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Optimal Control Problems over Large Time Intervals*

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Many finite time optimal control problems can be solved by patching together solutions to infinite time problems, and similar transients are seen at each end of the control interval.

Key Words—Optimal control; optimal systems; computational methods.

Abstract—The paper focuses on problems of optimal control over a large time interval with end states prescribed. For a wide class of systems and performance indices, an approximate solution is characterized by piecing together two solutions for two infinite time problems. These solutions exhibit similar transient behaviour.

1. INTRODUCTION

CONSIDER the following linear quadratic optimal control problem: for the time-invariant, completely controllable system

$$\dot{x} = Fx + Gu \tag{1.1}$$

with prescribed $x(0)$, $x(T)$, choose $u(\cdot)$ to minimize the performance index

$$J(x(0), u(\cdot), T) = \int_0^T [u'Ru + x'Qx]\,\mathrm{d}t \tag{1.2}$$

with $R > 0$, $Q \geqslant 0$ and where $[F, Q^{1/2}]$ is completely observable.

When T becomes large, the trajectories acquire several semiquantitative properties:

(i) most of the transient activity occurs near $t = 0$ and $t = T$;

(ii) $x(t)$ remains very small away from $t = 0$ and $t = T$;

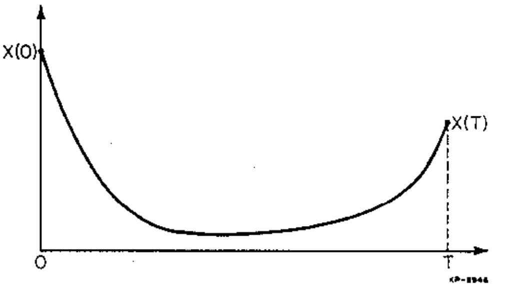

FIG. 1. Optimal trajectory.

(iii) the time constants associated with the trajectory near $t = 0$ are the same as those associated with the trajectory near $t = T$;

(iv) the optimal trajectory and control for the original problem can be approximately obtained by piecing together the optimal trajectory and control associated with the two indices

$$J_+ = \int_0^\infty [u'Ru + x'Qx]\,\mathrm{d}t$$

with $x(0)$ prescribed (the solution defines the part of the trajectory for the original problem near $x = 0$) and

$$J_- = \int_{-\infty}^T [u'Ru + x'Qx]\,\mathrm{d}t$$

with $x(T)$ prescribed, the value of $x(-\infty)$ being immaterial. (Here, the solution defines the part of the trajectory of the original problem near $t = T$.)

Properties (i)–(iii) are illustrated in Fig. 1 and property (iv) by Figs 1 and 2. For a precise formulation of the basic ideas, see Wilde and Kokotovic (1972), and for some extensions and additional remarks see Wilde and Kokotovic (1973), Kokotovic *et al.* (1976), Kokotovic (1984) and

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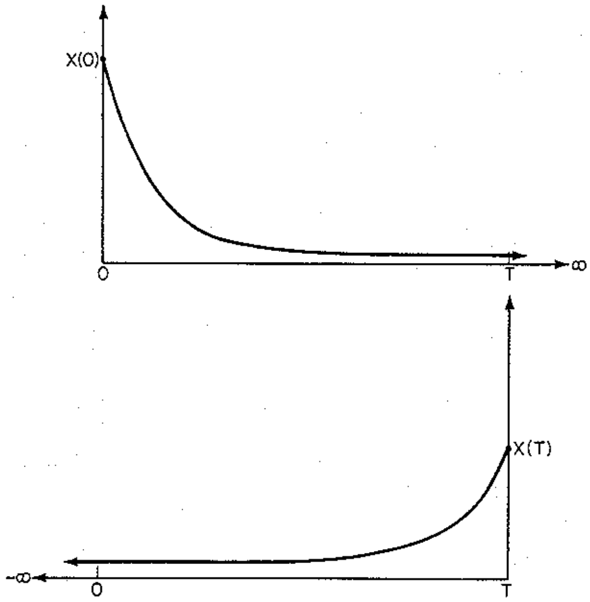

FIG. 2. (a) Optimal trajectory for V_+; (b) optimal trajectory for V_-.

Kokotovic *et al.* (1986).

The purpose of this paper is to explore extensions of these properties to problems which involve either a non-linear system or a non-quadratic performance index, or both. Roughly speaking, the class of systems that are encompased includes those which have been studied as extensions of linear-quadratic theory, in, for example, Rekasius (1964), Bass and Weber (1966), Anderson (1969), Asseo (1969), Moylan and Anderson (1973), Glad (1984) and Tsitsiklis and Athans (1984). In these references, extensive use is made of Hamilton–Jacobi theory, and this is one of the tools here. Another aid to our thinking has been the recognition [see Kokotovic *et al.* (1986)] that an optimal control problem with T large in the performance index in comparison to the dominant time constants of the system has some properties which are accessible with singular perturbation theory.

2. REVIEW OF CERTAIN LINEAR-QUADRATIC OPTIMIZATION IDEAS

In this section, those assertions made in Section 1 concerning two point linear-quadratic problems are summarized in a quantitative way. The treatment is effectively drawn from Wilde and Kokotovic (1972).

