polynomials of $f'(s, q)$

$$f'_1(s, q)=s^3+2s^2+8s+4;$$

$$f'_2(s, q)=s^3+2s^2+3s+2$$

$$f'_3(s, q)=s^3+2s^2+8s+2;$$

$$f'_4(s, q)=s^3+2s^2+3s+4$$

are Hurwitz, hence, $f(s, q)$ is Hurwitz invariant.

Appendix A

Proof of Theorem 3.1

Necessity is trivially proved by letting $k_1(q) = k_2(q) = 1$. *(Sufficiency):* When n is even, let

$$h(s^2, q)=\alpha_0(q)s^n+\alpha_2(q)s^{n-2}+\alpha_4(q)s^{n-4}+\cdots+\alpha_n(q);$$

$$sg(s^2, q)=\alpha_1(q)s^{n-1}+\alpha_3(q)s^{n-3}+\alpha_5(q)s^{n-5}+\cdots+\alpha_{n-1}(q)s.$$

Then

$$f'(s, q)=k_1(q)h(s^2, q)+k_2(q)sg(s^2, q)$$
$$=k_1(q)h(z, q)+sk_2(q)g(z, q).$$

It follows from the Hermite-Bieler theorem [6] that if $f'(s, q)$ is Hurwitz invariant over Q, then $k_1(q)h(z, q)$ and $k_2(q)g(z, q)$ form a positive pair for every $q \in Q$; i.e., the zeros of the polynomials $k_1(q)h(z, q)$ and $k_2(q)g(z, q)$ must be distinct, real, negative, and interlaced as follows:

$$u_1<v_1<u_2<v_2<\cdots<v_{m-1}<u_m<0$$

where $m = n/2$, u_i are zeros of $k_1(q)h(z, q)$ and v_i are zeros of $k_2(q)g(z, q)$. However, $k_1(q)h(z, q)$ and $h(z, q)$ have the same zeros, $k_2(q)g(z, q)$ and $g(z, q)$ have the same zeros. Hence, $h(z, q)$ and $g(z, q)$ also form a positive pair for every $q \in Q$. Thus, the Hermite-Bieler theorem guarantees that

$$f(s, q)=h(z, q)+sg(z, q)$$

is also Hurwitz invariant over Q.

When n is odd, similar proof leads to the same conclusion. □

Appendix B

Proof of Theorem 3.4

By assumption the four bounding polynomials of $f(s, q)$ are Hurwitz, it follows from the Kharitonov theorem that every polynomial

$$f(s)=\alpha_0s^n+\alpha_1s^{n-1}+\cdots+\alpha_n;\ \alpha_i \in [a_i, b_i]$$

is Hurwitz. Since

$$a_i \le a'_i,\ b'_i \le b_i,$$

then, every polynomial

$$f(s)=\alpha_0s^n+\alpha_1s^{n-1}+\cdots+\alpha_n;\ \alpha_i \in [a'_i, b'_i]$$

is also Hurwitz. Consequently, four bounding polynomials of $f'(s, q)$ are Hurwitz. □

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On Robust Hurwitz Polynomials

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Abstract—**In this note, Kharitonov's theorem on robust Hurwitz polynomials is simplified for low-order polynomials. Specifically, for n = 3, 4, and 5, the number of polynomials required to check robust stability is one, two, and three, respectively, instead of four. Furthermore, it is shown that for $n \geqslant 6$, the number of polynomials for robust stability checking is necessarily four, thus further simplification is not possible. The same simplifications arise in robust Schur polynomials by using the bilinear transformation. Applications of these simplifications to two-dimensional polynomials as well as to robustness for single parameters are indicated.**

I. Introduction

Since Kharitonov's results were first published [1], several papers have been devoted to extending and interpreting this work. In particular, Bose [2] advanced a network theoretic proof to Kharitonov's theorem. Bialas and Garloff [3] have extended the theorem to robustness as a function of a single parameter. Bose and Zeheb [4] have extended the robustness results to Schur polynomials. Bose, Jury and Zeheb have simplified the above results to determining only sufficient robustness conditions. A full discrete-time parallel of Kharitonov's results has not yet been obtained. However, some results in this direction were recently published by Hollot and Bartlett [6]. As further background results, it is pertinent to mention the works of Guiver and Bose [7], and Barmish [8]. The above indicates the importance of Kharitonov's theorem in its particular applications which undoubtedly will motivate further research in the future.

