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Physically Based Parameterizations for Designing Adaptive Algorithms*

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Special state variable and transfer function descriptions are developed for systems whose unknown parameters are a wide class of physical element values. The dependence of the state variable structures on these parameters is "rank-1" and that of the numerator and denominator polynomials of the transfer functions multilinear.

Key Words—Adaptive algorithms; physical paramters; rank-1; multilinear.

Abstract—Structural properties are examined of systems with physical component values as parameters. Both state variable realizations and transfer function descriptions are investigated. The transfer functions in particular are shown to be the ratios of polynomials with coefficients multilinear in the parameters. These structures prove useful in formulating adaptive algorithms.

1. INTRODUCTION

A FACTOR critical to the performance of adaptive algorithms is their underlying parameterizations. Parameterizations reflecting greater *a priori* knowledge about the unknown system can be expected to lead to improved performance. Frequently, adaptive systems (Lion, 1967; Narendra and Kudva, 1974; Lüders and Narendra, 1974; Kreisselmeier, 1977; Anderson, 1977) treat all the transfer function coefficients as the unknown parameters. In the process considerable information may be discarded. Sometimes, the only unknowns in a system are the values of certain physical elements or parameters. As far as transfer functions are concerned, this may mean that some of the coefficients are known *a priori*, as also are some relationships, possibly nonlinear, which exist between them. Accordingly, a parameterization is developed here involving unknown parameters which have direct physical relevance. Such a parameterization has been the basis of adaptive algorithms formulated by Dasgupta *et al.* (1983, 1984, 1986a, b).

Section 2 specifies the parameterization in question and demonstrates how RLC circuits fall within its ambit. Section 3 investigates the corresponding state variable realizations while Section 4 shows that the resulting minimal transfer functions are the ratios of polynomials having coefficients multilinear in the parameters. For example, for a system with two unkown parameters, k_1 and k_2, the transfer function could be

$$W(s, k_1, k_2) = \frac{p_0(s) + k_1p_1(s) + k_2p_2(s) + k_1k_2p_{12}(s)}{q_0(s) + k_1q_1(s) + k_2q_2(s) + k_1k_2q_{12}(s)} \quad (1.1)$$

where the polynomials $p_{(.)}(s)$ and $q_{(.)}(s)$ are known. Section 5 derives conditions under which systems with transfer functions such as (1) conform to the parameterization under study. All results in the last two sections apply only to single-input, single-output systems.

2. THE PARAMETERIZATION

Much of the background material for this section is contained in Anderson and Vongpanitlerd (1973, pp. 156–200). To understand how physical element values affect a wide class of systems, our attention will be restricted to electric circuits containing resistors, inductors and capacitors. The extensions to the corresponding chemical and mechanical analogues will of course be immediate.

Consider a resistor R appearing in an n-port circuit. Clearly the resistor can be extracted from the rest of the circuit in a manner depicted in Fig. 1.

Suppose that u_1, y_1 are the port voltage and current (or current and voltage) at the right hand port of the "circuit without the resistor", and U, Y are the input and output vectors of the terminated

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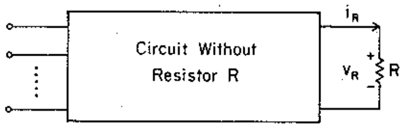

FIG. 1. Representation of a circuit with a resistor R.

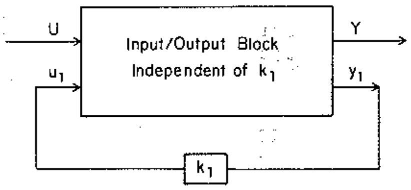

FIG. 2. Representation of a circuit with an unknown parameter k_1.

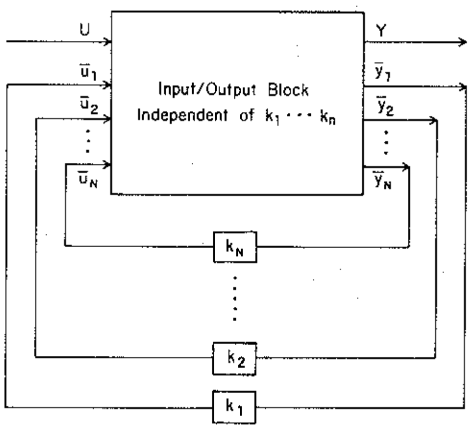

FIG. 3 Representation of systems satisfying Assumption 2.1.

network, with entries of U, Y corresponding to voltages or currents at the left port. If a port voltage appears in one of U, Y the port current must appear in the other. With k_1 then identified with R or R^{-1} depending whether u_1 is a voltage or current, redescription via Fig. 2 is possible. Of course, for this description to make sense, the hybrid matrix relating $[U^T u_1]^T$ to $[Y^T y_1]^T$ should exist.

More generally, when a circuit has N unknown physical components, with values $k_1, \ldots, k_N$, then in many cases an input–output description of the form in Fig. 3 exists. Assumption 2.1 formalizes this description and constitutes the standing assumption for this paper.

