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q-Markov covariance equivalent realizations

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This paper describes a class of reduced-order models that match both the first q-Markov parameters and the first q-output covariances for linear systems subject to white noise inputs. The class of reduced models contains the earlier models of Yousuff, Skelton and Wagie (1985) as a special case.

1. Introduction

Let the time-invariant linear stable system

$$\left.\begin{aligned} \dot{x}_1 &= A_1 x_1 + D_1 w, \quad && w \in \mathbf{R}^{n_w} \\ y_1 &= C_1 x_1, && y_1 \in \mathbf{R}^{n_y} \end{aligned}\right\} \tag{1.1}$$

with white noise $w(t)$ with unit intensity, be reduced to

$$\left.\begin{aligned} \dot{x}_2 &= A_2 x_2 + D_2 w \\ y_2 &= C_2 x_2, \quad && y_R \in R^{n_y} \end{aligned}\right\} \tag{1.2}$$

where x_1 is of dimension n and x_2 is of dimension $r < n$.

Definition

If the realization (A_2, D_2, C_2) has these two properties:

$$C_1 A_1^i D_1 = C_2 A_2^i D_2, \qquad i = 0, 1, \ldots, q-1 \tag{1.3 a}$$

$$C_1 A_1^i X_1 C_1^* = C_2 A_2^i X_2 C_2^*, \quad i = 0, 1, \ldots, q-1 \tag{1.3 b}$$

where

$$\left.\begin{aligned} &0 = X_1 A_1^* + A_1 X_1 + D_1 D_1^*, \quad 0 = X_2 A_2^* + A_2 X_2 + D_2 D_2^* \\ &X_1 > 0, \quad X_2 > 0 \end{aligned}\right\} \tag{1.4}$$

the realization (A_2, D_2, C_2) is called a 'q-Markov COVariance Equivalent Realization' of (A_1, D_1, C_1), or simply a 'q-Markov COVER.'

Systems (1.1) and (1.2) have the same first q-Markov parameters and the same output covariance and $(q-1)$ derivatives of output covariance when w is white. When $q = \infty$, the q-Markov COVER is a stochastically equivalent realization in the sense of Anderson (1969), since the entire autocorrelation is matched.

$$\mathbf{E}[y_1(t+\tau) y_1^*(\tau)] = \sum_{i=0}^{\infty} [C_1 A_1^i X_1 C_1^*] \frac{t^i}{i!}$$

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An algorithm is available (Yousuff *et al.* 1985) which constructs a q-Markov COVER for any finite q and this algorithm has been applied to the controller reduction problem of Yousuff and Skelton (1984). This procedure generates a specific member of a broader class of q-Markov COVERs which are described in this paper. Mullis and Roberts (1976) and Inouye (1983) present an approach similar to Levinson's algorithm. Their derivation is more complicated than ours, and they produce just one possibility rather than the broad class defined here.

One motivation for studying the q-Markov COVER problem is made manifest by our first result.

Let the controller (with controller input z and output u)

$$\left.\begin{aligned} u &= Gx_c + G_z z \\ \dot{x}_c &= A_c x_c + Fz \end{aligned}\right\} \tag{1.5}$$

driving the plant (with external input u_c, feedback input u, output y, measurement z, process noise w and measurement noise v)

$$\left.\begin{aligned} \dot{x} &= A_p x + B_p(u + u_c) + D_p w \\ y &= C_p x \\ z &= M_p x + v \end{aligned}\right\} \tag{1.6}$$

yield the closed-loop relation (setting v and w to zero)

$$\begin{bmatrix} y(s) \\ u(s) \end{bmatrix} = [Z(s)]u_c, \quad Z(s) = \sum_{i=0}^{\infty} \frac{\mathbf{CA}^i\mathbf{D}}{s^{i+1}} \tag{1.7}$$

where

$$\mathbf{C} = \begin{bmatrix} C_p & 0 \\ G_z M_p & G \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_p + B_p G_z M_p & B_p G \\ F M_p & A_c \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} B_p \\ 0 \end{bmatrix} \tag{1.8}$$

Theorem 1

Any reduction of any linear controller which preserves the first $(q-1)$-Markov parameters (of the controller) will also preserve the first q-Markov parameters of the closed-loop system between external inputs u_c, feedback inputs u and plant outputs y.

