\( (J + V(t), B(t) + C(t), U) \) is globally null controllable from 0 iff

\[
G(T) = \int_0^T \|(B^T(t) + C^T(t))(I + \hat{D}(t))e^{-J(t)(I + D^T(T))}y\|_2^2 \ dt + \infty
\]

for all nonzero vectors \( y \in \mathbb{R}^s \). (Note: \( G(0) \) and \( G(T) \) diverge or converge together.)

\( G(T) \) can be rewritten as

\[
G(T) = \sum_{k=k_0}^\infty \int_k^{k+o} \|(B^T(t) + \hat{C}^T(t))e^{-J(t)(I + D^T(T))}y\|_2^2 \ dt
\]

where

\[
\hat{C}^T(t) = B^T(t)\hat{D}(t) + C^T(t)(I + \hat{D}(t))
\]

and

\[
\hat{y} = (I + D^T(T))y.
\]

Letting \( T = k_0w \) where \( k_0 \) is a sufficiently large integer, we obtain

\[
G(T) \geq \sum_{k=k_0}^\infty k \int_k^{k+o} \|(B^T(t) + \hat{C}^T(t)) \exp(J*(k_0w + o - t)) \exp(-J*(k_0w + o - k_0w))\|_2^2 \ dt.
\]

Using Lemma 1, i.e., that \( \hat{R}(k_0w, k_0w + o) \) is positive definite for all \( k \geq k_0 \) sufficiently large, it follows that

\[
G(T) \geq \sum_{k=k_0}^\infty \mu \int_{k_0}^{k+o} \exp(-J*(k_0w + o - k_0w))\|_2^2 \ dt.
\]

where \( \mu > 0 \) depends only on \( o \) [also \( A(t) \) and \( B(t) \)].

Finally, using Lemma 2, we get

\[
G(T) \geq \sum_{k=k_0}^\infty k \mu \int_k^{k+o} \exp(-2o((k-k_0)w + o)((k-k_0)w + o)^2(a + a(k)).
\]

The last series is clearly divergent because \( \text{Re}(\lambda_i) \approx 0 \ i = 1, \ldots, s \).

It follows, therefore, that \( G(T) = G(0) = +\infty \) for all nonzero vectors \( y \in \mathbb{R}^s \) (hence, \( y \in \mathbb{R}^s \) since \( I + D^T(T) \) is nonsingular), which implies that \( (A(t) + V(t), B(t) + C(t), U_2) \) is globally null controllable from 0.

We end this note by giving a simple example of the applicability of the previous result. Consider a perturbed harmonic oscillator

\[
x(t) + (1 + g(t))x = (\cos t + f(t))u \]

where

\[
g(t) \in L^1[0, \infty) \]

and \( f(t) \to 0 \) as \( t \to \infty \). Conditions a), b), and d) of the theorem are clearly satisfied. Condition c)) also holds if we note that

\[
K(t) = \begin{bmatrix} \cos t & \sin t \\ \cos t & -\sin t \end{bmatrix}
\]

has rank 2 at any time \( t \) such that \( \cos t \neq 0 \). We conclude, therefore, that the oscillator is globally null controllable by means of \( U_2(t < q \leq \infty) \).

(Note that the assumption that \( g(t) \) be integrable is very sharp in the sense that we cannot replace it by \( g(t) \to 0 \) as \( t \to \infty \) and, in general, still preserve controllability; see [II] for more details.)

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Control of Decentralized Systems with Distributed Controller Complexity

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Abstract—An approach is presented which allows design of stabilizing decentralized controllers for linear systems with two scalar channels, such that each local controller is endowed with some dynamics while the sum of the orders is kept smaller than the system order. It is shown that for almost all arbitrary order systems with two scalar channels the local controllers can be chosen such that their orders \( d_1 \) and \( d_2 \) satisfy \( d_1 + d_2 \geq n - 2 \), and the orders \( d_1 \) and \( d_2 \) are close to \((n-1)/2, n-1-((2m)/2)\), where \( m \) is the number of stable zeros of the cross coupling transfer functions in the system.

The approach is to design one of the local controllers such that the McMillan degree of the resulting one-channel system is reduced. Then the other local controller only has to deal with a model of reduced dimension and can thus be chosen of lower order.