Assumption 2.1. Consider a system with an n-dimensional input vector U and m-dimensional output vector Y. Suppose it has N unknown physical components with values $k_1, \ldots, k_N$. Then defining $K \triangleq \text{diag}\{k_1, \ldots, k_N\}$ there exist two N-vectors $y_1(t)$ and $u_1(t)$ and an $(m+N) \times (n+N)$ dimensional transfer function matrix $T(s)$ such that

$$[Y^T(s)\, y_1^T(s)]^T = T(s)[U^T(s)\, u_1^T(s)]^T$$

and

$$u_1(t) = K y_1(t). \qquad \nabla\nabla\nabla$$

Remark 2.1. The elements which cannot be treated in this manner are mutual inductors—they allow cross-coupling between energy storage devices—and gyrators. Theorem 2.1 below shows how RLC circuits conform to this description. Before stating the theorem, a definition is required.

Consider an m port RLC network having m-dimensional input and output vectors $U(\cdot)$ and $Y(\cdot)$, respectively, with $Y(\cdot)$ finite for all finite $U(\cdot)$. Suppose also that each port is represented in the input vector by either but not both of its voltage or current and that if a port voltage (current) appears in the input then the corresponding current (voltage) appears in the output. Suppose N of the network components are $\Omega_1, \ldots, \Omega_N$. Then under these conditions $\Omega_1, \ldots, \Omega_N$ are extractable if the conditions of the following definition are met.

Definition 2.1. Consider the $m + N$ port network obtained by extracting $\Omega_1, \ldots, \Omega_N$. Let $\bar{U}(\cdot)$ and $\bar{Y}(\cdot)$ be N-dimensional vectors having elements $\bar{u}_i(\cdot)$ and $\bar{y}_i(.)$ such that

(i) $\bar{u}_i(\cdot)$ is either the voltage or the current at the port created by the removal of Ω_i;

(ii) if $\bar{u}_i(\cdot)$ is the relevant voltage, then $\bar{y}_i(\cdot)$ is the corresponding current and vice versa.

Then for the given $U(\cdot)$ and $Y(\cdot)$, $\Omega_1, \ldots, \Omega_N$ are extractable if there exist a choice of $\bar{U}$ and $\bar{Y}$ and a finite M such that

$$[Y^T(\cdot), \bar{Y}^T(\cdot)]^T = M[U^T(\cdot), \bar{U}^T(\cdot)]^T. \qquad (2.1)$$

$\nabla\nabla\nabla$

Observe that the definition requires the existence of a hybrid description relating the augmented vectors $[\bar{Y}^T(\cdot), Y^T(\cdot)]$ and $[\bar{U}^T(\cdot), U^T(\cdot)]^T$ to each other. As pointed out in Anderson and Vongpanitlerd (1973, pp. 171–198), the elements of an RLC network for a given input/output set will fail to satisfy the condition for extractability if one of the following hold.

(i) A particular element Ω_i does not affect the input/output relationship, in which case Ω_i can be removed from consideration.

(ii) The problem of finding a state variable description is ill posed. Since the motivation here is to consider parameterizations relevant to adaptive control, such a system is of no interest to us.

Thus, in dealing with RLC circuits, this paper will be restricted to situations where Definition 2.1 applies. Theorem 2.1 will now be stated and the proof given.

Theorem 2.1. Consider an m port LTI lumped RLC circuit with m inputs u_i and m outputs y_i which are the port voltages and currents. Suppose every port is represented in the input vector by either, but not both, of its voltage and current. Also, if a particular voltage appears in the input the corresponding current appears in the output and vice versa. Suppose the N components of the system are $\{\mu_i\}_{i=1}^N$ with component values $\{\mu_i\}_{i=1}^N$. Then a representation of the form of Assumption 2.1 exists with $k_i = \mu_i$ or $1/\mu_i$, if Ω_i are extractable in the sense of Definition 2.1.

Proof. With appropriate definitions and the extractability property there exist M_{ij}, $i = \{1, 2\}$, $j = \{1, 2\}$ such that

$$\begin{bmatrix} Y(s) \\ \bar{Y}(s) \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} U(s) \\ \bar{U}(s) \end{bmatrix}$$

where $\bar{Y}_i(s)$ is either the voltage across Ω_i, or the current through it; if $\bar{Y}_i(s)$ is a voltage then $\bar{U}_i(s)$ is the corresponding current and vice versa. Now the relations between the voltages, and currents across resistors, inductors and capacitors are respectively given by

$$V(s) = RI(s)$$

$$V(s) = sLI(s)$$

$$V(s) = \frac{1}{sc} I(s).$$

Thus, with appropriate ordering of Ω_i and $K \triangleq \text{diag}\, \{k_i\}_{i=1}^N$

$$\bar{U}(s) = \begin{bmatrix} I & 0 & 0 \\ 0 & sI & 0 \\ 0 & 0 & \frac{1}{s} I \end{bmatrix} K \bar{Y}(s).$$

Thus, with

$$U_1(s) = \begin{bmatrix} I & 0 & 0 \\ 0 & \frac{1}{s} I & 0 \\ 0 & 0 & sI \end{bmatrix} \bar{U}(s)$$

and $Y_1(s) = \bar{Y}(s)$ the result follows. ∇∇∇

Remark 2.2. The result trivially extends to mechanical and chemical analogs of RLC circuits. It also holds for a large number of active circuits having controlled sources and their analogs.

3. THE STATE VARIABLE REALIZATION

In this section it will be shown how N-parameter systems satisfying Assumption 2.1 are reflected in their state variable realizations. Before this is done, the following definition will be made.