The contents of this paper are devoted to Kharitonov's theorem where it is shown that for low-order polynomials some simplifications arise. Specifically, instead of having to test four polynomials for stability, in order to deduce robust stability, results show that for robust stability checking of polynomials of degree n = 3, 4, 5, one requires the testing of only one, two, or three polynomials, respectively. Furthermore, for $n \geqslant 6$, one normally requires the testing of all the four polynomials, as shown by Kharitonov, thus further simplification is not possible. We demonstrate the latter result by invoking the necessary and differing sufficiency conditions for stability of Lipatov and Sokolov [9].

By using the bilinear transformation as discussed by Bose and Zeheb

Manuscript received July 1, 1986; revised April 13, 1987. Paper recommended by Associate Editor, R. Sivan.

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IEEE Log Number 8716154.

0018-9286/87/1000-0909$01.00

[4] and Bose, Jury, and Zeheb [5], such simplification is also translated to polynomials associated with discrete-time systems. In many applications, low-order polynomials often arise, so that the indicated simplification plays an important role.

II. Simplifications for $n = 3, 4$

Before presenting a simplification for Kharitonov's theorem, we first recall the theorem, as follows.

Kharitonov's Theorem: The polynomials

$$F(s)=\sum_{k=0}^{n} a_k s^{n-k}$$

$$a_k \in [u_k, v_k] \tag{1}$$

where the real coefficients a_k take any arbitrary value in the closed interval $[u_k, v_k]$ are strictly Hurwitz *if and only if* the following four polynomials are strictly Hurwitz:

$$F_1(s)=v_0+u_1s^{n-1}+u_2s^{n-2}+v_3s^{n-3}+\cdots \tag{3}$$

$$F_2(s)=v_0+v_1s^{n-1}+u_2s^{n-2}+u_3s^{n-3}+\cdots \tag{4}$$

$$F_3(s)=u_0+u_1s^{n-1}+v_2s^{n-2}+v_3s^{n-3}+\cdots \tag{5}$$

$$F_4(s)=u_0+v_1s^{n-1}+v_2s^{n-2}+u_3s^{n-3}+\cdots. \tag{6}$$

We will indicate the simplification for monic polynomials; however, for nonmonic polynomials the same simplification holds. This was mentioned by Kharitonov [1] and proved by Bose [2]. For $n = 2$, it is trivial to show that the robust stability condition is $u_1, u_2 > 0$. For $n = 3$, we have

$$F(s)=s^3+a_1s^2+a_2s+a_3, \qquad a_i>0. \tag{7}$$

For robustness

$$a_1 \in [u_1, v_1],\ a_2 \in [u_2, v_2] \cdot a_3 \in [u_3, v_3]. \tag{8}$$

We assume that $u_i > 0$ for all i. Then a necessary and sufficient condition for (7) to be Hurwitz, given positivity of the a_i, is

$$a_1a_2-a_3>0. \tag{9}$$

Clearly, the above condition is satisfied for all coefficient values defined by (8) *if and only if* the following single polynomial is strictly Hurwitz:

$$p_1(s)=s^3+u_1s^2+u_2s+v_3. \tag{10}$$

For $n = 4$ we have:

$$F(s)=s^4+a_1s^3+a_2s^2+a_1s+a_4, \qquad a_i \in [u_i, v_i],\ a_i>0. \tag{11}$$

We assume that $u_i > 0$ for all i. In this case, we invoke the Lienard–Chipart theorem [10], [11], and we obtain the following necessary and sufficient stability condition:

$$a_1a_2a_3-a_1^2a_4-a_3^2>0. \tag{12}$$

(Of course, (12) is only necessary and sufficient given the *a priori* assumption that $a_i > 0$ for all i.)

