equality now being excluded.

Corollary 4.3

Let A be $m_1 \times m_1$ and stable, B, C^* $m_1 \times p$ and satisfying (4.2 *c*, *d*) with Σ as in (4.2 *a*, *b*), $0 < \Sigma_1 < I$. Let A, B, C and Σ be partitioned as in (4.7), (4.2 *b*) and $q = \text{rank } B_2$.

Then:

(1) Every all-pass matrix $E(s)$ such that $E_+(s) = C(sI - A)^{-1}B$ has $q = \text{rank } B_2$ strictly positive partial indices; they are determined by A, B and satisfy (4.24).

(2) For any set of $p - q$ non-positive integers $\beta_1, \ldots, \beta_{p-q}$ there exists an all-pass matrix $E(s)$ such that $E_+(s) = C(sI - A)^{-1}B$ and the non-positive partial indices of $E(s)$ are $\beta_1, \ldots, \beta_{p-q}$.

Note

Obviously when $q = p$ there are no β_i to choose.

Proof

Part 1 follows immediately from Theorem 4.4.

For part 2, by Theorem 4.4, all we need to do is show that when constructing an $E(s)$ from A, B, C as in Remark 5 following Theorem 4.1 we can choose $\tilde{C}_2$, $\tilde{A}_{22}$ to satisfy (4.9 *a*, *b*, *e*) and to have observability indices $-\beta_1, \ldots, -\beta_{p-q}, 0, \ldots, 0$ (q extra zeros, which must be added because $\tilde{C}_2$, $\tilde{A}_{22}$ has p observability indices. Of course some of the β_i may also be zero). This can be done as follows:

(i) Find a non-singular $p \times p$ matrix X such that $B_2X = [\bar{B}_2, 0]$, with $\bar{B}_2$ $r \times q$ and rank q.

(ii) Define $l = -(\beta_1 + \ldots + \beta_{p-q})$.

(iii) Choose a $(p - q) \times l$ matrix G and an $l \times l$ matrix $\bar{A}_{22}$ such that $\bar{A}_{22}$ is completely unstable and the observability indices of $(G, \bar{A}_{22})$ are $-\beta_1, \ldots, -\beta_{p-q}$. This is easily done using observability canonical forms—see e.g. Gevers and Wertz (1986).

(iv) Pick $p \times p$ matrix D_e such that $D_e^* C_2 + B_2^* = 0$ and $D_e^* D_e = I$.

(v) Define $p \times l$ matrix $\bar{C}_2 = D_e X[0, G^*]^*$.
Note the set of observability indices of $\bar{C}_2$, $\bar{A}_{22}$ is $\{-\beta_1, \ldots, -\beta_{p-q}, 0, \ldots, 0\}$.

(vi) Let $Q_2 = Q_2^*$ satisfy $\bar{A}_{22}^* Q_2 + Q_2 \bar{A}_{22} + \bar{C}_2^* \bar{C}_2 = 0$ and let $Q = -K^*K$ with K non-singular (note $Q_2 < 0$ since $\bar{A}_{22}$ is completely unstable and $\bar{C}_2$, $\bar{A}_{22}$ is observable).

(vii) Define

$$\tilde{A}_{22} = K\bar{A}_{22}K^{-1}$$

$$\tilde{C}_2 = \bar{C}_2 K^{-1}$$

$$\tilde{B}_2^* = -D_e^* \tilde{C}_2$$

It is easily seen that (4.9 *a*, *b*, *e*) are satisfied and that the set of observability indices of $\tilde{C}_2, \tilde{A}_{22}$ is $\{-\beta_1, \ldots, -\beta_{p-q}, 0, \ldots, 0\}$. □

Remark

Observe from (8.79) of Glover (1984) that the observability indices of $\tilde{C}_2, \tilde{A}_{22}$ are the observability indices of $K(s)$ in the characterization of $E_-(s)$ in Corollary 8.6 of Glover (1984). Thus the non-positive factorization indices of $E(s)$ are the negative of the $p-q$ largest observability indices of $K(s)$ in Corollary 8.6 of Glover (the remaining q observability indices are zero).

We now combine Lemma 4.1, Theorem 4.3 and Theorem 4.4 to obtain our main result for the phase-matching algorithm of Green and Anderson (1986).

Theorem 4.5

Let $E(s) \in L_\infty$ be all-pass with m_1 stable and m_2 unstable poles. Let r be defined by (2.7) and $k_j, j = 1, \ldots, p$ be the partial indices of $E(s)$. Then

$$E(s) = V(-s)^{-1} W(\bar{s})^* \tag{4.36}$$

with $V(s)$, $W(s)$ proper stable and minimum phase if and only if

$$m_2 = m_1 - r \tag{4.37}$$

Furthermore

(*a*) $V(\infty)$, $W(\infty)$ are non-singular if and only if $r = 0$ (so $m_1 = m_2$).

(*b*) $V(\infty)$, $W(\infty)$ are singular if and only if $E(s)$ is a minimal degree Nehari extension of $E_+(s)$.

(*c*) $E(s)$ is the unique Nehari estension of $E_+(s)$ if and only if $k_j > 0 \, \forall_j$.

Proof

By Theorem 4.3, the factorization (4.36) exists if and only if $k_j \geqslant 0 \, \forall_j$, or, equivalently,

$$\sum_{k_j \leqslant 0} k_j = 0 \tag{4.38}$$

Now, by Theorem 4.4, this is equivalent to $m_2 = m_1 - r$.

(*a*) $V(\infty)$, $W(\infty)$ are non-singular if and only if $k_j = 0 \, \forall_j$ by Theorem 4.3. Since we already have (4.38), this is equivalent to

$$\sum_{k_j \geqslant 0} k_j = 0$$

That is, $r = 0$ by Theorem 4.4.