Definition 3.1. A state variable realization described by the quadruple $\{F, G, H, J\}$ has a *rank-1* dependence on N parameters $k_1, \ldots, k_N$, if $\forall i \in \{1, \ldots, N\}$, there exist $a_i, b_i \in R$, independent of k_i, such that defining α_i as either

$$\alpha_i = \frac{1}{a_i + k_i b_i}$$

or

$$\alpha_i = \frac{k_i}{a_i + k_i b_i}$$

the following is true. There exist, $\forall i \in \{1, \ldots, N\}$, F_i, G_i, H_i, J_i, h_i and g_i all independent of k_i, such that

$$\left[\begin{array}{c|c} sI - F & G \\ \hline -H & J \end{array}\right] = \left[\begin{array}{c|c} sI - F_i & G_i \\ \hline -H_i & J_i \end{array}\right] + \alpha_i h_i g_i^{\mathrm{T}}$$

with g_i, h_i vectors. In other words, one can claim that

$$\frac{\partial}{\partial \alpha_i} \left[\begin{array}{c|c} sI - F & G \\ \hline -H & J \end{array}\right] = h_i g_i^{\mathrm{T}}. \qquad ∇∇∇$$

Observe that if a MIMO system obeys this definition, then so does any SISO subsystem contained by it. The following theorem shows that a system conforming to the requirements of Assumption 2.1 has at least one state variable realization which has a rank-1 dependence on the N parameters.

Theorem 3.1. Consider a system having an input output description satisfying Assumption 2.1. In other words, if u and y are input and output vectors, respectively, there exist N input and outputs, all elements of vectors u_1 and y_1, respectively, such that

$$\begin{bmatrix} y(s) \\ y_1(s) \end{bmatrix} = T(s) \begin{bmatrix} u(s) \\ u_1(s) \end{bmatrix}$$

and

$$y_1 = K^{-1}u_1$$

with $K \triangleq \text{diag}\{k_1,\ldots,k_N\}$. Assume $T(s)$ is proper. Then there exists at least one state variable realization, with input u and output y, which has a rank-1 dependence on the k_i for all but isolated values of k_i.

Proof. Since $T(s)$ is proper, matrices A, B, B_1, C, C_1, D_{11}, D_{12}, D_{21} and D_{22} exist such that

$$\dot{x} = Ax + [B \quad B_1][u^{\mathrm{T}} \quad u_1^{\mathrm{T}}]^{\mathrm{T}}$$

$$[y^{\mathrm{T}} \quad y_1^{\mathrm{T}}]^{\mathrm{T}} = [C^{\mathrm{T}} \quad C_1^{\mathrm{T}}]^{\mathrm{T}}x + \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}[u^{\mathrm{T}} \quad u_1^{\mathrm{T}}]^{\mathrm{T}}. \qquad (3.1)$$

As $u_1 = Ky_1$, a little manipulation shows that

$$\dot{x} = \{A + B_1K[I - D_{22}K]^{-1}C_1\}x + \{B + B_1K[I - D_{22}K]^{-1}D_{21}\}u$$

$$y = \{C + D_{12}K[I - D_{22}K]^{-1}C_1\}x + \{D_{11} + D_{12}K[I - D_{22}K]^{-1}D_{21}\}u.$$

Thus

$$\left[\begin{array}{c|c} sI - F & G \\ \hline -H & J \end{array}\right] = \left[\begin{array}{c|c} sI - A & B \\ \hline -C & D_{11} \end{array}\right] + \begin{bmatrix} B_1 \\ D_{12} \end{bmatrix} K[I - D_{22}K]^{-1}[C_1 \quad D_{21}]. \qquad (3.2)$$

Denote $K[I - D_{22}K]^{-1}$ by Γ. Suppose

$$A_i = I - D_{22}K + k_iD_{22}e_ie_i^{\mathrm{T}} \qquad (3.3)$$

where e_i is the unit vector with unity in the ith element and zero elsewhere. The matrix A_i is independent of k_i. Observe that matrices like Γ and A_i are invertible for all but isolated values of the k_i. Our analysis shall exclude such points in the k space. Also,

$$\Gamma = K\left[A_i^{-1} + k_i\frac{A_i^{-1}D_{22}e_ie_i^{\mathrm{T}}A_i^{-1}}{1 - k_ie_i^{\mathrm{T}}A_i^{-1}D_{22}e_i}\right].$$

Consider the jth row of Γ,

$$e_j^{\mathrm{T}}\Gamma = k_je_j^{\mathrm{T}}\left[A_i^{-1} + k_i\frac{A_i^{-1}D_{22}e_ie_i^{\mathrm{T}}A_i^{-1}}{1 - k_ie_i^{\mathrm{T}}A_i^{-1}D_{22}e_i}\right].$$