Proof

We must show that any two controllers with the same $q-1$ values of

$$GA_c^iF, \quad i = 0, 1, \ldots, q-2 \tag{1.9}$$

and the same G_z, will yield exactly the same first q values of

$$\mathbf{CA}^i\mathbf{D}, \quad i = 0, 1, \ldots, q-1 \tag{1.10}$$

To see this, note that if the first $q-1$ values of (1.9) are preserved, then so are the first $q-1$ values of

$$A_{12}A_{22}^iA_{21} = B_p G A_c^i F M_p \tag{1.11}$$

The final step is to observe that only the first n columns of A^i (denoted by $(A^i)_1$) are involved in the calculation

$$\mathbf{CA}^i\mathbf{D} = \begin{bmatrix} C_p & 0 \\ G_z M_p & G \end{bmatrix} A^i \begin{bmatrix} B_p \\ 0 \end{bmatrix} = \begin{bmatrix} C_p & 0 \\ G_z M_p & G \end{bmatrix} (A^i)_1 B_p \tag{1.12}$$

and that

$$\begin{bmatrix} C_p & 0 \\ G_z M_p & G \end{bmatrix} \left[\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^i \right]_1 = f(A_{12} A_{22}^j A_{21}), \quad j = 0, 1, 2, \ldots, i-1 \tag{1.13}$$

where the parameters (A_{12}, A_{22}, A_{21}) appear in $f(\cdot)$ *only* in this combination $A_{12}A_{22}^jA_{21}$. Hence, the first q values of $\mathbf{CA}^i\mathbf{D}$, $i = 0, 1, 2, \ldots, q-1$ will be modified by a change in controller iff the controller changes the first $q-1$ values of $A_{12}A_{22}^jA_{21}$, or equivalently, of GA_c^iF, $i = 0, 1, \ldots, q-2$. □

This theorem can also be proven for discrete-time systems. Theorem 1 provides strong motivation for using q-Markov COVER methods for controller reduction since this method preserves a specified number of Markov parameters.

Note that (1.3 *a*) is equivalent to requiring a realization of order r with transfer matrix $G_R(s)$ that

$$\frac{d^i}{ds^i}[G_R(s^{-1})]_{s=0} = \frac{d^i}{ds^i}[G(s^{-1})]_{s=0}, \quad i = 1, \ldots, q \tag{1.14}$$

$$G_R(s) = C_2(sI - A_2)^{-1}D_2 \tag{1.15}$$

$$G(s) = C_1(sI - A_1)^{-1}D_1 \tag{1.16}$$

In addition, the reduced realization (A_2, C_2, D_2) must match the covariance derivatives

$$\frac{d^i}{dt^i} E[y(t+\tau)y^*(\tau)]_{t\to 0+} = \frac{d^i}{dt^i} E[y_R(t+\tau)y^*(\tau)]_{t\to 0+}, \quad i = 0, 1, \ldots, q-1 \tag{1.17}$$

Section 2 defines a new set of coordinates from which one may generate a class of solutions to the q-Markov COVER problem. Section 3 describes a special covariance *control problem* which arises out of the *model reduction* discussion of § 2.

2. Structure of the *q*-Markov COVER problem

We define (A_2, D_2, C_2) to be a *projection* of (A_1, D_1, C_1) iff

$$A_2 = \mathbf{L}A_1\mathbf{R}, \quad D_2 = \mathbf{L}D_1, \quad C_2 = C_1\mathbf{R}, \quad \mathbf{LR} = I \tag{2.1}$$

We may construct an $\mathbf{L}$ and $\mathbf{R}$ satisfying $\mathbf{LR} = I$ by

$$\mathbf{L} = P^{-1}[I \quad 0]T^{-1}, \quad \mathbf{R} = T\begin{bmatrix} I \\ G \end{bmatrix}P \tag{2.2}$$

for some (P, T, G) where the $n \times n$ matrix T serves to place the system (A_1, C_1, D_1) in a convenient basis prior to the projection $\begin{bmatrix} I \\ G \end{bmatrix}[I \quad 0]$, and the $r \times r$ matrix P plays the role of a similarity transformation of the *reduced system*.