With appropriate controllability and observability assumptions, the steady-state Riccati equation associated with (1.1) and (1.2), viz.

$$XF + F'X - XGR^{-1}G'X + Q = 0 \qquad (2.1)$$

has a maximum solution P with $P > 0$ and a minimum solution N with $N < 0$. If λ denotes the costate variable in the associated Hamiltonian system

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} F & -GR^{-1}G' \\ -Q & -F' \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}, \qquad (2.2)$$

then the transformation

$$\begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} I & I \\ P & N \end{bmatrix} \begin{bmatrix} y \\ \theta \end{bmatrix} \qquad (2.3)$$

is non-singular and transforms (2.2) to

$$\begin{bmatrix} \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} F - GR^{-1}G'P & 0 \\ 0 & F - GR^{-1}G'N \end{bmatrix} \begin{bmatrix} y \\ \theta \end{bmatrix}. \qquad (2.4)$$

Further, from the fact that

$$PF + F'P - PGR^{-1}G'P + Q = 0$$
$$NF + F'N - NGR^{-1}G'N + Q = 0,$$

it follows by subtraction that

$$[P - N][F - GR^{-1}G'P] + [F' - NGR^{-1}G'][P - N] = 0 \qquad (2.5)$$

so that the eigenvalues of $F - GR^{-1}G'N$ are seen to be the negative of those of $F - GR^{-1}G'P$, which are known to have negative real parts. So (2.4) displays the occurrence of decaying and expanding modes, and it is easy to establish properties (i)–(iii) stated in the Introduction quantitatively. Property (iv) is explained at greater length in Wilde and Kokotovic (1972). It rests on the fact that

$$x'(0)Px(0) = \inf_{u(\cdot)} \int_0^{\infty} [u'Ru + x'Qx]\,dt$$

$$x'(0)Nx(0) = \inf_{u(\cdot)} \int_{-\infty}^{0} [u'Ru + x'Qx]\,dt.$$

3. NON-LINEAR SYSTEMS NON-QUADRATIC INDICIES: INFINITE TIME RESULTS

The systems

$$\dot{x} = f(x) + G(x)u \qquad (3.1)$$

with $x \in R^n$, $u \in R^m$, $f(0) = 0$, $f(\cdot)$ and $G(\cdot)$ smooth will be considered. It is assumed that solutions of (3.1) exist for all $x(0)$ and all $u(\cdot)$ suitably restricted, e.g. $u(\cdot)$ has a one-sided limit at all points. One way to ensure this is to make the *Global Lipschitz assumption.* $f(\cdot)$ and $G(\cdot)$ satisfy a global Lipschitz condition.

It is further assumed that (3.1) is controllable from any initial state to the origin and from the origin to any prescribed state. More precisely, the following requirements are set forth.

Controllability condition. There exists η such that for any $x(0)$ with $\|x(0)\| \leqslant \xi$, for arbitrary positive ξ, there exists a control taking $x(0)$ to $x(\eta) = 0$ with $\sup_{0 \leqslant t \leqslant \eta} [\|u(t)\|, \|x(t)\|] \leqslant h_c(\xi)$. Here, $h_c(\xi)$ is a continuous monotone increasing function of ξ with $h_c(0) = 0$, $h_c(\xi) \neq 0$ for $\xi \neq 0$. A similar statement holds for control from $x(0) = 0$ to some $x(\eta)$ with $\|x(\eta)\| \leqslant \xi$, and without loss of generality, the same $h_c(\xi)$ can be used.

Consideration will be given to the performance index

$$J[x(0), u(\cdot), T] = \int_0^T [u'u + m(x)]\,dt \quad (3.2)$$

in which $m(x) \geqslant 0\ \forall x$, with $m(\cdot)$ continuous in x and zero for $x = 0$. It will also be assumed that Hamilton–Jacobi theory can be used to calculate the optimal control, and that for the infinite time problem, $m(\cdot)$ observes the states of (3.1) in such a manner that for no initial state $x(0) = x_0 \neq 0$ will the unforced trajectory of (3.1) lead to $m(x(t)) \equiv 0$.

For reasons that will become apparent subsequently, this condition in fact needs to be strengthened.

Observability condition. There exists η such that for any $x(0)$ with $\|x(0)\| \geqslant \xi$, for arbitrary positive ξ,

$$\int_0^\eta m(x)\,dt \geqslant h_0(\xi)$$

$$\int_{-\eta}^0 m(x)\,dt \geqslant h_0(\xi),$$

where $x(t)$ is evaluated along trajectories of the unforced system $\dot{x} = f(x)$ and h_0 is a continuous monotone increasing function of ξ with $h_0(0) = 0$; $h_0(\xi) \neq 0$ for $\xi \neq 0$.

Notice that as a consequence of this condition, no unforced trajectory of (3.1) computed backwards in time from $x(0) \neq 0$ satisfies $m(x(t)) \equiv 0$; equivalently, no trajectory of $\dot{x} = -f(x)$ computed forwards from $x(0) \neq 0$ satisfies $m(x(t)) \equiv 0$. Notice also that the observability condition will be automatically fulfilled if $m(x)$ is positive definite.