If $i \neq j$ then

$$e_j^{\mathrm{T}}\Gamma = k_je_j^{\mathrm{T}}A_i^{-1} + \frac{k_ik_j}{1 - k_ie_i^{\mathrm{T}}A_i^{-1}D_{22}e_i}e_j^{\mathrm{T}}A_i^{-1}D_{22}e_ie_i^{\mathrm{T}}A_i^{-1}$$

whence, with $\alpha_i = k_i/(1 - k_ie_i^{\mathrm{T}}A_i^{-1}D_{22}e_i)$,

$$e_j^{\mathrm{T}}\Gamma = v_{ji}^{\mathrm{T}} + \alpha_iw_{ji}e_i^{\mathrm{T}}A_i^{-1}$$

with $v_{ji}, w_{ji}, (e_i^{\mathrm{T}}A_i^{-1})$ independent of k_i and v_{ji} a vector and w_{ji} a scalar. If $i = j$

$$e_i^{\mathrm{T}}\Gamma = \alpha_ie_i^{\mathrm{T}}A_i^{-1}.$$

Thus

$$\Gamma = V_i + \alpha_i\begin{bmatrix} w_{1j} \\ \vdots \\ w_{i-1,i} \\ 1 \\ w_{i+1,i} \\ \vdots \\ w_{N,i} \end{bmatrix} e_i^{\mathrm{T}}A_i^{-1}$$

with V_i, $e_i^{\mathrm{T}}A_i^{-1}$ and w_{ji} independent of k_i. Hence, the result follows. ∇∇∇

Presented below is a non-electrical example where the state variable realization has a rank-1 dependence on most element values. The dynamics pertinent to the attitude control of the communications technology satellite, *Hermes* (Diduch and Balasubramaniam, 1982), are as follows:

$$\dot{x} = Ax + Bu$$

$$Y = Cx$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -w_0h/I_1 & 0 & 0 & w_0 - h/I_1 \\ 0 & 0 & 0 & 1 \\ 0 & h/I_2 - w_0 & -w_0h/I_2 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ F_1L_1G_1\cos\alpha/I_1 & 0 \\ 0 & 0 \\ -F_1L_1G_1\sin\alpha/I_2 & F_2L_2G_2/I_2 \end{bmatrix}$$

$$X = \begin{bmatrix} \phi \\ \dot{\phi} \\ \psi \\ \dot{\psi} \end{bmatrix}$$

$$c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Here ϕ is the roll, x is the yaw, I_1 is the moment of inertia about the roll axis, I_2, that about the yaw axis, w_0 is the orbital rate, h the nominal wheel angular momentum, α the offset angle, L_1 and L_2 the offset and yaw thruster moment arms, respectively, and G_1 and G_2, the inpulse bit factors. The inputs u_1 and u_2 provide a guide for the level of consumed fuel.

It is evident that the parameters $I_1, I_2, F_1, F_2, L_1, L_2, G_1$ and G_2 appear in the state variable representation in a rank-1 fashion. Also, although

$$\frac{\partial}{\partial\alpha}\left[\begin{array}{c|c} sI - A & B \\ \hline -c & 0 \end{array}\right]$$

has rank 1, the system matrix is obviously not linear in α. Thus α does not quite conform to the definition of rank-1 dependence. The parameters w_0 and h, on the other hand, clearly do not appear in a rank-1 fashion. But, by definition (one is the orbital rate and the other the wheel angular momentum), one can see that they must allow cross-coupling between energy storage elements. They thus fall in the same category as mutual inductors, which as the authors have emphasized, do not fall within the requirements of Assumption 2.1.

4. SISO TRANSFER FUNCTIONS

In this section, it will be shown that SISO systems which have rank-1 state variable realizations necessarily have minimal transfer function descriptions which are the ratios of two polynomials multilinear in the unknown parameters.

At the outset the following definition of coprimeness of polynomials in more than one variable will be introduced.

Definition 4.1. Consider $p_i(x_1,\ldots,x_n,x_{n+1},\ldots,x_m)$, $i = 1,2,\ldots,r$, which are r polynomials in the indeterminates $x_1,\ldots,x_m$. Then p_i *are coprime with respect to the variables* $x_1,\ldots,x_n$ if there exists no nontrivial f which is a polynomial in $x_1,\ldots,x_n$, but *rational* in $x_{n+1},\ldots,x_m$, such that

$$ff_i = p_i, \qquad \forall\, i \in \{1,\ldots,r\}$$

with f_i polynomials in $x_1,\ldots,x_n$ and rationals in $x_{n+1},\ldots,x_m$, as well.

The following theorem shows that a system having a state variable realization which has a rank-1 dependence on a single parameter α has a transfer function whose numerator and denominator are affine in α.

Theorem 4.1. An n-dimensional SISO system represented by

$$\dot{x} = F(\alpha)x + g(\alpha)u$$
$$y = h^{\mathrm{T}}(\alpha)x + j(\alpha)u$$

with a system matrix of the form

$$\left[\begin{array}{c|c} sI - F(\alpha) & g(\alpha) \\ \hline -h^{\mathrm{T}}(\alpha) & j(\alpha) \end{array}\right] = \left[\begin{array}{c|c} sI - F_0 & g_0 \\ \hline -h_0^{\mathrm{T}} & j_0 \end{array}\right] + \alpha\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}[g_1^{\mathrm{T}} \mid g_2] \qquad (4.1)$$

where $F_0, g_0, h_0, j_0, h_1, h_2, g_1$ and g_2 are independent of α, has a transfer function

$$W(s) = \frac{a(s) + \alpha b(a)}{c(s) + \alpha d(s)} \qquad (4.2)$$

for every α. The polynomials $a(s)$, $b(s)$, $c(s)$ and $d(s)$ are independent of α and obey the following restrictions:

(a) $\delta[c(s)] = n$, $\delta[d(s)] < n$, $\delta[a(s)] \leqslant n$ and $\delta[b(s)] \leqslant n$;

(b) $a(s)d(s) - b(s)c(s)$ is factorizable into two polynomials of degree not exceeding n.