I. INTRODUCTION

Consider a system given by

\[
\begin{pmatrix}
\dot{y}_1(s) \\
\dot{y}_2(s) \\
\end{pmatrix} = \begin{pmatrix}
\frac{n_1(s)}{d_1(s)} & \frac{n_2(s)}{d_2(s)} \\
\frac{n_3(s)}{d_3(s)} & \frac{n_4(s)}{d_4(s)} \\
\end{pmatrix} \begin{pmatrix}
u_1(s) \\
\nu_2(s) \\
\end{pmatrix} = W(s) \begin{pmatrix}
u_1(s) \\
\nu_2(s) \\
\end{pmatrix}
\]

(1.1)

where \( \nu_1(s) \) and \( y_2(s) \) are scalar input and output signals, the \( n_i(s) \) are scalar polynomials \((i = 1, 2, d(s) \) is the characteristic polynomial of \( W(s) \), and \( W(s) \) is strictly proper. In this note we consider the problem of constructing feedback controllers of the form

\[
\nu_i(s) = \frac{p_i(s)}{q_i(s)} y(s) + \nu_i(s), \quad i = 1, 2
\]

(1.2)

such that the closed-loop system is stable and the complexity of the two local controllers is roughly of equal size. More specifically, the orders of the proper rational functions \( p_i(s)/q_i(s) \) and \( p_2(s)/q_2(s) \) should be close to being equal while their sum is kept small.


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It is known [1] that the system (1.1) can be stabilized by local controllers of the form (1.2), provided there are no unstable fixed modes. Moreover, using the approach in [2], one of the controllers \( p_1(s)/q_1(s) \) and \( p_2(s)/q_2(s) \) can be chosen of order 0 and the other can then be of order \( n - 1 \). It is clearly of practical importance to avoid this unsymmetric distribution of controller complexity.

For the special case where (1.1) represents two so-called interconnected systems, results in this direction are obtained in [3]. The general case, however, seems to be much more difficult to analyze. But this is understandable, since even in the centralized case there are only crude bounds available for the minimal dimension of a stabilizing controller (e.g., [4]). Moreover, many of these centralized results are only true for generic systems [5, 6]. In this note it will be shown that for almost all systems (1.1) stabilizing local controllers of the form (1.2) can be found such that

\[
\text{deg } q_1 + \text{deg } q_2 \leq \text{deg } d - 2
\]

where \( \mu_0 \) is the number of stable zeros of \( n_1(s)n_2(s) \).

The approach is illustrated as follows. Suppose that we first select \( p_2(s)/q_2(s) \) and then select \( p_1(s)/q_1(s) \). After the channel 2 controller has been inserted, the closed-loop system has degree \( d + q_2 \). But the transfer function \( t_1(s) \) from \( u(s) \) to \( y(s) \) (which depends on \( p_2(s) \) and \( q_2(s) \)) may have smaller degree due to pole-zero cancellations. The main achievement of this note is to show which pole-zero cancellations are possible in \( t_1(s) \) by a proper choice of \( p_2(s)/q_2(s) \). Roughly speaking, any collection of \( \mu \) zeros of the polynomial \( n_1(s)n_2(s) \) can be cancelled in \( t_1(s) \) by a controller of degree \( (\mu - 1)/2 \) (assuming for simplicity \( \mu \) odd).

In order to achieve internal stability, one should only cancel the stable zeros of \( n_1(s)n_2(s) \). After such a cancellation, the degree of \( t_1(s) \) is deg \( d + (\mu - 1)/2 = \mu + 1/2 \) and the controller \( p_2(s)/q_2(s) \) has to deal only with a model of reduced dimension. In particular, \( p_2(s)/q_2(s) \) of degree \( \mu + 1/2 - 1 \) suffices in order to stabilize \( t_1(s) \). This means that it is possible to stabilize the overall system by two local controllers of degrees \( (\mu - 1)/2 \) and deg. \( d - (\mu + 3)/2 \), respectively, where \( \mu \) is less or equal to the number of stable zeros of \( n_1(s)n_2(s) \). So if, for instance, \( \mu \) is even and \( n_1(s)n_2(s) \) has at least \( \mu = \text{deg } d - 1 \) stable zeros, then the system can be stabilized by two local controllers, each of degree \( (\text{deg } d)/2 - 1 \).

A preliminary version of this note appeared in [7].

This note is organized as follows. In Section II the basic problem of reducing the degree of \( t_1(s) \) is studied, and in Section III the generic case is analyzed. Section IV is concerned with the implications of the previous results to decentralized control, and in Section V some conclusions are drawn.

II. Model Reduction by Feedback

Consider the system (1.1). Since \( d(s) \) is the characteristic polynomial of \( W(s) \), it is the least common multiple of the denominators of each entry of \( W(s) \) and det \( W(s) \). This means that

\[
l(s) := \frac{n_1(s)n_2(s) - n_2(s)n_1(s)}{d(s)}
\]

is polynomial and

\[
d(s)l(s) = n_1(s)n_2(s) - n_2(s)n_1(s)
\]

gcd \((n_1(s), n_2(s), n_3(s), d(s), l(s), l(s)) = 1\). Suppose that \( p_2(s)/q_2(s) \) has been selected and the channel 2 controller inserted. Then the transfer function from \( u(s) \) to \( y(s) \) is given by

\[
t_1(s) = \frac{l(s)p_2(s) + n_1(s)q_2(s)}{n_2(s)p_2(s) + d(s)q_2(s)}
\]