Now the four Kharitonov polynomials are

$$p_1(s)=s_4+u_1s^3+u_2s^2+v_3s+v_4 \tag{13}$$

$$p_2(s)=s_4+v_1s^3+u_2s^2+u_3s+v_4 \tag{14}$$

$$p_3(s)=s_4+u_1s^3+v_2s^2+v_3s+u_4 \tag{15}$$

$$p_4(s)=s_4+v_1s^3+v_2s^2+u_3s+u_4. \tag{16}$$

Now it is straightforward to see that if we satisfy the pertinent Hurwitz condition for $p_1(s)$, then the related condition for $p_3(s)$ is also satisfied, that is, if:

$$u_1u_2v_3-u_1^2v_4-v_3^2>0 \tag{17}$$

then

$$u_1v_2v_3-u_1^2u_4-v_3^2>0 \tag{18}$$

is also satisfied. Hence, the polynomial $p_3(s)$ is redundant for the purposes of checking robust stability. Similarly, if the pertinent Hurwitz condition for $p_2(s)$ is satisfied, then the related condition for $p_4(s)$ is satisfied. Thus, the polynomial $p_4(s)$ is redundant. Hence, the two pertinent polynomials are $p_1(s)$ and $p_2(s)$ in (13) and (14).

Remark 1: In an earlier and different approach by Guiver and Bose [7], it is indicated that for a quartic equation under coefficient perturbations, one requires the stability of four polynomials. Apart from coefficient positivity in these polynomials, conditions of the form of (17) for each polynomial are needed. Based on the above results, we see that only the stability conditions for two polynomials are needed (apart from coefficient positivity). Specifically, the positivity of the forms in [7, (5b) and (5c)] are the pertinent ones.

III. Simplification for $n = 5$

For the case $n = 5$, we will show that one needs to check for stability only *three* of the Kharitonov polynomials. The proof of this fact is based on the Hermite–Bieler theorem [11] which is used in network theory.

Theorem: If

$$0<u_i \leqslant v_i, \qquad i=1, \cdots, 5 \tag{19}$$

then the stability of

$$s^5+u_1s^4+u_2s^3+v_3s^2+v_4s+u_5 \tag{20}$$

$$s^5+v_1s^4+u_2s^3+u_3s^2+v_4s+v_5 \tag{21}$$

$$s^5+v_1s^4+v_2s^3+u_3s^2+u_4s+v_5 \tag{22}$$

implies the stability of

$$s^5+u_1s^4+v_2s^3+v_3s^2+u_4s+u_5. \tag{23}$$

By virtue of results in [11], the theorem can be reformulated in terms of properties of the following four transfer functions:

$$Z_1(\lambda)=\frac{\lambda^2+u_2\lambda+v_4}{u_1\lambda^2+v_3\lambda+u_5},\quad Z_2(\lambda)=\frac{\lambda^2+u_2\lambda+v_4}{v_1\lambda^2+u_3\lambda+v_5} \tag{24}$$

$$Z_3(\lambda)=\frac{\lambda^2+v_2\lambda+u_4}{v_1\lambda^2+u_3\lambda+v_5},\quad Z_4(\lambda)=\frac{\lambda^2+v_2\lambda+u_4}{u_1\lambda^2+v_3\lambda+u_5} \tag{25}$$

as follows.

Suppose that the poles and zeros of $Z_1(\lambda)$, $Z_2(\lambda)$, $Z_3(\lambda)$ are all real and negative, and possess the interlacing property with a pole closest to the origin; then the poles and zeros of $Z_4(\lambda)$ (which are clearly negative) possess the same interlacing property.

Remark 2: It is known, see [11], that the above properties of $Z_1(\lambda)$, $Z_2(\lambda)$, and $Z_3(\lambda)$ are equivalent to the Hurwitz properties of the three polynomials in (20)–(22).