Conversely, any transfer function of the form (4.2) has a state variable realization of the form (4.1) provided that conditions (a) and (b) hold.

Proof. (i) From (4.1), some calculation shows that

$$W(s) = [a(s)c(s) + \alpha\{a(s)d(s) + g_2h_2c^2(s) - h_2\gamma(s)c(s) + g_2\beta(s)c(s) - \beta(s)\gamma(s)\}]/\{c^2(s) + \alpha c(s)d(s)\} \qquad (4.3)$$

with $c(s) = \det(sI - F_0)$ and a, β, γ, d defined by (4.4):

$$\bar{W}(s) = \begin{bmatrix} a(s)/c(s) & \beta(s)/c(s) \\ \gamma(s)/c(s) & d(s)/c(s) \end{bmatrix} = \begin{bmatrix} h_0^{\mathrm{T}} \\ g_1^{\mathrm{T}} \end{bmatrix}[sI - F_0]^{-1}[g_0 \quad h_1]. \qquad (4.4)$$

Because $\bar{W}(s)$ has an nth order realization, its Macmillan degree is not greater than n. Thus,

$ad - \beta\gamma$ is divisible by $c(s)$. Define $b(s)$ by

$$b(s)c(s) = a(s)d(s) - [\gamma(s) - g_2c(s)][\beta(s) + h_2c(s)]. \quad (4.5)$$

Then (4.3) has the same form as (4.2) and by (4.5) and (4.4) conditions (a) and (b) are satisfield.

(ii) Suppose that (a) and (b) hold for (4.2). Let

$$a(s)d(s) - b(s)c(s) = f_1(s)f_2(s) = [\gamma(s) - g_2c(s)][\beta(s) + h_2c(s)]$$

with $\delta[\gamma(s)] < n$ and $\delta[\beta(s)] < n$. Then $a(s)d(s) - f_1(s)f_2(s)$ is divisible by $c(s)$ whence

$$\hat{W}(s) = \begin{bmatrix} a(s)/c(s) & f_2(s)/c(s) \\ f_1(s)/c(s) & d(s)/c(s) \end{bmatrix}$$

has Macmillan degree no greater than n. Hence, $\hat{W}(s)$ can be expressed as

$$\begin{bmatrix} h_0^{\mathrm{T}} \\ g_1^{\mathrm{T}} \end{bmatrix}[sI - F_0]^{-1}[g_0 \quad h_1] + \begin{bmatrix} \gamma_0 & h_2 \\ -g_2 & 0 \end{bmatrix}$$

so that by reversing the argument in the first part of this theorem a state variable realization of (4.2) exists in the form displayed in (4.1). ∇∇∇

Remark 4.1. Given any $W(s)$ of the form (4.2), with condition (a) holding, it may be that (b) cannot be satisfied, because $ad - bc$ has all complex roots and n is odd. But then one can multiply both the numerator and denominator in (4.2) by $(s + \eta)$, for any η to ensure that the resulting a, b, c and d polynomials satisfy (b). The resulting state variable realization conforming to (4.1) is then *non minimal.* Thus, a system having a non minimal state variable realization of the form in (4.1) *need not* have a minimal realization of the same form. Notice also, that, if $\delta[c(s)] < \delta[d(s)]$, then (a) will hold by replacing α with $1/\alpha$. Thus all transfer functions of the form (4.2) which are proper at almost all values of α, have state variable descriptions like (4.1). However, the latter need not be minimal.

Remark 4.2. If the state variable realization has a rank-1 dependence on k_1 then $\alpha_1 = k_1/(a_1 + k_1b_1)$ or $1/(a_1 + k_1b_1)$, whence both the numerator and denominator of the transfer function are linear in k_1, even though the degree requirements of (a) may have to be replaced by $\delta[d(s)] \leqslant n$. If $a_1 = 0$, k_1 would need to be replaced by $1/k_1$.

The result of the theorem is now extended to include the situation where the number of parameters exceeds one. To this end Lemmas 4.1–4.3 are needed, which derive some results for the coprimeness of polynomials in more than one variable.

Lemma 4.1. Consider a transfer function

$$W(s,k_1,k_2) = \frac{a(s,k_2) + k_1b(s,k_2)}{c(s,k_2) + k_1d(s,k_2)}$$

where a, b, c, d are polynomials in s and k_2. Suppose

(i) $a(s,k_2)$, $b(s,k_2)$, $c(s,k_2)$ and $d(s,k_2)$ are coprime with respect to s and k_2;

(ii) $a(s,k_2)d(s,k_2) - b(s,k_2)c(s,k_2) \not\equiv 0$.

Then $a(s,k_2) + k_1b(s,k_2)$ and $c(s,k_2) + k_1d(s,k_2)$ are coprime with respect to s, k_1 and k_2.