where \( n_2(s)p_2(s) + d(s)q_2(s) \) is the characteristic polynomial of the resulting system. From (2.4) it is clear that \( t_1(s) \) will generally have degree \( d + q_2 \) minus the number of pole/zero cancellations. In this section, we discuss how \( p_2(s)/q_2(s) \) may be selected so as to minimize the McMillian degree of \( t_1(s) \), without unstable pole-zero cancellations being introduced in (2.4). Put another way, we are seeking \( p_2(s)/q_2(s) \) so as to maximize the number of asymptotically stable uncontrollable-from-\( u(s) \), and/or unobservable-from-\( y_1(s) \) modes. This has the effect that the channel 1 controller has to deal with a model of reduced dimension.

It is obvious that a cancellation occurs in (2.4) at the common zeros of \( l(s), n_1(s), n_2(s), d(s), q_2(s) \), no matter how \( p_2(s)/q_2(s) \) is chosen. This means that the zeros of the greatest common divisor of \( l(s), n_1(s), n_2(s), d(s), q_2(s) \), denoted by \( \sigma(s) \), are decentralized fixed modes [1]. In fact, all decentralized fixed modes must be zeros of \( \sigma(s) \). Define the polynomials \( f_1(s) := l(s)/\sigma(s), f_2(s) := n_1(s)/\sigma(s), f_3(s) := n_2(s)/\sigma(s), f_4(s) := d(s)/\sigma(s) \), and assume \( \sigma(s) \) is stable (has all its zeros in the left-half plane). Then the problem can be restated to choose \( p_2(s) \) such that

\[
t_1(s) = \frac{l(s)p_2(s) + n_1(s)q_2(s)}{n_2(s)p_2(s) + d(s)q_2(s)}
\]

has least degree without unstable pole-zero cancellations being introduced in (2.5).

By definition of \( \sigma(s) \), there is an \( \sigma(s)^2 \) divides \( n_1(s)n_2(s) \), and the remaining theorem states that the remaining zeros of \( n_1(s)n_2(s) \), i.e., the zeros of the polynomial \( n_1(s)n_2(s)/\sigma(s)^2 \), are crucial for further cancellations to occur in (2.5).

Theorem 2.1: Consider the system (1.1) and suppose \( \sigma(s) \) is of degree \( r \). Let \( \mu \) be any odd (even) nonnegative integer. Then there exist coprime polynomials \( p_2(s) \) and \( q_2(s) \) such that \( \sigma(s) \neq 0, \deg p_2, \deg q_2 \leq (\mu - 1)/2 (\mu/2 \text{ in case } \mu \text{ even}) \) and

\[
t_1(s) = \frac{\sigma(s)\delta(s)}{\sigma(s)\delta(s)}
\]

where \( \delta, \gamma \leq \text{deg } d - r - (\mu + 1)/2 (\text{deg } d - (\mu + 3)/2 \text{ in case } \mu \text{ even}) \) and \( \gamma \leq \text{deg } u \) and if only if \( u(s) \) is a divisor of the polynomial \( n_2(s)n_2(s)/\sigma(s)^2 \).

The above theorem gives a complete characterization of those factors \( u(s) \) that may be cancelled in \( t_1(s) \) by a controller whose degree is at least \( d \) of \( v(s) \). For the closed-loop system to be stable, \( \sigma(s) \) and \( v(s) \) have to be stable. So, for instance, if \( n_2(s)n_2(s)/\sigma(s)^2 \) contains \( \mu = n - r - 1 \) stable zeros (\( \mu \) odd), then \( p_2, q_2 \) will be of degree \( (n - r)/2 - 1 \) and \( p_1, q_1 \) can be chosen of degree \( (n - r)/2 - 1 \).

Theorem 2.1 has two possible disadvantages. It might be necessary that \( p_2, q_2 \) is not proper and \( \delta(s), \gamma(s) \) may not be coprime, as extra unplanned and unstable cancellations in \( t_1(s) \) may occur. These points are illustrated by [7, Examples 3.2 and 3.3]. Also, the examples show that without further assumptions on the system the statement of Theorem 2.1 cannot be strengthened. In this sense Theorem 2.1 gives a complete solution to the problem posed in Section III. In this note it will be shown that these disadvantages are not present for generic systems, i.e., it will be shown that generically \( p_2, q_2 \) will be proper and \( \delta(s), \gamma(s) \) will be coprime.

The constructive proof of Theorem 2.1 follows from the lemmas and remarks given next.

Lemma 2.2: With quantities as defined above, suppose that in (2.5), \( p_2(s) \) and \( q_2(s) \) are coprime. Suppose \( \rho(s) \) is the greatest common divisor of numerator and denominator in (2.5). Then \( \rho(s) \) divides \( n_1(s)n_2(s)/\sigma(s)^2 \).