Connection between two infinite time problems

Define

$$V_+[x(0)] = \inf_{u(\cdot)} \int_0^\infty [u'u + m(x)]\,dt \quad (3.3a)$$

$$V_-[x(0)] = -\inf_{u(\cdot)} \int_{-\infty}^0 [u'u + m(x)]\,dt. \quad (3.3b)$$

Noting that

$$\min \{u'u + m(x) + \nabla V'_+(x)[f(x) + G(x)u]\}$$

is achieved by $u = -\tfrac{1}{2}G'(x)\nabla V_+(x)$, it follows that the Hamilton–Jacobi equation for V_+ is

$$\nabla V'_+(x)[f(x) - \tfrac{1}{2}GG'\nabla V_+(x)] = -m(x) - \tfrac{1}{4}\nabla V'_+(x)GG'\nabla V_+(x). \quad (3.4)$$

Now to obtain the equation for V_-, set $s = -t$. We are then concerned with the system

$$\frac{dx}{ds} = -f(x) - G(x)u \quad (3.5)$$

and finding the infimum

$$W(x(0)) = \inf_{u(\cdot)} \int_0^\infty [u'u + m(x)]\,ds. \quad (3.6)$$

The assumption that (3.1) is controllable from the origin to an initial state implies that (3.5) is controllable from an initial state to the origin, which is important in guaranteeing the existence of $W(x)$. In terms of $W(x)$, the optimal control is now given by

$$u = \tfrac{1}{2}G'\nabla W(x) \quad (3.7)$$

and the Hamilton–Jacobi equation is

$$\nabla W'(x)[-f(x) - \tfrac{1}{4}GG'\nabla W(x)] = -m(x) - \tfrac{1}{4}\nabla W'(x)GG'\nabla W(x). \quad (3.8)$$

Now recognize that $V_-(x) = -W(x)$ for all x, by virtue of the definition of these two quantities. Consequently,

$$\nabla V'_-(x)[f(x) - \tfrac{1}{2}GG'\nabla V_-(x)] = -m(x) - \tfrac{1}{4}\nabla V'_-(x)GG'\nabla V_-(x). \quad (3.9)$$

Accordingly, the following has been established:

Property 3.1. $V_+(x)$ and $V_-(x)$ satisfy the same Hamilton–Jacobi equation.

There is naturally a distinction between $V_+(x)$ and $V_-(x)$;

Property 3.2. $V_+(x)$ and $V_-(x)$ are respectively positive and negative definite functions of x.

Proof. Equation (3.3a) shows that $V_+(x)$ is certainly a non-negative function of x. Along optimal trajectories, it results that

$$\dot{x} = f(x) - \tfrac{1}{2}GG'\nabla V_+(x) \tag{3.10}$$

and (3.4) then shows that

$$\frac{\mathrm{d}V_+(x(t))}{\mathrm{d}t} = -m(x) - \tfrac{1}{4}\nabla V'_+(x)GG'\nabla V_+(x), \tag{3.11}$$

so $V_+(x(t))$ is non-increasing. If $V_+(x_0) = 0$ for some $x_0 \neq 0$, then the following should hold along the optimal trajectory $x(t)$ commencing at x_0: $\dot{V}_+(x(t)) \equiv 0$, $\nabla V_+(x(t)) \equiv 0$, $m(x(t)) \equiv 0$. Because $\nabla V_+(x(t)) \equiv 0$, such a trajectory $x(t)$ would be a solution of $\dot{x} = f(x)$, contradicting the assumption that $m(x)$ observes unforced trajectories of (3.1).

A similar argument to that showing $V_+(x)$ is positive definite shows that $V_-(x)$ is a negative definite function of x.

The fact that V_+ and V_- satisfy the same Hamilton–Jacobi equation, with V_+ positive definite and V_- negative definite, parallels the results for the linear-quadratic case, concerning the matrices P, N or functions $x'Px$, $x'Nx$ of the previous section.

Now in the linear quadratic case, there is a *unique* positive definite solution of the steady-state Riccati equation. Observe for completeness that the same property holds true for the Hamilton–Jacobi equation:

$$\nabla X(x)'[f(x) - \tfrac{1}{2}GG'\nabla X(x)]' = -m(x) - \tfrac{1}{4}\nabla X(x)'GG\nabla X(x). \tag{3.12}$$

(This uniqueness result will be needed subsequently.)

Property 3.3. $V_+(x)$ is the only positive definite solution of the steady-state Hamilton–Jacobi equation (3.12) with $V_+(0) = 0$.

Proof. Suppose that $W_+(x)$ is a second and different positive definite solution of (3.12) with $W_+(0) = 0$. It is readily checked that $W_+(x)$ is monotone non-increasing along trajectories of

$$\dot{x} = f(x) - \tfrac{1}{2}G(x)G'(x)\nabla W_+(x) \tag{3.13}$$

and that $\dfrac{\mathrm{d}W_+(x)}{\mathrm{d}t}$ is not identically zero. Hence (3.13) is asymptotically stable. Now consider (3.1) with control $u(t) = u_1(t) - \tfrac{1}{2}G'(x)\nabla W_+(x)$, and the associated value of (3.2):

$$\begin{aligned} J[x(0), u(\cdot), T] &= \int_0^T [u'u + m(x)]\,\mathrm{d}t \\ &= \int_0^T [u_1'u_1 - u_1'G'(x)\nabla W_+(x) \\ &\quad + \tfrac{1}{4}\nabla W'_+(x)G(x)G'(x)\nabla W_+(x) + m(x)]\,\mathrm{d}t \\ &= \int_0^T \{u_1'u_1 - \nabla W'_+(x)[f(x) \\ &\quad - \tfrac{1}{2}G(x)G'(x)\nabla W_+(x) + G(x)u_1]\}\,\mathrm{d}t \\ &= \int_0^T u_1'u_1\,\mathrm{d}t + W_+(x(0)) - W_+(x(T)). \end{aligned}$$