Proof. According to Hodge and Pedoe (1953, p. 36), the ring of polynomials in the variables s, k_1, k_2 over the field of real numbers is a unique factorization domain. Let

$$\begin{aligned} a(s,k_2) + k_1b(s,k_2) &= m(s,k_1,k_2)p(s,k_1,k_2) \\ c(s,k_2) + k_1d(s,k_2) &= m(s,k_1,k_2)q(s,k_1,k_2) \end{aligned} \quad (4.6)$$

with m, p, q polynomials in s, k_1 and k_2 and m not trivial. Consider the following cases.

Case I: m *is independent of* k_1. It is virtually immediate that (i) is violated.

Case II: m *is dependent on* k_1. Then m must be affine in k_1 and p, q are independent of k_1. Suppose $m(s,k_1,k_2) = r_1(s,k_2) + k_1r_2(s,k_2)$. Then

$$\begin{aligned} a(s,k_2) &= r_1(s,k_2)p(s,k_2) \\ b(s,k_2) &= r_2(s,k_2)p(s,k_2) \\ c(s,k_2) &= r_1(s,k_2)q(s,k_2) \\ d(s,k_2) &= r_2(s,k_2)q(s,k_2) \end{aligned} \quad (4.7)$$

whence $ad - bc \equiv 0$, which too contradicts the hypothesis here. ∇∇∇

Remark 4.3. Violation of (ii) implies that W is independent of k_1. We now show that the transfer function is expressible as ratio of polynomials multilinear in k_i. Moreover, from (4.7) one can see that if the numerator and denominator of W are not coprime w.r.t. k_1, then

$$W(s,k_1,k_2) = \frac{a(s,k_2)}{c(s,k_2)} = W(s,0,k_2).$$

Lemma 4.2. Suppose that the transfer function $W(s,k_1,\ldots,k_N)$ is expressible as

$$W(s,k_1,\ldots,k_N) = \frac{a_i(s,k^{(i)}) + k_ib_i(s,k^{(i)})}{c_i(s,k^{(i)}) + k_id_i(s,k^{(i)})} \quad \forall\, i \in \{1,\ldots,N\}$$

where $k^{(i)} \triangleq \{k_1,\ldots,k_N\} - \{k_i\}$. Suppose $a_id_i - b_ic_i \not\equiv 0$ and a_i, b_i, c_i, d_i are coprime with respect to s and $k^{(i)}$.

Then

$$W(s,k_1,\dots,k_N)=\frac{P(s,k_1,\dots,k_N)}{Q(s,k_1,\dots,k_N)}$$

where P and Q are multilinear in k_i.

Proof. The case when $N=2$ will be proved. The more general case follows along the same lines. Suppose

$$W(s,k_1,k_2)=\frac{a_1(s,k_2)+k_1b_1(s,k_2)}{c_1(s,k_2)+k_1d_1(s,k_2)}$$
$$=\frac{a_2(s,k_1)+k_2b_2(s,k_1)}{c_2(s,k_1)+k_2d_2(s,k_1)}$$

and suppose that the other hypotheses specialized to $N=2$ holds. Then

$$[a_1(s,k_2)+k_1b_1(s,k_2)][c_2(s,k_1)+k_2d_2(s,k_1)]=[a_2(s,k_1)+k_2b_2(s,k_1)]\times[c_1(s,k_2)+k_1d_1(s,k_2)].$$

By Lemma 4.1, $a_1(s,k_2)+k_1b_1(s,k_2)$ and $c_1(s,k_2)+k_1d_1(s,k_2)$ are coprime with respect to s,k_1 and k_2. Thus, $a_1(s,k_2)+k_1b_1(s,k_2)$ divides $a_2(s,k_1)+k_2b_2(s,k_1)$. Thus, $a_1(s,k_2)$ and $b_1(s,k_2)$ can be at most linear in k_2. Similarly, $c_1(s,k_2)$ and $d_1(s,k_2)$ can be at most linear in k_2. Hence the transfer function $W(s,k_1,k_2)$ can be written as

$$\frac{p_0(s)+k_1p_1(s)+k_2p_2(s)+k_1k_2p_{12}(s)}{q_0(s)+k_1q_1(s)+k_2q_2(s)+k_1k_2q_{12}(s)}$$

whence the result follows. ∇∇∇

It is of interest here to consider minimality with respect to s alone. That is, it is to be shown that if the hypothesis of Lemma 4.3 holds, then P and Q have no common factors which are *polynomials* in s but *rational* in the k_i. The following lemma which follows from Youla and Gnavi (1979) shows that this is indeed the case.

Lemma 4.3. If $P(s,k_1,k_2,\dots,k_N)$ and $Q(s,k_1,\dots,k_N)$, polynomials in $s,k_1,\dots,k_N$, have no common factors which are polynomials in $s,k_1,\dots,k_N$, then they have no common factors which are polynomials in s but rational in $k_1,\dots,k_N$.

Proof. If P and Q have no common factors which are polynomials in $s,k_1,\dots,k_N$ then P and Q are *minor coprime* as well (see Youla and Gnavi for a definition). Thus by Youla and Gnavi there exist polynomials x, y and ψ, with ψ nontrivial and independent of s, for which

$$P(s,k_1,\dots,k_N)x(s,k_1,\dots,k_N)+Q(s,k_1,\dots,k_N)y(s,k_1,\dots,k_N)=\psi(k_1,\dots,k_N).$$

Thus, dividing both sides by ψ the result is immediate. ∇∇∇

Now Theorem 4.1 and Lemmas 4.1–4.3 together yield the following main result of this section.