Proof: By hypothesis, there exist coprime polynomials \( \alpha(s), \beta(s) \) such that

\[
\rho(s) + \alpha(s) = \mu_0 + \mu_2 + \nu_2 = \mu_2
\]

hence

\[
(\nu - \alpha_1\alpha_2)\nu_2 = \rho(\beta - \beta_1\alpha_1, (\nu - \alpha_1\alpha_2)\nu_2 = \rho(\beta - \alpha_2)\nu_2.
\]

Since \( p_2, q_2 \) are coprime, \( \gamma \alpha + \delta_2 = 1 \) for some \( \gamma, \delta \). Hence,

\[
-\alpha_2\alpha_2 = \beta_1 - \beta_1\alpha_2 = \rho(\gamma \alpha - \beta_1) + \rho(\beta - \alpha_2(\nu_2))
\]
Lemma 2.2 shows that stable pole-zero cancellations in (2.4) are restricted to the decentralized fixed modes (these cancel irrespective of the choice of $p_2$ and $q_2$) and to the stable part of $n_2(s)/n_1(s)/\sigma(s)$\cite{1}. This proves necessity of the condition in Theorem 2.1.

Lemma 2.3: Let $u(s)$ be a divisor of $n_2(s)/n_1(s)/\sigma(s)$. Suppose that $v$ is factored as $v(s) = u(s)/p_2(s)$, where $gcd(\eta_1, \eta_2, d) = 1$, $gcd(v_2, n_1, n)$ = 1, and $gcd(v_1, u_2) = 1$. (This is always possible, since $gcd(\eta_1, \eta_2, d)$ = 1.) If further $v_2(s)$, $n_2(s)$ are coprime, $v_2(s)$, $l(s)$ are coprime, and $p_2(s), q_2(s)$ are chosen such that

$$n_2(s)/p_2(s) + d(s)/q_2(s) = v_1(s)/\sigma(s)\quad (2.8)$$

for some $\alpha(s)$, $\beta(s)$, then $u(s)$ is a common factor of $n_2(s)/p_2(s) + d(s)/q_2(s)$ and $l(s)/p_2(s) + n_1(s)/q_2(s)$.\[Lemma 2.4: In the above proof we heavily exploit the fact that $v_1(s), v_2(s)$ are coprime. There is, however, no guarantee that this condition is as priori satisfied. Fortunately, we can enforce this condition by employing a preliminary control: Suppose the feedforward control $u_2(s) = -(p_1(s)/d(s))v_2(s)$ is modified into

$$l'(s) = l(s) + fn_1(s)\quad (2.10)$$

and the polynomial $l(s)$ is modified into

$$l'(s) = l(s) + fn_1(s)$$

From this and the definition of $u_1(s), u_2(s)$, it is clear that we can always choose $f$ such that $v_1(s), n_2(s)$\[Lemma 2.5: Let all quantities be defined as before and let $\beta := \deg v$. There exists a nontrivial solution $p_2(s), q_2(s)$ to (2.8), (2.9) for some $\alpha(s), \beta(s)$, where the polynomials $p_2(s), q_2(s)$ satisfy $\deg p_2, \deg q_2 \leq \mu/2$ in $\mathbf{Z}$, $\alpha$ is even and $\deg p_2, \deg q_2 \leq (\mu - 1)/2$ in $\mathbf{Z}$ odd. Furthermore, $\beta$ is not identically zero in $\mathbf{Z}$, $v_1(s), n_2(s)$ are coprime.\]

Proof: Assume first that $\mu$ is odd and define $n := \deg \beta$. One may view (2.8), (2.9) to be equations in the unknown coefficients of $p_2, q_2, \alpha, \beta$. Since the left-hand side of (2.8) has degree at most $n + (\mu - 1)/2, \alpha$ has degree at most $n + (\mu - 1)/2 - \deg v_1$. Similarly, $\beta$ has degree at most $n + (\mu - 1)/2 - \deg v_2$. Hence, the equations (2.8), (2.9) contain unknown coefficients. Comparing now the coefficients of equal powers in (2.8), (2.9), one obtains $2n + (\mu - 1)/2 + 1 = 2n + \mu + 1$ homogeneous linear equations in the unknown coefficients. Since there is one more variable than equation, there always exists a nontrivial solution. Hence there exists a solution $p_2, q_2$ of (2.8), (2.9), where $\deg p_2, \deg q_2 \leq (\mu - 1)/2$ and at least one of the polynomials $p_2, q_2, \alpha, \beta$ is nonzero. To show that $q_2$ is nonzero, assume the contrary. Then $n_2(s)/p_2(s) = u_2(s)/\sigma(s)$ and $l(s)/p_2(s) = v_2(s)/\sigma(s)$. Since $n_2(s), n_2(s)$ are coprime, $v$ is a divisor of $p_2(s)$. But since $\deg p_2 < \deg v$, this is only possible for $p_1(s)$ = 0. This immediately implies $\alpha = 0, \beta = 0$, which is a contradiction. Hence, $q_2$ is a nonzero polynomial. The case $\mu$ even can be proven following the same lines as above.