Now attention is temporarily restricted to controls $u_1(\cdot)$ with the property that $x(T) \to 0$ as $T \to \infty$. Denote the set of $u_1(\cdot)$ by U_1. Then

$$\lim_{T\to\infty} J[x(0), u(\cdot), T] = \int_0^\infty u_1'u_1\,\mathrm{d}t + W_+[x(0)]$$

and

$$\begin{aligned} &\inf_{u(\cdot)\,\text{with}\,x(T)\to 0}\ \lim_{T\to\infty} J[x(0), u(\cdot), T] \\ &= \inf_{u_1(\cdot)\in U_1} \int_0^\infty u_1'u_1\,\mathrm{d}t + W_+[x(0)] \\ &= W_+[x(0)]. \end{aligned}$$

On the other hand, it is known that $\lim_{T\to\infty} V[x(0), u(\cdot), T]$ has a minimum value of $V_+[x(0)]$, and this is attained by a $u(\cdot)$ which results in $x(T) \to 0$. Hence

$$\begin{aligned} &\inf_{u(\cdot)\,\text{with}\,x(T)\to 0}\ \lim_{T\to\infty} J[x(0), u(\cdot), T] \\ &\qquad = \inf_{u(\cdot)} \lim_{T\to\infty} J[x(0), u(\cdot), Y] = V_+[x(0)], \end{aligned}$$

i.e.

$$V_+[x(0)] = W_+[x(0)].$$

Of course, it can also be proved that $V_-(x)$ is the only negative definite solution with $V_-(0) = 0$. Another characterization of $V_+(x)$ which can be easily proved, but which will not be used here, is that among the solutions of (3.12), the only one for which $\dot{x} = f(x) - \tfrac{1}{2}GG'\nabla X(x)$ is asymptotically stable is $X(x) = V_+(x)$.

The last points to be made in this section concern the parallel with the linear quadratic result (2.5) and the associated remarks.

Property 3.4.

$$[\nabla V_+ - \nabla V_-]'[f(x) - \tfrac{1}{2}G(x)G'(x)\nabla V_+(x)]$$
$$= -[\nabla V_+ - \nabla V_-]'[f(x) - \tfrac{1}{2}G(x)G'(x)\nabla V_-(x)] \quad (3.14)$$
$$= -\tfrac{1}{2}[\nabla V_+(x) - \nabla V_-(x)]'G(x)G'(x)[\nabla V_+(x) - \nabla V_-(x)]. \quad (3.15)$$

Proof. Consider (3.12) with $X(x)$ replaced by $V_+(x)$, $V_-(x)$ and subtract one equation from the other. There results

$$[\nabla V_+(x) - \nabla V_-(x)]'f(x) = +\tfrac{1}{4}\nabla V_+'(x)G(x)G'(x)\nabla V_+(x) - \tfrac{1}{4}\nabla V_-'(x)G(x)G'(x)\nabla V_-(x).$$

Simple manipulation then establishes that

$$[\nabla V_+(x) - \nabla V_-(x)]'[f(x) - \tfrac{1}{2}G(x)G'(x)\nabla V_+(x)] = -\tfrac{1}{2}[\nabla V_+(x) - \nabla V_-(x)]'G(x)G'(x)[\nabla V_+(x) - \nabla V_-(x)]. \quad (3.16)$$

The right-hand side of (3.15) is invariant under interchange of $V_+(x)$ and $V_-(x)$ [as are the individual equations for $V_+(x)$, $V_-(x)$]. It follows that the left-hand side of (3.15) is equal to the expression obtained by interchanging $V_+(x)$ and $V_-(x)$ on the left-hand side, i.e. (3.14) holds.

Several observations follow. The properties established for $V_+(x)$, $V_-(x)$ show that $V_+(x) - V_-(x)$ is a positive definite function of x, zero at $x = 0$, and (3.15) shows that it is monotone decreasing along trajectories of $\dot{x} = f(x) - \tfrac{1}{2}G(x)G'(x)\nabla V_+(x)$. Thus $\bar{V}(x) \triangleq V_+(x) - V_-(x)$ is a Lyapunov function for (3.10), which is known to be asymptotically stable.

Further, $\bar{V}(x)$ is also a Lyapunov function for

$$\dot{x} = -f(x) + \tfrac{1}{2}G(x)G'(x)\nabla V_-(x) \quad (3.17)$$

and (3.14) shows that the rate of decay of $\bar{V}(x)$ at a point x on (3.10) and (3.16) is the same. Thus it can be asserted that

Property 3.5. The transient behaviours of (3.10) and (3.16) are the same, in the sense that there exists a single Lyapunov function $\bar{V}(x)$ for both systems such that $d\bar{V}(x)/dt$ evaluated for the same point x on a trajectory of either system is the same.