Theorem 4.2. If the state variable realization of a SISO linear time-invariant finite dimensional system has a rank one dependence on N parameters $k_1,\dots,k_N$, then it has a minimal (with respect to s) (see Definition 4.1 for coprimeness) transfer function description whose numerator and denominator polynomials are multilinear in the k_i.

5. TRANSFER FUNCTIONS CONFORMING TO THE STANDING ASSUMPTION

In this section, conditions under which scalar transfer functions with numerator and denominator multilinear in the unknown parameters, correspond to systems satisfying Assumption 2.1 are derived. Theorem 5.1 below summarizes these conditions. The theorem can be understood by observing that a scalar system satisfying this assumption can be expressed in terms of the following input–output description:

$$\begin{aligned}Y(s)&=\frac{a(s)}{c(s)}U(s)+\frac{g^{\mathrm T}(s)}{c(s)}U_1(s)\\ Y_1(s)&=\frac{h(s)}{c(s)}U(s)+\frac{D(s)}{c(s)}U_1(s)\\ U_1(s)&=KY_1(s)\\ K&\triangleq\operatorname{diag}\{k_1,\dots,k_N\}\end{aligned}\qquad(5.1)$$

where $a(\cdot)$ and $c(\cdot)$ are scalar polynomials with $c(\cdot)$ the characteristic polynomial of the transfer function relating $[U,U_1]^{\mathrm T}$ to $[Y,Y_1]^{\mathrm T}$, $h(\cdot)$ and $g(\cdot)$ are N-dimensional polynomial vectors and $D(\cdot)$ is an $N\times N$ polynomial matrix. It is not hard to see then, that the transfer function relating U to Y is

$$W(s)=\frac{a(s)}{c(s)}+\frac{g^{\mathrm T}(s)}{c(s)}K[c(s)I-D(s)K]^{-1}h(s).\qquad(5.2)$$

Equation (5.2) forms the basis for Theorem 5.1.

Theorem 5.1. Consider a system with a proper transfer function

$$W(s,k)=P(s,k)/Q(s,k)\qquad(5.3)$$

where

$$k \triangleq [k_1, \ldots, k_N]^T$$

$$P(s,k) = p_0(s) + \sum_{r \subset S} \left(\prod_{i \in r} k_i \right) p_r(s) \qquad (5.4)$$

$$Q(s,k) = q_0(s) + \sum_{r \subset S} \left(\prod_{i \in r} k_i \right) q_r(s)$$

with $S = \{1, \ldots, N\}$ and $\delta[q_0(s)] \geq \delta[q_r(s)]$. Suppose $P(s,k)$ and $Q(s,k)$ are coprime with respect to s and the elements of k. Then the corresponding system satisfies Assumption 2.1, with a proper $T(s)$, iff there exist a scalar polynomial $f(s)$, N-dimensional vector polynomials $g(s)$ and $h(s)$ and an $N \times N$ matrix polynomial $D(s)$, with $\delta[g(s)]$, $\delta[h(s)]$ and $\delta[D(s)]$ all less than or equal to $\delta[f(s)q_0(s)]$, such that the following hold $\forall r \subset S$:

(i) $\det(-D_r) = f(s)q_r(s)[q_0(s)f(s)]^{|r|-1}$
(ii) $[q_0(s)f(s)]^{|r|}p_r(s)f(s) = a(s)\det D_r(s) - g_r^T(s)\{\operatorname{Adj} D_r(s)\}h_r(s)$

where D_r, g_r and h_r are defined as below.

(a) $D_r(s)$ is a $|r| \times |r|$ matrix consisting of the ith rows and ith columns of $D(s)\ \forall\, i \in r$.
(b) $g_r(s)$ and $h_r(s)$ are $|r|$-dimensional vectors consisting, respectively, of the ith elements of $g(s)$ and $h(s)\ \forall\, i \in r$.

The ordering of elements in all cases is consistent.

Remark 5.1. Since P and Q are coprime w.r.t s and $k_1, \ldots, k_N$, by Lemma 4.3 they are also coprime w.r.t. s alone.

Proof.(i) First the "only if" part will be proved. Suppose Assumption 2.1 is satisfied with proper $T(s)$. Then (5.1) holds and $W(s,k)$ is given by (5.2), with all quantities obviously defined. Thus to complete the proof the $p_r(s)$ and $q_r(s)$ in (5.4) must be related to the quantities in (5.2).