Remark 2.6: Lemma 2.5 does not state that $p_2(s) + q_2(s)$ are coprime. Indeed, there are examples where the constructive proof of Lemma 2.5 leads to noncoprime polynomials $p_2, q_2$. Of course, there is no sense in implementing a noncoprime controller. Therefore, let us consider this case in more detail. Assume $p_2(s)$ and $q_2(s)$ satisfy (2.8), (2.9) for some $\alpha, \beta$ and $p_2(s) = e(s)p_2(s), q_2(s) = e(s)q_2(s)$, where $deg e \geq 1$ and $p_2', q_2'$ are coprime. Using the controller given by $p_2', q_2'$, there is only a cancellation of order $\deg e$ in (2.4). However, the degree of the controller is also reduced by $deg e$. Hence, the effective degree of $t_{11}(s)$ is unaffected by using the controller $p_2'/q_2'$ instead of $p_2/q_2$. The proof of Theorem 2.1 now follows from Lemmas 2.2, 2.3, 2.5 and Remarks 2.4 and 2.6.

Remark 2.7: Note that generally it cannot be recommended that one applies Theorem 2.1 with $\mu$ even. If $\mu$ is even and $v(s)$ is a divisor of $v(s)$ of degree $\mu - 1$, and $v(s)$ is cancelled in $t_{11}(s)$ using a controller of order $(\mu - 1)/2 = \mu/2 - 1$ (the odd version of Theorem 2.1), then the McMillan degree of $t_{11}(s)$ is $deg d - r - m/2$. Hence, the same goal as can be accomplished by a controller of dimension $\mu/2$ can be accomplished with one of degree $\mu/2 - 1$. Remark 2.8: We have seen that the equations (2.8), (2.9) are crucial for the solution of the decentralized control problem. In [7] it has been shown that these equations are equivalent to a rational interpolation problem.

Example 2.9: This example illustrates Theorem 2.1. The reader is invited to follow the proof with this example. Consider

$$W = \frac{1/s}{(s+3)/n_3(s+1)}$$

Then $d = s^3(s+1), I_1 = 1 - (s+2)(s+3), n_1 = s^2(s+1), n_2 = s^3(s+3), h_2 = s = 1$. Let us take $u = (s+1)/n_2(s+1)$ and $\mu = 3$ and $p_2, q_2$ will be of degree $\mu - 1/2 = 1$, $t_{11}$ will be of degree $n = (\mu - 1)/2 = 2$, so that $p_1/q_1$ will also be of degree 1. First note that $gcd(v_2, n_1) = 1$ so that we can take $v_1 = v_2 = 1$ in Lemma 2.3 and (2.8) reads ($p_2(s) = p_2 + p_21s, q_2(s) = q_2 + q_21s, e(s) = g_0 + g_1s + g_2s^2$):

$$(s^2 + p_21s + s)(s^2 + q_21s + s) = (s + 1)(s + 2)(s + 3)(g_0 + g_1s + g_2s^2).$$

(2.11)

Comparing coefficients of equal powers in $s$, (2.11) is equivalent to a set of linear equations. Assuming $q_21 = 1$, one obtains the unique solution $p_2 = 365/5, p_21 = -365/5, q_2 = 365/5, q_21 = 1, a_0 = -6/5, a_1 = 0$. With this choice of the controller we get

$$t_{11}(s) = \frac{(s+1)(s+2)(s+3)(s+6)}{(s+1)(s+2)(s+3)(s-6/5) = s^3(s+1)}.$$
the set $S$ of all systems of the form

$$W = \begin{bmatrix} n_{11} & n_{12} \\ d & d' \\ n_{21} & n_{22} \\ d & d' \end{bmatrix}$$

(3.1)

where $\deg d = n, \deg d' \leq n - 1$ and the leading coefficient of $d'$ is one. If we consider the coefficients of the polynomials $n_{i,j}$ as variables, then $S$ is isomorphic to $\mathbb{R}^{2n}$, and is thus a topological space. All subsets of $S$ which we shall define henceforth will thus also be topological spaces with topology induced from $S$.

If $W \in S$, then $d$ will in general not be the characteristic polynomial of $W$. This is true only in the following subset of $S$:

$$S := \{ W \in S : d \text{ is characteristic polynomial of } W \}$$

$$= \{ W \in S : d \text{ divides } n_{11}q_2 - n_{12}p_2 \}$$

and $gcd[(d, n_{11}, n_{21}, n_{22}, d')] = 1$.