Now define the matrices $\mathbf{M}_{iq}$, Θ_{iq}, by

$$\mathbf{M}_{iq} \triangleq \Theta_{iq}[D_i \quad X_i C_i^*], \quad \Theta_{iq}^* \triangleq [C_i^* \quad A_i^* C_i^* \quad \dots \quad A_i^{q-1*} C_i^*] \quad (i=1,2) \tag{2.3}$$

Hence, from our first definition, (A_2, D_2, C_2) is a q-Markov COVER iff

$$\mathbf{M}_{2q} = \mathbf{M}_{1q} \tag{2.4 a}$$

$$\mathbf{M}_{2q} \triangleq \Theta_{2p}[D_2 \quad X_2 C_2^*], \quad \Theta_{2q} \triangleq [C_2^* \quad A_2^* C_2^* \quad \dots \quad A_2^{q-1*} C_2^*] \tag{2.4 b}$$

The matrix $\mathbf{M}_{1q}$ plays an important role in linear systems. Any realization (A_2, D_2, C_2) which preserves the first q-Markov parameters of (A_1, D_1, C_1) satisfies

$$\mathbf{M}_{1q}\begin{bmatrix} I_{n_w} \\ 0 \end{bmatrix} = \mathbf{M}_{2q}\begin{bmatrix} I_{n_w} \\ 0 \end{bmatrix}, \quad I_{n_w} \triangleq \text{identity matrix } (n_w \times n_w) \tag{2.5}$$

and any realization which preserves the first q-output covariances satisfies

$$\mathbf{M}_{1q}\begin{bmatrix} 0 \\ I_{n_y} \end{bmatrix} = \mathbf{M}_{2q}\begin{bmatrix} 0 \\ I_{n_y} \end{bmatrix} \tag{2.6}$$

Hence, the stochastically equivalent realizations of Anderson (1969) satisfy (2.6) for $q=\infty$. Realizations preserving both q-Markov parameters and q covariances satisfy (2.4 a). This implies that if the original system were described by state x, then

$$\left.\begin{aligned} & x = T\mathbf{x}, \quad \mathbf{x}^* \triangleq (\mathbf{x}_R^*, \mathbf{x}_T^*), \quad x^* \triangleq (x_R^* x_T^*), \\ & \begin{bmatrix} \dot{\mathbf{x}}_R \\ \dot{\mathbf{x}}_T \end{bmatrix} = \begin{bmatrix} A_R & A_{RT} \\ A_{TR} & A_T \end{bmatrix}\begin{bmatrix} \mathbf{x}_R \\ \mathbf{x}_T \end{bmatrix} + \begin{bmatrix} D_R \\ D_T \end{bmatrix} w \\ & y = [C_R \quad C_T]\begin{bmatrix} \mathbf{x}_R \\ \mathbf{x}_T \end{bmatrix} \end{aligned}\right\} \tag{2.7}$$

and applying the projection (2.2) to the reduced model

$$\left.\begin{aligned} A_2 &= \mathbf{L}A_1\mathbf{R} = P^{-1}[A_R + A_{RT}G]P = P^{-1}\hat{A}_2 P \\ D_2 &= \mathbf{L}D_1 = P^{-1}D_R = P^{-1}\hat{D}_2 \\ C_2 &= C_1\mathbf{R} = [C_R + C_T G]P = \hat{C}_2 P \end{aligned}\right\} \tag{2.8}$$

which has the transfer function

$$\begin{aligned} G_2(s) &= [C_R + C_T G][sI - (A_R + A_{RT}G)]^{-1}D_R \\ &= \hat{C}_2(sI - \hat{A}_2)^{-1}\hat{D}_2 = C_2(sI - A_2)^{-1}D_2 \end{aligned} \tag{2.9}$$

where (2.8) defines $(\hat{A}_2, \hat{D}_2, \hat{C}_2)$ to be the reduced model when $P = I$. The effect of P on the state covariance of the reduced model is clearly present,

$$X_2 = P^{-1}\hat{X}_2 P^{-*} \tag{2.10}$$

where $\hat{X}_2$ is the state covariance of $(\hat{A}_2, \hat{D}_2)$. Of course, P has *no* effect on the matrix $\mathbf{M}_{2q}$ and hence on our ability to match $\mathbf{M}_{1q}$. To wit

$$\mathbf{M}_{2q} = \Theta_{2q}[D_2 \quad X_2 C_2^*] = \hat{\Theta}_{2q} P[P^{-1}\hat{D}_2 \quad P^{-1}\hat{X}_2 P^{-*} P^* \hat{C}_2^*] = \hat{\Theta}_{2q}[\hat{D}_2 \quad \hat{X}_2 \hat{C}_2^*] \tag{2.11}$$