The proof is based on the following two observations:

Observation 1: Consider the two polynomials

$$\lambda^2+\alpha\lambda+\beta \tag{26}$$

and

$$\lambda^2+\gamma\lambda+\delta. \tag{27}$$

Then there exist positive u_1, v_1, u_3, v_3, u_5, and v_5 with

$$0<u_i \leqslant v_i \tag{28}$$

such that

$$\lambda^2+\alpha\lambda+\beta=u_1^{-1}[u_1\lambda^2+v_3\lambda+u_5] \quad (29)$$

$$\lambda^2+\gamma\lambda+\delta=v_1^{-1}[v_1\lambda^2+u_3\lambda+v_5] \quad (30)$$

if and only if α, β, γ, δ are all positive with

$$\alpha \geqslant \gamma \quad (31)$$

and

$$\alpha\delta \geqslant \beta\gamma. \quad (32)$$

Proof: Assume (28), (29), and (30). Positivity of α, β, γ, and δ is immediate. Also,

$$\alpha=\frac{v_3}{u_1} \geqslant \frac{u_3}{v_1}=\gamma \quad (33)$$

$$\alpha\delta=\frac{v_3}{u_1}\frac{v_5}{v_1} \geqslant \frac{u_3u_5}{u_1v_1}=\left(\frac{u_5}{u_1}\right)\left(\frac{u_3}{v_1}\right)=\beta\gamma. \quad (34)$$

Conversely, assume (31) and (32), and positivity of α, β, γ, and δ. Select $v_1 = 1$, and determine u_3, v_5 by (30).

Select u_1 such that

$$\frac{\gamma}{\alpha} \leqslant u_1 \leqslant \min(1, \delta/\beta) \quad (35)$$

(inequalities (31) and (32) ensure this is possible). This determines v_3, u_5, by (29) and we must prove $u_1 \leqslant v_1$, $u_3 \leqslant v_3$, $u_5 \leqslant v_5$. From (35) and $v_1 = 1$, we readily have $u_1 \leqslant v_1$. Also, from (35)

$$\frac{u_1}{v_1} \geqslant \frac{\gamma}{\alpha}=\frac{u_3/v_1}{v_3/u_1}=\left(\frac{u_3}{v_3}\right)\left(\frac{u_1}{v_1}\right) \quad (36)$$

hence

$$1 \geqslant \frac{u_3}{v_3}. \quad (37)$$

Finally, from (35)

$$\frac{u_1}{v_1} \leqslant \frac{\delta}{\beta}=\frac{v_5/v_1}{u_5/u_1}=\left(\frac{v_5}{u_5}\right)\left(\frac{u_1}{v_1}\right) \quad (38)$$

hence

$$1 \leqslant \frac{v_5}{u_5}. \quad (39)$$

Remark 3: The main point of observation 1 is that it simplifies the comparison of the pole positions of $Z_1(\lambda)$ with those of $Z_2(\lambda)$, $Z_3(\lambda)$, in that we can study just monic polynomials.

Observation 2: Consider two polynomials

$$\lambda^2+\alpha\lambda+\beta \quad (40)$$

and

$$\lambda^2+\gamma\lambda+\delta \quad (41)$$

for α, β, γ, δ all positive. Assume both polynomials have distinct negative real zeros $-x_1$, $-x_2$ and $-x_1 - \epsilon_1$, $-x_2 + \epsilon_2$, where without loss of generality,

$$0<x_1<x_2 \quad (42)$$

$$0<x_1+\epsilon_1<x_2-\epsilon_2. \quad (43)$$

Then

$$\alpha \geqslant \gamma,\ \alpha\delta \geqslant \beta\gamma \quad (44)$$

imply

$$0 \leqslant \epsilon_1 \leqslant \epsilon_2. \quad (45)$$

Remark 4: The conclusion of the above observation is that in passing from $\lambda^2 + \alpha\lambda + \beta$ to $\lambda^2 + \gamma\lambda + \delta$ under the constraint (44), the right-most zero moves *left*, and the left-most zero moves *right* by a larger amount. This will allow us to study the relation between the zeros of $u_1\lambda^2 + u_3\lambda + v_5$ and $v_1\lambda^2 + u_3\lambda + v_5$.