(*b*) Follows from (*a*), or as follows: by Theorem 4.3, $V(\infty)$, $W(\infty)$ are singular *if* $k > 0$. That is, in view of (4.38) that

$$r = \sum_{k_j \geqslant 0} k_j > 0 \text{ by Theorem 4.4}$$

Thus, $E(s)$ is a Nehari extension of $E_+(s)$ by Lemma 4.1 and it is minimal because of (4.37).

(*c*) By Lemma 4.1, $E(s)$ is the unique Nehari extension of $E_+(s)$ if and only if rank $B_2 = p$ (B_2 as in Theorem 4.1). By Theorem 4.4, this is equivalent to $E(s)$ having p strictly positive partial indices, i.e. $k_j > 0, j = 1, \ldots, p$. □

Remarks

Part (*c*) can be written as the promised generalization of Lemma 3.2.

Corollary 4.4

Let $E(s) \in L_\infty$ be all-pass. Then $E(s)$ can be factored as $E(s) = V(-s)^{-1} W(\bar{s})^*$ with $V(s)$, $W(s)$ strictly proper, stable and minimum phase if and only if $E(s)$ is the unique Nehari extension of its stable part $E_+(s)$.

Proof

By Theorem 4.5 we need to show we can choose V, W strictly proper if and only if $k_j > 0 \; \forall_j$. This follows from Theorem 4.3. □

Remarks

(1) Corollary 4.4 does not say that every proper, stable, minimum-phase factorization of a unique Nehari extension is strictly proper. It only says that a strictly proper factorization exists.

(2) Observe that Lemma 4.1 concerns the additive decomposition of an all-pass matrix $E(s)$, whilst Theorem 4.5 concerns the multiplicative decomposition of $E(s)$. Note the curiously prominent role of rank B_2 in Theorem 4.4 and Lemma 4.1, and the importance of r in both decompositions.

(3) State-space formulas for pairs $V(s)$, $W(s)$ satisfying (4.36) can be given, and this will be the subject of a subsequent paper.

We now discuss the relationship of our results to those of Dym and Gohberg (1983 b) ((1983 a) deals with the discrete-time case—the results of (1983 b) are in fact proved from those of (1983 a) via the bilinear transformation).

Firstly note that the numbers n_i in their (1983 b) paper are dim F_i, with F_i as in (4.28 *a*). According to Theorem 1.1 of Dym and Gohberg (1983 b), the number of strictly positive indices is $n_0 - n_1$. This is in the proof of Theorem 4.4 just above (4.30). We, however, identify $n_0 - n_1$ as the mysterious rank B_2 of Glover (1984) in (4.30). Thus Theorem 1.1 of Dym and Gohberg (1983 b) is Corollary 4.3. The proof of Corollary 4.3, however, actually constructs an all-pass matrix with the β_i as its non-positive indices.

Theorem 1.2 of Dym and Gohberg (1983 b) is Theorem 4.5 part (*c*), using the identification of $n_0 - n_1$ as rank B_2 and Lemma 4.1.

Theorem 1.3 is Theorem 4.5 part (*a*). Theorem 1.3 also says that there is a one-to-one correspondence between all-pass matrices with zero partial indices and given stable part and the set of unitary matrices. This is seen from Theorem 4.1 by the observation that an all-pass matrix $E(s)$ with $m_2 = m_1 - r$ (i.e. all partial indices non-negative) is uniquely determined by $E_+(s)$ and $E(\infty)$ ($E(\infty)$ being the unitary matrix). Thus, given $E_+(s)$, there is a one-to-one correspondence between $E(s)$ and $E(\infty)$ when $m_2 = m_1 - r$. When, in addition, $r = 0$ the one-to-one correspondence is also onto, since B_2, C_2 have no rows, implying we can pick D_e $(= E(\infty))$ to be an unitary matrix in Remark 5 following Theorem 4.1.

Equations (4.24) and (4.25) do not appear in Dym and Gohberg (1983 b), although (4.24) can be deduced from Theorem 2.5 of their (1983 a) paper.

Thus this paper can be seen as paralleling Dym and Gohberg (1983 b), giving state-space conditions for the factorization properties of all-pass matrices. This provides, for example, the link between the uniqueness conditions of Nehari extensions given in Glover (1984) and Dym and Gohberg (1983 b) via the identification $n_0 - n_1$ as rank B_2. Additionally, the proof of Corollary 4.3 provides a procedure for constructing all-pass extensions with specified non-positive indices and shows that these indices are directly related to the choice of $K(s)$ in Corollary 8.6 of Glover (1984), whilst Theorem 1.1 of Dym and Gohberg (1983 b) gives existence only. Of course our results, specifically Theorem 4.5, are also tailored for the application to stochastic model reduction by phase matching (Opdenacker and Jonckheere 1985, Jonckheere and Helton 1985, Harshavadhana and Jonckheere 1987, Green and Anderson 1986).

6. Conclusion

We have shown that the factorization properties of all-pass matrix functions are closely related to their properties as Nehari extensions. In particular, an all-pass Nehari extension is factorable into proper stable minimum-phase factors if and only if it is a minimal degree extension (i.e. one characterized by the construction of Theorem 6.3 of Glover (1984)). Moreover there exists a strictly proper factorization if and only if the Nehari extension is unique.

More generally, we have shown there to be significant connections between the Hankel singular values of an all-pass matrix function and its partial factorization indices, e.g. equations (4.24) and (4.25). This work thus parallels the operator-theoretic developments of Dym and Gohberg (1983 a, b) with state-space techniques.

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