4. NON-LINEAR SYSTEMS, NON-QUADRATIC INDICES AND FINITE TIME RESULTS

The aim in this section is to show that a finite time optimization problem requiring a transition from initial state $x(0)$ to final state $x(T)$, with T large, can be approximately solved by piecing together solutions for the two infinite time problems. The system equation is (3.1), and

$$V_+[x(0)] = \inf_{u(\cdot)} \int_0^{\infty} [u'u + m(x)]\,dt \quad (4.1a)$$

$$V_-[x(T)] = -\inf_{u(\cdot)} \int_{-\infty}^{T} [u'u + m(x)]\,dt \quad (4.1b)$$

$$V[x(0), x(T)] = \inf_{u(\cdot)} \int_0^{T} [u'u + m(x)]\,dt \quad (4.1c)$$

are set, where the arguments $x(0)$ and/or $x(T)$ denote values to be assumed by the state of (3.1) at time 0 and/or T.

Let $\varepsilon > 0$ be small and arbitrary. Choose T_0 sufficiently large (and certainly greater than $\sigma\eta$, η being defined in the controllability condition of the last section) that the optimal trajectory of (4.1a) for all $x(0)$ with $\|x(0)\| \leqslant 1$ has $\|x(t)\| < \varepsilon$ for all $t \geqslant T_0$ and the optimal trajectory of (4.1b) with $T = T_0$ for all $x(T_0)$ with $\|x(T_0)\| \leqslant 1$ has $\|x(t)\| < \varepsilon$ for all $t \leqslant 2T_0/3$. That such a T_0 can be found is a consequence of the asymptotic stability results described in the previous section. The following property is asserted.

Property 4.1. Suppose $\|x(0)\| \leqslant 1$ and $\|x(T)\| \leqslant 1$. With arbitrary $\varepsilon > 0$ and with $T_0(\varepsilon)$ chosen as above, there holds for all $T \geqslant T_0$

$$V[x(0), x(T)] \leqslant V_+[x(0)] - V_-[x(T)] + k_1(\varepsilon) \quad (4.2)$$

where

$$k_1(\varepsilon) = 2\eta[|h_c(\varepsilon)|^2 + \sup_{\|x\| \leqslant h_c(\varepsilon)} m(x)] \quad (4.3)$$

and is a monotone increasing function of ε with $\lim_{\varepsilon \downarrow 0} k_1(\varepsilon) = 0$.

Proof. Define a control and trajectory achieving the transition from $x(0)$ to $x(T)$ by concatenating five pieces as follows:

- (i) over $[0, T/3]$, the control is the optimal control for $V_+[x(0)]$;
- (ii) over $[T/3, T/3 + \eta]$, the control takes $x(T/3)$ to the origin;
- (iii) over $[T/3 + \eta, (2T/3) - \eta]$, the control is zero;
- (iv) over $[(2T/3) - \eta, (2T/3)]$, the control takes the state from the origin to $x(2T/3)$, where $x(2T/3)$ is a point on the optimal state trajectory associated with $V_-[x(T)]$; and
- (v) over $[2T/3, T]$, the control is the optimal control for $V_-[x(T)]$.

It is trivial to conclude (4.2): the contributions to the non-optimal cost from the five pieces are overbounded respectively by $V_+[x(0)]$, $\frac{1}{2}k_1(\varepsilon)$, 0, $\frac{1}{2}k_1(\varepsilon)$ and $-V_-[x(T)]$, and the non-optimal cost overbounds the optimal cost.

In the next property, a bound in the opposite direction is demonstrated; taken in conjunction with (4.2), it allows the conclusion that $V[x(0), x(T)]$ and $V_+[x(0)] - V_-[x(T)]$ are close for large T and have a difference approaching zero as $T \to \infty$.

Property 4.2. Suppose that $\|x(0)\| \leqslant 1$ and $\|x(T)\| \leqslant 1$. With arbitrary $\varepsilon > 0$, there exists a $T_1(\varepsilon)$ such that for all $T \geqslant T_1(\varepsilon)$,

$$V_+[x(0)] - V_-[x(T)] \leqslant V[x(0), x(T)] + k_2(\varepsilon), \tag{4.4}$$

where $k_2(\varepsilon)$ is monotone increasing in ε, with $\lim_{\varepsilon \downarrow 0} k_2(\varepsilon) = 0$.

The proof of Property 4.2 makes use of the following lemma, the proof of which is in the Appendix.

Lemma A. Consider equation (3.1) over a fixed interval $[a, b]$ with initial condition $x(a)$. Let $x_0(\cdot)$ denote the trajectory with $u \equiv 0$ and $x_1(\cdot)$ denote the trajectory with $u(\cdot)$ not identically zero. Suppose that

$$\int_a^b u'u \, dt \leqslant \varepsilon^2. \tag{4.5}$$

Then

$$\sup_{t \in [a,b]} \|x_1(t) - x_0(t)\| \leqslant L\epsilon \tag{4.6}$$

from some L.

Proof of Property 4.2. The first step is to show that with T suitably large, optimal trajectories for $V[x(0), x(T)]$ attain very small values at intermediate points.