Remark 5: We could have chosen the zeros of $\lambda^2 + \gamma\lambda + \delta$ to be $-x_1 + \epsilon_1$ and $-x_2 - \epsilon_2$, with

$$0<x_1+\epsilon_1<x_2+\epsilon_2. \quad (46)$$

Proof: Observe that

$$\alpha=x_1+x_2,\ \beta=x_1x_2,\ \gamma=x_1+x_2+\epsilon_1-\epsilon_2 \quad (47)$$

$$\delta=(x_1+\epsilon_1)(x_2-\epsilon_2). \quad (48)$$

One can check immediately that $\alpha \geqslant \gamma$ implies $\epsilon_2 \geqslant \epsilon_1$. Also, simple calculation gives

$$\alpha\delta-\beta\gamma=\epsilon_1x_2(x_2-\epsilon_2)-\epsilon_2x_1(x_1+\epsilon_1). \quad (49)$$

From (42) and (43), we see that

$$x_1(x_1+\epsilon_1)<x_2(x_2-\epsilon_2). \quad (50)$$

If $\epsilon_1 < 0$, then

$$\alpha\delta-\beta\gamma<\epsilon_1x_1(x_1+\epsilon_1)-\epsilon_2x_1(x_1+\epsilon_1)=(\epsilon_1-\epsilon_2)x_1(x_1+\epsilon_1) \leqslant 0 \quad (51)$$

which is a contradiction. Hence, $\epsilon_1 \geqslant 0$.

With these observations in hand, we turn now to the theorem.

Proof of Theorem: Compare first the zeros of $Z_1(\lambda)$ and $Z_2(\lambda)$ with those of $Z_3(\lambda)$, i.e., the zeros of $\lambda^2 + u_2\lambda + v_4$ and $\lambda^2 + v_2\lambda + u_4$. Notice that

$$v_2 \geqslant u_2 \quad (52)$$

$$v_2v_4 \geqslant u_2u_4. \quad (53)$$

Hence, by Observation 2, the zeros of $Z_3(\lambda)$ lie outside the zeros of $Z_1(\lambda)$, $Z_2(\lambda)$ (see Fig. 1). Next, compare the poles of $Z_1(\lambda)$ with those of $Z_2(\lambda)$ and $Z_3(\lambda)$. These are given by the zeros of $u_1\lambda^2 + v_3\lambda + u_5$ and of $v_1\lambda^2 + u_3\lambda + v_5$. By combining Observations 1 and 2, we see that the poles of $Z_2(\lambda)$, $Z_3(\lambda)$ lie inside the poles of $Z_1(\lambda)$, see Fig. 1 again.

Fig. 1 also reflects the interlacing property of the poles and zeros of $Z_1(\lambda)$, $Z_2(\lambda)$, and $Z_3(\lambda)$. Note now that the zeros and poles of $Z_4(\lambda)$ are the zeros of Z_3 and poles of Z_1, respectively. It follows, as depicted in the figure, that these zeros and poles acquire the interlacing property. This proves the theorem.

Utilizing the above approach, one can readily show, as indicated earlier with a different argument, that for $n = 4$, only two Kharitonov polynomials are required.

Remark 6: Based on the works of Bose and Zeheb [4], the simplification for low-order polynomials can be also translated into discrete-time polynomials, using the bilinear transformation. However, for the discrete case only sufficient conditions are thereby obtained. Also, from this work simplified sufficient conditions for two-dimensional systems can be obtained.

Remark 7: Based on the works of Bose, Jury, and Zeheb [5], the simplification for low-order polynomials can be translated into the problem of polynomial robustness as a function of a single parameter.