For a fixed positive integer $\mu$ define

$$S_\mu := \{ W \in S : n_{12}p_2 \text{ has a stable divisor of degree } \mu \}.$$  

Then $S_\mu$ is an open subset of $S$. In the following theorem we shall assume $\mu$ odd, which is no restriction in view of Remark 2.7.

**Theorem 3.1:** There exists an open and dense subset $S_\mu$ of $S_\mu$ such that for all $W(s) \in S_\mu$ there exist polynomials $p_2(s)$ and $q_2(s)$ with

$$t_1 = \frac{\beta_2 + n_1q_2}{n_2p_2 + dq_2} = \frac{u_\gamma}{\gamma} = \gamma \quad \text{stable of degree } \mu$$

$$\deg p_2 \leq \frac{\mu - 1}{2}, \quad \deg q_2 = \frac{\mu - 1}{2}, \quad p_2 \text{ and } q_2 \text{ coprime}$$

$$\deg \delta \leq n - \frac{\mu + 1}{2}, \quad \deg \gamma = n - \frac{\mu + 1}{2}, \quad \delta \text{ and } \gamma \text{ coprime}.$$  

The proof is omitted since it is quite technical and gives no further insight into the theorem.

Note that $p_2/q_2$ and $t_1$ are proper and strictly proper, respectively, and there are no cancellations in $t_1$, except those caused by $u$.

The following necessary and sufficient condition for $\mu$ stable pole/zero cancellations to occur in $t_1$ follows from Theorem 3.1 and Lemma 2.2.

**Corollary 3.2:** There exists an open and dense subset $S_\mu$ of $S$ such that for $W(s) \in S_\mu$ there exist $p_2$ and $q_2$ with the properties specified in Theorem 3.1 if and only if $n_{12}p_2$ contains a stable divisor of degree $\mu$.

**Proof:** Take

$$S = \{ S_{2\mu - 2} \cap S_{2\mu - 3} \cap \ldots \cap S_1 \}$$

$$\cup \{ S_{2\mu - 3} \cap S_{2\mu - 4} \cap \ldots \cap S_1 \} \\cup \{ S_1 \}$$

$$\text{closure } S_{2\mu - 2}$$

$$\cup \{ S \cap \text{closure } S_1 \}$$

$$\cup \{ S \cap \text{closure } S_{2\mu} \}$$

$$\cup \{ W \in S : \Re(s) > 0 \}$$

for all zeros of $n_{12}p_2$.

**IV. THE DECENTRALIZED CONTROL PROBLEM**

We will now return to the decentralized control problem, and consider the extent to which the controller complexity can be distributed among the two channels. Theorem 3.1 implies the following main result of this note.

**Theorem 4.1:** Let $S$ be the set of all strictly proper $n$th order systems with two scalar channels as defined in Section III. There exists an open and dense subset $S$ of $S$ that every system in $S$ can be stabilized by proper local controllers (1.2) of degree $d_1$ and $d_2$, respectively, where $d_1$ and $d_2$ satisfy i)-iv). Here, $\mu_0$ is the number of stable zeros of $n_{12}p_2$.

i) $d_1 = d_2 = n/2 - 1$ in case $\mu_0 \geq n$ and $n$ is even.

ii) $d_1 = (n - 1)/2, d_2 = (n - 1)/2 - 1$ in case $\mu_0 \geq n$ and $n$ is odd.

iii) $d_1 = n - (\mu_0 - 1)/2 - 2, d_2 = (\mu_0 - 1)/2$ in case $\mu_0 < n$ and $\mu_0$ is odd.

iv) $d_1 = n - \mu_0/2 - 1, d_2 = \mu_0/2 - 1$ in case $\mu_0 < n$ and $\mu_0$ is even.

**Proof:** Define $S$ as in the proof of Corollary 3.2. Then the orders $d_1$ and $d_2$ in i), ii), iii), and iv) are obtained by applying Theorem 3.1 with $\mu = n - 1, n = \mu_0 = \mu = \mu_0 - 1,$ respectively.

**Corollary 4.2:** The statement of Theorem 4.1 is also true for orders $d_1$ and $d_2$ satisfying

$$d_1 + d_2 \leq n - 2$$

$$\max \{ d_1, d_2 \} \leq \left\{ \frac{\mu_0 - 1}{2}, \frac{n - n - 1 \mu_0}{2} \right\}.$$  