Equation (4.2) shows that $V[x(0), x(T)]$ for fixed $x(0), x(T)$ is bounded as a function of T, since the right-hand side is independent of T. Let $\delta > 0$. (For the moment, δ will be arbitrary. Later, it will be restricted in terms of an arbitrary $\varepsilon > 0$.) Now there exists an integer $N(\delta)$ such that

$$V[x(0), x(T)] \leqslant \frac{N\delta^2}{3} \quad \text{for all } T. \tag{4.7}$$

Hence along the optimal trajectory

$$\sum_{k=0}^{N-1} \int_{kT/N}^{(k+1)T/N} [u'u + m(x)] \, dt \leqslant \frac{N\delta^2}{3} \tag{4.8}$$

and consequently in the interval $(T/3, 2T/3)$, there exists at least one choice of k (between $N/3$ and $2N/3$), depending in general on P, such that

$$\int_{kT/N}^{(k+1)T/N} u'u \, dt \leqslant \delta^2. \tag{4.9}$$

Let $x_0(t)$ denote the trajectory of (3.1) over $I_k \triangleq \left(\frac{kT}{N}, \frac{(k+1)T}{N}\right)$ defined by initial condition $x(kT/N)$ (lying on the optimal trajectory), and zero input, for $t \geqslant kT/N$, while $x(t)$ continues to denote the optimal trajectory. Then the lemma implies that

$$\sup_{t \in I_k} \|x_0(t) - x(t)\| \leqslant L\delta. \tag{4.10}$$

It is remarked that L depends on $x(kT/N)$. It is possible to find an L that works for all $x(0), x(T)$ in a compact set, say $\|x(0)\| \leqslant 1$, $\|x(T)\| \leqslant 1$. (For if $x(0)$, $x(T)$ lie in compact sets, so must $x(kT/N)$. Then an argument based on the Heine–Borel theorem and the construction of L in the lemma proof yield the claim.)

Now (4.9) also implies that

$$\int_{kT/N}^{(k+1)T/N} m(x) \, dt \leqslant \delta^2, \tag{4.11}$$

and the continuity of $m(\cdot)$ and (4.10) then yield that

$$\int_{kT/N}^{(k+1)T/N} m(x_0(t)) \, dt \leqslant \delta^2 + l(\delta), \tag{4.12}$$

where $l(\delta)$ is monotone increasing in δ and $l(0) = 0$. Pick $\varepsilon > 0$ and choose δ so that $\delta^2 + l(\delta) < h_0(\varepsilon)$, which determines N, and choose T so that $T \geqslant T_1$, where $T_1/N > \eta$. Then the observability condition ensures that $\|x(kT/N)\| < \varepsilon$, i.e. for some $t_1 \in [T/3, 2T/3]$, $\|x(t_1)\| < \varepsilon$ where $x(t_1)$ is a point on the optimal trajectory joining $x(0)$ to $x(T)$; such a point always exists provided $T \geqslant T_1$.

Now observe that

$$V_+[x(0)] \leqslant \int_0^{t_1} [u'u + m(x)] \, dt + V_+[x(t_1)].$$

Here, $V_+[x(t_1)] = \inf_{u(\cdot)} \int_{t_1}^{\infty} [u'u + x(x)] \, dt$ with (3.1) possessing initial condition $x(t_1)$ at time t_1. If $x_1 = x(t_1)$, time invariance guarantees this is $\inf \int_0^{\infty} [u'u + m(x) \, dt$ with $x_1 = x(0)$.

The first integral on the right is computed using the optimal trajectory for $V[x(0), x(T)]$, and the inequality simply says that the optimal trajectory for $V_+[x(0)]$ will achieve a lower cost than a trajectory which is non-optimal over $[0, t_1]$, then optimal over $[t_1, \infty]$.

Similarly,

$$-V_-[x(T)] \leqslant \int_{t_1}^{T} [u'u + m(x)]\,dt - V_-[x(t_1)]$$

and adding leads to

$$V_+[x(0)] - V_-[x(T)] \leqslant V[x(0), x(T)] + V_+[x(t_1)] - V_-[x(t_1)].$$

Now $\|x(t_1)\| < \varepsilon$ and $V_+[x(t_1)], V_-[x(t_1)]$ are continuous in $x(t_1)$. Hence there exists a $k_2(\varepsilon)$, monotone increasing with ε and zero at the origin for which

$$\sup_{\|x(t_1)\| \leqslant \varepsilon} |V_+[x(t_1)] - V_-[x(t_1)]| \leqslant k_2(\varepsilon)$$

and then Property 4.2 follows.

Together, Properties 4.1 and 4.2 show that $V[x(0), x(T)]$ and $V_+[x(0)] - V_-[x(T)]$ can be made as close as desired by choosing T large enough. Further, the control strategy suggested in Property 4.1 (use the optimal control for $V_+[x(0)]$ until $x(t)$ is small, control to the origin and attach this to a control from the origin to a small $x(t)$ on the $V_-[x(T)]$ trajectory) achieves a good approximation to the optimal control for $V[x(0), x(T)]$. This is because the cost of such a control is overbounded, as proved in establishing Property 4.1 by

$$V_+[x(0)] - V_-[x(T)] + k_1(\varepsilon)$$

and by (4.4), it is in turn overbounded (provided T is suitably large) by

$$V[x(0), x(T)] + k_1(\varepsilon) + k_2(\varepsilon).$$

$k_1(\varepsilon)$ and $k_2(\varepsilon)$ can be made as small as required by choosing ε arbitrary small (of course ensuring that T is appropriately large).

Note that all the above conclusions have apparently required an assumption that $\|x(0)\| \leqslant 1$, $\|x(T)\| \leqslant 1$. Obviously, any compact sets can be assumed for $x(0)$, $x(T)$. The only effect of changing the sets is that the size of interval T to achieve a given closeness of $V[x(0), x(T)]$ and $V_+[x(0)] - V_-[x(T)]$ will be affected.