IV. Requirement for All Four Kharitonov Polynomials When $n \geqslant 6$

With $u_i < v_i$, consider the four polynomials

$$P_1(s)=s^6+u_1s^5+u_2s^4+v_3s^3+v_4s^2+u_5s+u_6 \quad (54)$$

$$P_2(s)=s^6+v_1s^5+u_2s^4+u_3s^3+v_4s^2+v_5s+u_6 \quad (55)$$

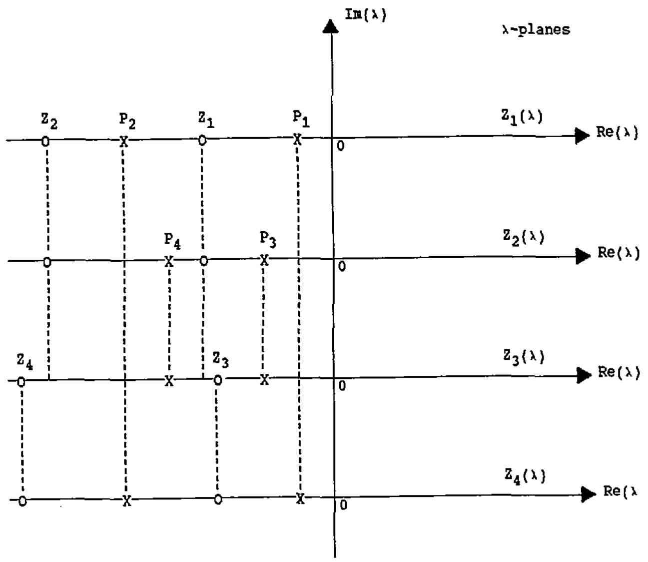

Fig. 1. Pole-zero patterns of $Z_1(\lambda)$, $Z_2(\lambda)$, $Z_3(\lambda)$, and consequently of $Z_4(\lambda)$.

$$P_3(s)=s^6+v_1s^5+v_2s^4+u_3s^3+u_4s^2+v_5s+v_6 \tag{56}$$

$$P_4(s)=s^6+u_1s^5+v_2s^4+v_3s^3+u_4s^2+u_5s+v_6. \tag{57}$$

We aim to show that it is always possible to choose the u_i, v_i with $0 < u_i < v_i$ such that any three of these polynomials are stable, while the fourth is unstable.

To do this, we shall make use of the following result of Lipatov and Sokolov. Consider the polynomial

$$P(s)=s^6+a_1s^5+a_2s^4+\cdots+a_6 \tag{58}$$

and define

$$\lambda_1=\frac{a_3}{a_1a_2},\ \lambda_2=\frac{a_1a_4}{a_2a_3},\ \lambda_3=\frac{a_2a_5}{a_3a_4},\ \lambda_4=\frac{a_3a_6}{a_4a_5}. \tag{59}$$

Then $\lambda_i > 1$ for some i is a sufficient condition for instability, and $\lambda_i \leqslant 0.4$ for all i is a sufficient condition for stability.

We shall show now how to choose u_i, v_i so that P_j is unstable, while the other polynomials are stable.

There are always two successive coefficients of $P_j(s)$ for any j of the form u_{i+1}, v_{i+2}. Observe then that the associated λ_i, call it $\lambda_{i,j}$ (the second index referring to the polynomial), satisfy

$$\lambda_{i,j}=\frac{v_{i-1}}{u_i}\frac{v_{i+2}}{u_{i+1}}>\lambda_{i,k} \quad \text{for } k\neq j. \tag{60}$$

Thus, if we choose $j = 2$, considering $P_2(s)$, we see that u_3, v_4 are among the coefficients and

$$\lambda_{2,2}=\frac{v_1v_4}{u_2u_3}>\lambda_{2,1},\ \lambda_{2,3} \text{ and } \lambda_{2,4}. \tag{61}$$