**Proof:** In all four cases of Theorem 4.1 the above conditions are satisfied.

**Remark 4.3:** Let us discuss the issue of robustness. Certainly the design procedure in Section II is highly sensitive to changes in the system. More specifically, suppose the system $W$ changes to $W + W'$, where $W'$ is a perturbation in a state-space realization of $W$. Then the controller $p_2/q_2$ (designed for $W$) will not cause pole/zero cancellations in $t_1$ and $\Delta t_1$. But our main concern is not these cancellations but is stability of the closed-loop system with both local controllers inserted. And here we have robustness. If $p_2/q_2$ is chosen for $W$ as to achieve the cancellations in $t_1$ and $p_1/q_1$ is chosen to stabilize $t_1$, then the overall closed-loop system consisting of $W, p_2/q_2$, and $p_1/q_1$ is asymptotically stable. Now, since asymptotic stability is a robust property for the above class of perturbations, the perturbed closed-loop system will also be stable for $\Delta W$ small enough. This means that the overall design procedure is robust subject to small changes in the system. Obviously, the same is true for perturbations in the controller. The stability margin in general will be greater when the desired pole-zero cancellations are not close to the imaginary axis. In practice then, one might wish to rule out cancellations in the left half plane with damping ratio less than a prescribed value.

**Remark 4.4:** In [3], for systems of the form

$$y_1 = \frac{n_1}{d_1} (u_1 + k_1 y_2), \quad y_2 = \frac{n_2}{d_2} (u_2 + k_2 y_1),$$

it has been shown that the stable zeros $n_{12}$ can always be cancelled by $p_2/q_2$. The results of this note show that generically one can even cancel the stable zeros of $n_{12}p_2$ by $p_2/q_2$.

**V. CONCLUSIONS**

We have analyzed the task of designing a controller to connect around one channel of a two-channel system to achieve a reduction in the McMillan degree of the resulting one channel system, through the introduction of stable pole-zero cancellations. It turns out that exactly the stable zeros of the cross-coupling transfer function in the system can be cancelled by an appropriate controller. The design task proceeds by solving either linear polynomial equations or a rational interpolation problem. Generically, the controller will be proper and only stable pole/zero cancellations are introduced. However, examples show that in unigenetic situations nonproper controllers might be necessary and that undesired unstable pole/zero cancellations might occur.

For vector channels, some preliminary results have been obtained in the realm of this note, but using a state-space approach, cf. [7]. For systems with more than two channels, the procedures of the note are generally not possible. When they are possible, there is essentially no difference from the two-channel case, see [7].

**REFERENCES**


A Geometric Approach to Eigenstructure Assignment for Singular Systems

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Abstract—Moore's well-known result on the feedback assignment of eigenstructure in linear state-space systems [5] was generalized to descriptor systems in [9]. In this note we make some useful extensions to fill out the theory. To begin with, we do not assume controllability of the plant. It is shown that the maximum number of finite closed-loop eigenvalues is equal to dim (EL*) where L* is the supremal (A, E, R(B))-invariant subspace. In order to relate the possible closed-loop structure to the open-loop plant structure, we define a set of indexes for descriptor systems which are in many respects similar to the controllability indexes of state-space systems. In terms of these "controllability" indexes, we give a systematic approach to selecting the closed-loop eigenvectors.

A major contribution of this note is the fact that it is geometric in nature, and hence avoids any decomposition of the system matrices into a special form (e.g., Weierstrass form [1]). The geometric setting introduced here should constitute a basis for further research in generalizing the well-known results in geometric system theory (disturbance decoupling, input/output decoupling, etc.) to singular systems.

I. INTRODUCTION

In this note, we consider the linear time-invariant descriptor system

\[ DX : E x(t) = Ax(t) + Bu(t) \]  

(1.1)

where \( x(t) \in R^n \), \( u \in R^m \), and \( A, B \) are real matrices with appropriate dimensions. It will be assumed that (1.1) is regular, that is, \( \lambda E - A \neq 0 \) [1]. We shall be concerned with the problem of eigenstructure assignment for \( DX \) using a linear descriptor variable feedback.

We call (1.1) reachable if for all \( z \in R^n \) there is a control \( u(t) \) such that the solution \( x(t) \) is continuously differentiable and satisfies \( x(0) = 0 \), \( x(T) = z \) for some \( T > 0 \) [15]. The reachable subspace \( R \) is the subspace of all \( x(T) \) reachable from \( x(0) = 0 \). We call (1.1) controllable if for all \( z \in R^n \) there is a control \( u(t) \) such that the solution \( x(t) \) is continuously differentiable and satisfies \( x(0) = z \), \( x(T) = 0 \) for some \( T > 0 \). This is equivalent [2] to modal controllability in [16]. The triple \( (E, A, B) \) is reachable if and only if the reachability pencil

\[ R(s) = [sE - A \ B] \]

has no finite zeros and

\[ \text{rank } [E \ B] = n. \]

The triple is controllable if and only if \( R(s) \) has no finite or infinite zeros. \( R(s) \) has no infinite zeros if \( R(\lambda) \) has no zeros at \( \lambda = 0 \).