However, P plays a role in the choice of coordinates of the reduced model and hence we note its presence here. Coordinate transformations T on the higher dimensional model have no effect on $\mathbf{M}_{1q}$. Hence,

$$\mathbf{M}_{1q}=\Theta_{1q}[D_1 \quad X_1C_1^*]=\hat{\Theta}_{1q}[\hat{D}_1 \quad \hat{X}_1\hat{C}_1^*]=\hat{\mathbf{M}}_{1q}$$

where

$$\hat{\Theta}_{1q}=\Theta_{1q}T, \quad \hat{X}_1=T^{-1}X_1T^{-*}, \quad \hat{D}_1=T^{-1}D_1, \quad \hat{C}_1=C_1T$$

More specific results can be stated if we *begin* with the following special coordinates, which we define as q-normalized Hessenberg coordinates.

$$\hat{A}_1=T^{-1}A_1T=\begin{bmatrix} A_R & A_{RT} \\ A_{TR} & A_T \end{bmatrix}$$

$$\hat{C}_1=C_1T=[C_{11} \quad 0 \quad 0]=[C_R \quad C_T], \quad |C_{11}|\neq 0 \tag{2.12 a}$$

$$\hat{D}_1=T^{-1}D_1=\begin{bmatrix} D_R \\ D_T \end{bmatrix}$$

$$A_R=\begin{bmatrix} A_{11} & A_{12} & 0 & 0 & 0 & \cdots & 0 \\ A_{21} & A_{22} & A_{23} & 0 & 0 & \cdots & 0 \\ A_{31} & A_{32} & A_{33} & A_{34} & 0 & \cdots & 0 \\ \vdots & & & & & & 0 \\ A_{q-1,1} & & & & & & A_{q-1,q} \\ A_{q1} & & & & & \cdots & A_{qq} \end{bmatrix}, \quad A_{RT}=\begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ A_{q,q+1} & \cdots & A_{q,n} \end{bmatrix}$$

A_{TR}, A_T have no structural constraints, save the property

$$\hat{X}_1 \triangleq T^{-1}X_1T^{-*}=I \tag{2.12 b}$$

and

$$\operatorname{rank} A_{i,i+1}=n_{i+1}\leqslant n_i, \quad i=1,2,\ldots,q-1 \tag{2.12 c}$$

$$\sum_{i=1}^{p} n_i=n, \quad n_1=n_y, \quad A_{11}\in\mathbf{R}^{n_1\times n_1}, \quad r=\sum_{i=1}^{q} n_i, \quad A_R\in\mathbf{R}^{r\times r}$$

To simplify the notation, we define

$$B_q \triangleq [A_{q,q+1} \quad \cdots \quad A_{q,n}] \tag{2.12 d}$$

The form (2.12), including the $\hat{X}_1=I$ property, may be constructed from (A_1, C_1, D_1) in steps (setting the row block of zeros after A_{12} in A on Step 1 and the row blocks of zeros after A_{23} on Step 2, etc.) to give the form (2.12). An algorithm is given in the Appendix of Yousuff and Skelton (1984). This form is slightly less restrictive than block Hessenberg (since rows of zeros are assigned after the blocks $A_{i,i+1}$ only up to $i=q-1$. However, we simply stop the block Hessenberg construction of Yousuff and Skelton (1984) after Step $q-1$, not wishing to require the entire matrix $\hat{A}_1=T^{-1}A_1T$ to be block Hessenberg, only the first $q-1$ block of rows.

In these coordinates (2.12), note that if (A_R, D_R) are partitioned as follows:

$$A_R = \begin{bmatrix} A_{11} & \Gamma & 0 \\ \Psi & \Lambda & \Phi \\ A_{q1} & \Xi & A_{qq} \end{bmatrix} \quad D_R = \begin{bmatrix} D_a \\ D_b \\ D_q \end{bmatrix} \tag{2.12 e}$$

where A_{11} is $n_1 \times n_1$, Γ is $n_1 \times p$, Ψ is $p \times n_1$, Λ is $p \times r$, Φ is $p \times n_q$, A_{q1} is $n_q \times n_1$, Ξ is $n_q \times p$, A_{qq} is $n_q \times n_q$, and $p \triangleq n_2 + \ldots + n_{q1}$. Then the first q upper left blocks $n_1, n_2, \ldots, n_q$ of

$$0 = \hat{A}_1 \hat{X}_1 + \hat{X}_1 \hat{A}_1^* + \hat{D}_1 \hat{D}_1^*, \quad \hat{X}_1 = I \tag{2.12 f}$$