This section will be concluded with a discussion of optimization with state constraints. Suppose we still have (3.1) and it is required to minimize a performance index

$$V_c[x(0), x(T)] = \int_0^T [u'u + m(x)]\,dt \quad (4.13)$$

(the subscript c denoting constraint), and at some time $t_1 \in [0, T]$, it is required that $x(t_1)$ lie in some constraint state Ω. Suppose that $0 \ll t_1 \ll T$. It is not normally the case that Hamilton–Jacobi ideas can be applied to constrained state problems. However, they can be brought to bear indirectly by introducing extra approximations. Consider temporarily the problem with the constraint $x(t_1) = x_1$, where $x_1 \in \Omega$. Then clearly this can be regarded as a juxtaposition of two problems:

Find

$$V[x(0), x(t_1)] = \inf_{u(\cdot)} \int_0^{t_1} [u'u + m(x)]\,dt$$

and

$$V[x(t_1), x(T)] = \inf_{u(\cdot)} \int_{t_1}^{T} [u'u + m(x)]\,dt,$$

and the approximate value of $V_c[x(0), x(T)]$ with $x(t_1) = x_1$ will be

$$V_+[x(0)] - V_-[x_1] + V_+[x_1] - V_-[x(T)].$$

Here the approximations explained by Properties 4.1 and 4.2 are being used twice. It follows that the approximate value of $V_c[x(0), x(T)]$ with $x(t_1) \in \Omega$ will be

$$V_+[x(0)] - V_-[x(T)] + \inf_{x_1 \in \Omega} [V_+(x_1) - V_-(x_1)],$$

provided that the set Ω is compact, for otherwise the approximation argument is at risk, as noted in the earlier discussion. Suppose the infimum is achieved at $\bar{x}_1$. Then an approximation to the optimal control will be achieved by piecing together controls optimum for $V_+[x(0)]$, $V_\pm[\bar{x}_1]$, $V_-[x(T)]$, and controls driving small norm states to and from the origin.

5. EXAMPLE

As an illustration of the properties established in the preceding sections and their application to trajectory optimization problems, the scalar nonlinear system

$$\dot{x} = x^3 + u, \quad x(0) = a, \quad x(T) = b, \quad (5.1)$$

and the performance index

$$J = \int_0^T (u^2 + x^2)\,dt, \quad (5.2)$$

are considered, where a, b and T are given. The

Hamiltonian necessary condition for a trajectory of (5.1) minimizing (5.2) is

$$\dot{x} = x^3 - \tfrac{1}{2}\lambda, \qquad x(0) = a$$
$$\dot{\lambda} = -2x - 3x^2\lambda, \quad x(T) = b \tag{5.3}$$

and the optimal control is, in terms of the costate λ,

$$u = -\tfrac{1}{2}\lambda. \tag{5.4}$$

The results of this paper allow the solution of the two point boundary value problem (5.3) to be approximated via the two initial value problems associated with the Hamilton–Jacobi equation

$$\left(\frac{dV}{dx}\right)^2 - 4x^3\left(\frac{dV}{dx}\right) - 4x^2 = 0. \tag{5.5}$$

In terms of dV/dx, the optimal control is

$$u = -\tfrac{1}{2}\frac{dV}{dx}. \tag{5.6}$$

The two roots of the quadratic equation (5.5)

$$\frac{dV}{dx} = 2x^3 \pm 2x\sqrt{1 + x^4} \tag{5.7}$$

yield the positive and the negative solutions of (5.4) as

$$V_+(x) = \tfrac{1}{2}x^2(\sqrt{1 + x^4} + x^2) + \tfrac{1}{2}\ln(\sqrt{1 + x^2} + x^2) \tag{5.8}$$

$$V_-(x) = -\tfrac{1}{2}x^2(\sqrt{1 + x^4} - x^2) + \tfrac{1}{2}\ln(\sqrt{1 + x^4} + x^2). \tag{5.9}$$

While $V_+(x)$ and $V_-(x)$ illustrate the properties discussed in Section 3, only dV_+/dx and dV_-/dx given by (5.7) are needed to solve the forward and reverse initial value problems:

$$\frac{dx}{dt} = x^3 - \tfrac{1}{2}\frac{dV_+}{dx} = -x\sqrt{1 + x^4}, \quad x(0) = a \tag{5.10}$$

$$\frac{dx}{ds} = -x^3 + \tfrac{1}{2}\frac{dV_-}{dx} = -x\sqrt{1 + x^4}, \quad x(0) = b, \quad s = T - t. \tag{5.11}$$