Let \( \gamma_i \) = \( \text{deg } |E - A| \) and let \( \lambda_i \) be the \( \gamma_i \) th eigenvalues, \( \lambda_i \) be the set of (relative) eigenvalues of \( E \) and \( A \). It suffices for our purposes to consider only the case of distinct \( \lambda_i \)'s. If \( u_i \in \mathcal{C} \) denotes the (relative) eigenvector of \( E \) and \( A \) corresponding to \( \lambda_i \), \( \in \{E(A, E, R(B)) \} \) (i.e., if \( \lambda_i E - A \) \( u_i \) = 0 and \( \neq 0 \)), then \( \{v_{1i}, v_{2i}, \cdots, v_{\gamma_i}\} \) as well as \( \{v_{1i}, v_{2i}, \cdots, v_{\gamma_i}\} \) is linearly independent in \( \mathcal{C} \) [2].

Let a feedback law of the form \( u(t) = F x(t) + z(t) \) with \( F: R^n \rightarrow R^m \) linear and \( z(t) \in R^m \) an auxiliary input be applied to (1.1) to yield

\[ CLDS : E x(t) = (A + BF)x(t) + Bz(t). \]

(1.2)

Instrumental in answering the question of how to choose \( F \) so that \( CLDS \) has two prescribed sets \( \{v_1, v_2, \cdots, v_{\gamma_i}\} \) (where \( p \leq \text{rank } E \)) as its eigenvalues and eigenvectors will be \( L^* \), the supremal \( (A, E, R(B)) \)-invariant subspace of \( R^* \) [3], which is the (unique) subspace to which the subspace recursion

\[ L_{k+1} = A^{-1}[E_L + R(B)] : L_0 = R^* \]

(1.3)

converges. (A superscript \(-1 \) on a linear operator will denote its preimage.) The reader should note that (see [3]): 1) dim \( E \) (E + R(B)) = dim \( L^* \), and 2) \( D S \) is reachable (respectively, controllable) if \( \lambda \) is \( R^* \) (respectively, \( L^* + N(E) \) = \( R^* \)).

Now, let \( S_e = : \lim \{S_t\} \) where

\[ S_{k+1} = E^{-1}[A S_k + R(B)]; S_0 = 0. \]

(1.4)

If \( S_e \neq R^* \), then there exists a nonempty subset \( \sigma_0(E, A) \) of \( \sigma(E, A) \) defined by

\[ \sigma_0(E, A) = \{\lambda \in \sigma(E, A) : \text{rank } [\lambda E - A \ B]<R \}. \]

(1.5)

It follows from the analysis in [3] that the number of elements of \( \sigma_0(E, A) \), i.e., the number of uncontrollable (finite) eigenvalues of \( E \) and \( A \) (counting multiplicities) is dim \( (S_2^*) \) where \( (S_2^*) \) is the orthogonal complement of \( S_e \) in \( R^* \). It will be our standing assumption that \( \sigma_0(E, A) \) consists of distinct elements. It is trivial to show that \( \sigma_0(E, A) \) is closed under complex conjugation.

We may show [2] that reachable subspace \( R = L^* \cap S_e \) and controllable subspace \( C = (L^* + N(E)) \cap S_e \). Thus, (1.1) is reachable (respectively, controllable) if \( R = R^* \) (respectively, \( C = R^* \) [3]. (Note that this seems to be the first characterization of these subspaces in terms of subspace recursions.)

In this note, we show that the maximum number of closed-loop eigenvalues is dim \( EL^* \) and proceed to present necessary and sufficient conditions for two sets, each with dim \( EL^* \) elements, to be assigned as the eigenvalues and eigenvectors of (1.2) using the feedback \( u(t) = Fx(t) + z(t) \). A feedback map \( F \) which accomplishes this task is constructed in Section II. In Section III, a systematic approach to choosing the closed-loop eigenvectors is given by introducing a set of controllability indexes for (1.1). Section IV complements Section III by eliminating the pseudodependence of the procedure discussed in Section III on the Weierstrass canonical form of (1.1). This is accomplished by showing how to generate the controllability indexes using subspace recursions.

We close this section with a few remarks on notation and terminology. \( R \) and \( C \) will denote the fields of real and complex numbers; \( R^* \) and \( C^* \) will denote \( n \)-dimensional real and complex Euclidean spaces, respectively. Subspaces of \( R^* \) and \( C^* \) will be written in boldface. If \( S \) is a subspace, then its dimension will be written as dim \( (S) \). If \( V \) and \( L \) are two subspaces of \( R^* \) (or of \( C^* \) such that \( V \subset L \), then by "component of \( V \) in \( L \)" we mean a subspace \( V' \) of \( L \) satisfying \( L = V + V' \) and \( V' \neq 0 \), i.e., \( L = V + V' \) where \( V' \) denotes direct sum. The range and null spaces of a linear operator \( A \) will be denoted by \( R(A) \) and \( N(A) \), respectively. \( A^T \) denotes the transpose of \( A \). Depending on the context,