By definition, $T(s)$ and all submatrices thereof have realizations with characteristic polynomial $c(s)$. Thus $\forall\, r \subset S$ scalar polynomials $\tilde{q}_r(s)$ and $\tilde{p}_r(s)$ exist such that

$$c^{(|r|-1)}(s)\,\tilde{q}_r(s) = \det(-D_r(s)) \qquad (5.5)$$

and

$$c^{|r|}(s)\tilde{p}_r(s) = a(s)\det D_r(s) - g_r^T(s)\operatorname{Adj} D_r(s)h_r(s). \qquad (5.6)$$

Now,

$$W(s,k) = \bar{P}(s,k)/\bar{Q}(s,k)$$

where

$$\bar{P}(s,k) = a(s)\det(c(s)I - D(s)K) + g^T(s)K\operatorname{Adj}(c(s)I - D(s)K)h(s) \qquad (5.7)$$

and

$$\bar{Q}(s,k) = c(s)\det(c(s)I - D(s)K). \qquad (5.8)$$

Observe that the coefficients of $\bar{P}$ and $\bar{Q}$ can easily be shown to be multilinear in the k_i. Define $\lambda_i = 1/k_i$ and $\Lambda = K^{-1}$. Then the polynomials multiplying $\prod_{i \in r} k_i$ in $\bar{P}$, $\bar{Q}$ can be obtained as follows:

(i) set $k_j = 0$, $\forall j \notin r$;
(ii) divide by $\prod_{i \in r} k_i$;
(iii) set $\lambda_i = 0$ $i \in r$.

With K_r, Λ_r obviously defined and k_r a vector of elements $k_i \forall\, i \notin r$, for $\bar{P}$ the three steps respectively, translate to

$$\text{(i)}\quad \bar{P}(s,k_r) = \det\left[c(s)I - D(s)\begin{Bmatrix} K_r & 0 \\ 0 & 0 \end{Bmatrix}\right]\left[a(s) + g^T(s)\begin{bmatrix} K_r & 0 \\ 0 & 0 \end{bmatrix}\left[c(s)I - D(s)\begin{Bmatrix} K_r & 0 \\ 0 & 0 \end{Bmatrix}\right]^{-1} h(s)\right]$$
$$= \det(c(s)I_{|r|} - D_r(s)K_r)c^{N-|r|}(s)[a(s) + g_r^T(s)K_r(c(s)I_{|r|} - D_r(s)K_r)^{-1}h_r(s)];$$

$$\text{(ii)}\quad \frac{\bar{P}(s,k_r)}{\prod_{i \in r} k_i} = c^{N-|r|}(s)\det(c(s)\Lambda_r - D_r(s))[a(s) + g_r^T(s)(c(s)\Lambda_r - D_r(s))^{-1}h_r(s)];$$

$$\text{(iii)}\quad \bar{p}_r(s) = c^{N-|r|}(s)\det(-D_r(s))[a(s) - g_r^T(s)D_r^{-1}(s)h_r(s)]$$
$$= c^{N-|r|}(s)[a(s)\det(-D_r(s)) - g_r^T(s)\operatorname{Adj}[D_r(s)]h_r(s)]. \qquad (5.9)$$

Similarly,

$$\bar{q}_r(s) = \overset{N-|r|+1}{c(s)} \det(-D_r(s)). \quad (5.10)$$

By (5.5) and (5.6),

$$\bar{q}_r(s) = \overset{N-|r|+1}{c(s)} c(s)^{|r|-1}\tilde{q}_r(s) = \overset{N}{c(s)}\tilde{q}_r(s) \quad (5.11)$$

and

$$\bar{p}_r(s) = \overset{N}{c(s)}\tilde{p}_r(s). \quad (5.12)$$

Define $\tilde{p}_0(s) = a(s)$ and $\tilde{q}_0(s) = c(s)$. Thus,

$$W(s,k) = \frac{\tilde{P}(s,k)}{\tilde{Q}(s,k)}$$

with $\tilde{P}$ and $\tilde{Q}$ obviously defined. Consider now, the two following cases.

Case I: $\tilde{P}$ *and* $\tilde{Q}$ *are coprime with respect to every element of* k. Since P, Q, $\tilde{P}$ and $\tilde{Q}$ are all multilinear in the elements of k,

$$f(s)P(s,k) = \tilde{P}(s,k)$$

and

$$f(s)Q(s,k) = \tilde{Q}(s,k),$$

whence,

$$f(s)p_0(s) = a(s)$$

$$f(s)q_0(s) = c(s)$$

$$f(s)q_r(s) = \tilde{q}_r(s) \qquad \forall r \subset S$$

and

$$f(s)p_r(s) = \tilde{p}_r(s) \qquad \forall r \subset S.$$

Thus, conditions (i) and (ii) are satisfied due to (5.5) and (5.6).

Case II: $\tilde{P}$ *and* $\tilde{Q}$ *are not coprime with respect to* $k_i \forall i \in \bar{S} \subset S$. Then, by cancelling factors involving k_i, it can be seen that

$$W(s,k) = \frac{a(s)}{c(s)} + \frac{g^T(s)}{c(s)}\begin{bmatrix} K_r & 0 \\ 0 & 0 \end{bmatrix}$$

$$\times \left[c(s)I - D(s)\begin{bmatrix} K_r & 0 \\ 0 & 0 \end{bmatrix}\right]^{-1} h^T(s)$$

$$= \frac{a(s)}{c(s)} + \frac{g_r^T(s)}{c(s)} K_r(c(s)I_{|r|} - D_r(s)K_r)^{-1} h_r^T(s)$$

where r excludes the elements corresponding to i. The first equation is obtained by using the equation in Remark 4.3. Thus the proof can be established easily.

(ii) The "iff" part follows by a trivial reversal of the above arguments. ∇∇∇

6. CONCLUSION

Certain structural aspects of systems which have a parameterization involving physical component values as the parameters have been established. The systems investigated include RLC circuits and their chemical and mechanical analogs.

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