We shall now explain how to choose all u_i, v_i so that

$$\lambda_{2,2}>1 \tag{62}$$

but otherwise

$$\lambda_{i,j}<0.4. \tag{63}$$

In the process, we shall ensure that

$$u_1=v_1,\ u_2=v_2,\ u_3\neq v_3,\ u_4\neq v_4,\ u_5=v_5,\ u_6=v_6. \tag{64}$$

Choose

$$u_1=v_1=u_2=v_2=1. \tag{65}$$

Choose

$$v_3=0.4. \tag{66}$$

Notice that

$$\lambda_{1,1}=\frac{v_3}{u_1u_2}=0.4 \tag{67}$$

$$\lambda_{1,2}=\frac{u_3}{v_1u_2}\leqslant 0.4 \tag{68}$$

$$\lambda_{1,3}=\lambda_{1,2}\leqslant 0.4 \tag{69}$$

$$\lambda_{1,4}=0.4. \tag{70}$$

Now observe that

$$\lambda_{2,1}=\frac{u_1v_4}{u_2v_3}=\frac{v_4}{0.4} \tag{71}$$

$$\lambda_{2,2}=\frac{v_1v_4}{u_2u_3}=\frac{v_4}{u_3} \tag{72}$$

$$\lambda_{2,3}=\frac{v_1u_4}{v_2u_3}=\frac{u_4}{u_3} \tag{73}$$

$$\lambda_{2,4}=\frac{u_1u_4}{v_2v_3}=\frac{u_4}{v_3}. \tag{74}$$

Now choose

$$v_4=(0.4)^2 \tag{75}$$

$$u_3=\frac{1}{2}(0.4)^2<0.4=v_3 \tag{76}$$

$$u_4=0.4u_3. \tag{77}$$

Then

$$\lambda_{2,1}=0.4,\ \lambda_{2,2}>1,\ \lambda_{2,3}=0.4, \tag{78}$$

$$\lambda_{2,4}<0.4. \tag{79}$$

Choose

$$u_5=v_5<0.4\,\frac{u_3u_4}{v_2} \tag{80}$$

$$u_6=v_6<0.4\,\frac{u_4u_5}{v_3}. \tag{81}$$

In conjunction with (60), these choices ensure that

$$\lambda_{3,j}<0.4,\ \lambda_{4,j}<0.4\ \forall j. \tag{82}$$

It is clear that this argument extends to cover polynomials of arbitrary order. Hence, the four Kharitonov polynomials are required for $n \geqslant 6$.

V. Conclusion

In this note it is shown that Kharitonov's theorem for robust Hurwitz polynomials can be simplified for low-order polynomials. Specifically, for $n = 3$, 4, and 5, the polynomials needed are one, two, and three instead of four, given positivity of all coefficients. Also, for $n \geqslant 6$, the number of polynomials required in general is four. The proof for $n = 3, 4$ is algebraic, while for $n = 5$ is based on a network theoretic approach. Also, based on certain necessary and differing sufficiency conditions of stability as obtained by Lipatov and Sokolov, we showed that no further simplification of Kharitonov's results is possible for $n \geqslant 6$. With these new results, we clarified and extended the results of Kharitonov into a useful conclusion.

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Frequency Domain Conditions for the Stability of Perturbed Polynomials

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Abstract—**New necessary and sufficient conditions for the stability of perturbed polynomials of continuous systems are given in the frequency domain. The conditions are equivalent and in some respects more powerful than the well-known Kharitonov conditions. The new conditions allow considerable freedom in distributing the available uncertainty margin among the different coefficients of a polynomial and provide an indication as to whether the maximum allowable margin of uncertainty for a given polynomial has been reached.**

I. Introduction

A fundamental problem in robustness of linear systems is to find upper bounds on the allowable perturbations in the coefficients of the characteristic polynomial of a linear system such that the polynomial remains Hurwitz. The interest in this problem has greatly increased since the publication of Kharitonov's celebrated conditions [1] in 1978. These conditions assert that the perturbed polynomial will remain Hurwitz if and only if four other extreme polynomials of the same order are also Hurwitz. While a certain set of perturbation bounds can be checked for stability using Kharitonov's test, there is no indication as to what extent those bounds can be increased before the polynomial becomes unstable. In order to find the maximum allowable perturbation bounds, Barmish [2] and Bialas and Garloff [3] formulated the bounds in terms of one parameter, thus restricting the "structure" of the admissible region of perturbations.