Their solutions, denoted respectively by $x_f(t)$ and $x_r(s)$ are

$$x_f(t) = \frac{ae^{-t}}{\sqrt{1 + y(a)(1 - e^{-4t})}} \tag{5.12}$$

$$x_r(t) = \frac{be^{-s}}{\sqrt{1 + y(b)(1 - e^{-4s})}} \quad s = T - t, \tag{5.13}$$

where $y(z) = \tfrac{1}{2}z^4(1 + \sqrt{1 + z^4})^{-1}$. These solutions,

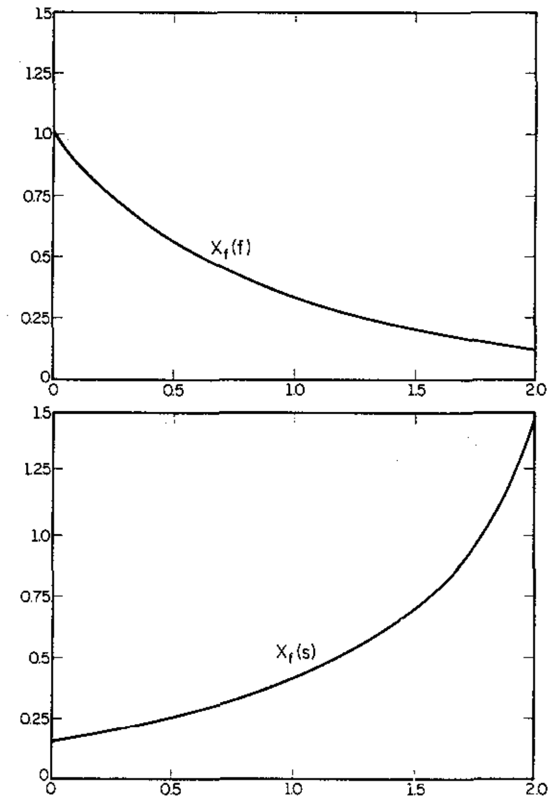

FIG. 3. (a) Solution of (5.10), $a = 1$; (b) solution of (5.11), $b = 1.5$.

shown in Figs 3a and 3b, are as predicted in Figs 2a and 2b. The simplest approximation $\tilde{x}(t)$ of the optimal trajectory would be

$$\tilde{x}(t) = \begin{cases} x_f(t), & 0 \leqslant t \leqslant t_c \\ x_r(T - t), & t_c < t \leqslant T, \end{cases} \tag{5.14}$$

where t_c is defined by $x_f(t_c) = x_r(T - t_c)$. For $T = 2$, the result is shown in Fig. 4 and compared with the exact optimal trajectory. Although $T = 2$ is not "sufficiently large" as required by the theory, the approximation is good and the maximum error near the centre of the interval is about 0.2. For larger T this error is much smaller.

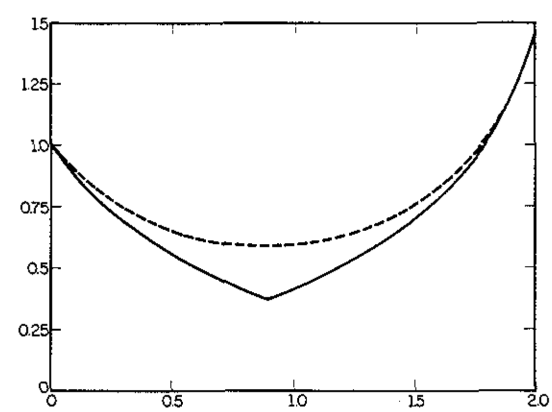
FIG. 4. Approximation (solid) and optimal (dotted) solutions of (5.1) and (5.2). $a = 1$, $b = 1.5$, $T = 2$.

6. CONCLUSION

There are several questions which the main ideas of this paper raise. First, will the results extend to problems where the Hamilton–Jacobi theory cannot be used, but the maximum principle still yields a two point boundary value problem? Second, do the results extend readily to tracking problems? In this connection, see Asseo (1969). Third, will the results allow simplificiation of the task of actually computing an optimal control in applications problems? The answer the authors suggest to each of these questions is yes, although the method of proof for the first question is far from clear; the real task is to specify a broad enough class of problems for which this is indeed the answer that one can regard the conclusions as reasonably general. Another problem is to relate the length of interval to approximation accuracy. Even in the linear-quadratic case, this is not a simple matter computationally, unless one is prepared to assume knowledge of the steady-state problem solution.

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APPENDIX: PROOF OF LEMMA A

Observe that for $a \leqslant t \leqslant b$,

$$x_1(t) - x_0(t) = \int_a^t (f[x_1(s)] - f[x_0(s)])\,ds + \int_a^t (G[x_1(s)] - G[x_0(s)])u(s)\,ds + \int_a^t G[x_0(s)]u(s)\,ds$$

$$\|x_1(t) - x_0(t)\| \leqslant \int_a^t k\|x_1(s) - x_0(s)\|[1 + \|u(s)\|]\,ds + l\int_a^t \|u(s)\|\,ds,$$

with k, l positive constants with existence guaranteed by the global Lipschitz condition and boundedness on finite intervals of $x_0(\cdot)$. In turn, using the L_2 bound on $u(\cdot)$,

$$\|x_1(t) - x_0(t)\| \leqslant \int_a^t k\|x_1(s) - x_0(s)\|[l + \|u(s)\|]\,ds + l(b-a)^{1/2}\varepsilon.$$

By the Bellman–Gronwall lemma, on $a \leqslant t \leqslant b$ there holds

$$\|x_1(t) - x_0(t)\| \leqslant l(b-a)^{1/2}\varepsilon \exp\left[\int_a^t k[1 + \|u(s)\|]\,ds\right]$$

$$\leqslant l(b-a)^{1/2}\varepsilon \exp[+\mathrm{k}(\mathrm{b}-\mathrm{a})^{1/2}\varepsilon].$$