Using a Nyquist-type argument Yeung [4] developed a sufficient condition for the perturbed polynomial to be stable. Other conditions based on the "principle of the argument" were also recently obtained by Argoun [5], [6].

Manuscript received November 13, 1986; revised February 4, 1987.
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IEEE Log Number 8716155.

The main result of this note is given by Theorem 3 which can be summarized as follows. The perturbed polynomial with known upper bounds on the coefficient perturbations will remain Hurwitz if and only if two polynomials of order $n/2$ whose coefficients are the perturbed odd and even coefficients of the original polynomial have no roots in common. This condition amounts to the necessary and sufficient condition that certain real frequency bands containing the roots of the above two polynomials do not overlap. It is shown that the frequency bands related to the even and odd coefficient perturbed polynomials alternate and the distance between successive bands is an indication of the margin of stability for the perturbed polynomial. Theorem 2 of this note gives another version of the necessary and sufficient condition which allows the designer maximum freedom in distributing the available error margin among the polynomial coefficients. The conditions are illustrated by a numerical example.

II. Formulation of the Problem

Consider the nominal Hurwitz polynomial $P_0(s)$ given by

$$P_0(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0. \tag{1}$$

Let the coefficient $a_0, a_1, \cdots, a_n$ be subject to unknown but bounded perturbations δa_i with known upper bounds Δa_i, i.e.,

$$|\delta a_i| \le \Delta a_i, \qquad i = 0, 1, \cdots, n. \tag{2}$$

We are interested in finding conditions under which the perturbed polynomial $P(s)$ with coefficients $\tilde{a}_i = a_i + \delta a_i$, $i = 0, \cdots, n$ remains Hurwitz. Substituting $s = j\omega$, the polynomial $P_0(j\omega)$ can be written as

$$P_0(j\omega) = R_0(\omega) + j\omega Q_0(\omega) \tag{3}$$

where

$$R_0(\omega) = a_0 - a_2\omega^2 + a_4\omega^4 + \cdots + (-1)^{m/2} a_m \omega^m$$

$$Q_0(\omega) = a_1 - a_3\omega^2 + a_5\omega^4 + \cdots + (-1)^{(l-1)/2} a_l \omega^{(l-1)}$$

$$\cdot \begin{cases} m = n,\ l = n-1,\ n \text{ even} \\ l = n,\ m = n-1,\ n \text{ odd}, \end{cases} \tag{4}$$

i.e., $R_0(\omega)$ contains the even-power terms of $P_0(j\omega)$ and $Q_0(\omega)$ contains the odd-power ones. Similar to (4), let $R(\omega)$ and $Q(\omega)$ represent the perturbed even-power and odd-power polynomials, respectively, with $\tilde{a}_i$ replacing a_i in (4). $R(\omega)$ and $Q(\omega)$ will in general be given by

$$R(\omega) = R_0(\omega) + \delta R(\omega) \tag{5}$$

$$Q(\omega) = Q_0(\omega) + \delta Q(\omega) \tag{6}$$

where $\delta R(\omega)$ and $\delta Q(\omega)$ are some perturbation polynomials given by

$$\delta R(\omega) = \delta a_0 + \delta a_2 \omega^2 + \cdots + \delta a_n \omega^m \tag{7}$$

$$\delta Q(\omega) = \delta a_1 + \delta a_3 \omega^2 + \cdots + \delta a_l \omega^{l-1} \tag{8}$$

where the coefficient perturbations $\delta a_0, \cdots, \delta a_n$ satisfy (2). Note that δ_{ai}, $i = 0, \cdots, n$ can be either positive or negative.

Consider now the four extreme polynomials

$$R_1(\omega) = R_0(\omega) + \Delta R(\omega) \tag{9}$$

$$R_2(\omega) = R_0(\omega) - \Delta R(\omega) \tag{10}$$

and

$$Q_1(\omega) = Q_0(\omega) + \Delta Q(\omega) \tag{11}$$

$$Q_2(\omega) = Q_0(\omega) - \Delta Q(\omega) \tag{12}$$