satisfy

$$0 = A_{11} + A_{11}^* + D_a D_a^* \tag{2.12 g}$$

$$0 = \Gamma + \Psi^* + D_a D_b^*, \quad \Gamma \triangleq [A_{12} \quad 0] \tag{2.12 h}$$

$$0 = A_{q1}^* + D_a D_q^* \tag{2.12 i}$$

$$0 = \Lambda + \Lambda^* + D_b D_b^* \tag{2.12 j}$$

$$0 = \Phi + \Xi^* + D_b D_q^*, \quad \Phi^* \triangleq [0 \quad A_{q-1,q}^*] \tag{2.12 k}$$

$$0 = A_{qq} + A_{qq}^* + D_q D_q^* \tag{2.12 l}$$

These expressions will be needed later.

Define the projection (2.8)

$$\left.\begin{aligned} &\mathbf{L}\hat{A}_1\mathbf{R} = A_2 \triangleq P^{-1}[A_R + BG]P = P^{-1}\hat{A}_2 P, \quad B \triangleq A_{RT} = \begin{bmatrix} 0 \\ B_q \end{bmatrix} \\ &\mathbf{L}\hat{D}_1 = D_2 \triangleq P^{-1} D_R = P^{-1}\hat{D}_2 \\ &\hat{C}_1 \mathbf{R} = C_2 \triangleq [C_R + C_T G]P = \hat{C}_2 P, \quad \hat{C}_2 = C_R + C_T G = [C_{11} \quad 0] \end{aligned}\right\} \tag{2.13}$$

Lemma 1

The coordinates (2.12) and the projection (2.13) have these properties:

(i) $\hat{\Theta}_{1q} = [\hat{\Theta}_{2q} \quad 0]$, $\hat{\Theta}_{2q}^* \triangleq [\hat{C}_2^* \quad \hat{A}_2^* \hat{C}_2^* \quad \ldots \quad \hat{A}_2^{q-1*} \hat{C}_2^*]$ (2.14)

(ii) $C_R A_R^i B = 0, \quad i = 0, 1, \ldots, q-1$ (2.15)

(iii) $\hat{\Theta}_{2q}$ is independent of the choice G

Proof

(i) By construction

$$\hat{\Theta}_{1q} = \begin{bmatrix} C \\ \vdots \\ CA^{q-1} \end{bmatrix} = \begin{bmatrix} d_1 & 0 & 0 & \ldots & 0 & 0 \\ x & d_2 & 0 & \ldots & 0 & 0 \\ x & x & d_3 & \ldots & 0 & 0 \\ \vdots & & & & \vdots & \vdots \\ x & & & \ldots & d_q & 0 \end{bmatrix} = [\hat{\Theta}_{2q} \quad 0]$$

where the matrices

$$d_2 \triangleq C_{11}A_{12}, \quad d_1 = C_{11}$$
$$d_3 \triangleq C_{11}A_{12}A_{23}$$
$$\vdots$$
$$d_q \triangleq C_{11}A_{12}A_{23}A_{34} \dots A_{q-1,q}$$

all have *full column rank* by property (2.12 *c*). Hence the rank of $\hat{\Theta}_{1q}$ is equal to the rank of $\hat{\Theta}_{2q}$ which is $(n_1 + n_2 + \dots + n_q) \triangleq r$, the order of the reduced model. (ii) and (iii) both follow from

$$\hat{\Theta}_{2q} = \begin{bmatrix} [C_{11} \quad 0] \\ [C_{11} \quad 0][A_R + BG] \\ \vdots \\ [C_{11} \quad 0][A_R + BG]^{q-1} \end{bmatrix} = \begin{bmatrix} [C_{11} \quad 0] \\ [C_{11} \quad 0][A_R] \\ \vdots \\ [C_{11} \quad 0]A_R^{q-1} \end{bmatrix}$$

where the essential property is

$$C_R A_R^i B = [C_{11} \quad 0 \quad 0 \quad 0] \begin{bmatrix} A_{11} & A_{12} & 0 & \dots & 0 \\ A_{21} & A_{22} & A_{23} & \dots & 0 \\ A_{31} & A_{32} & A_{33} & \dots & \vdots \\ \vdots & \vdots & \vdots & & A_{q-1,q} \\ A_{q1} & A_{q2} & A_{q3} & \dots & A_{qq} \end{bmatrix}^i \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ B_q \end{bmatrix} = 0$$

$$i = 0, 1, \dots, q\text{-}1$$

Lemma 2

The projection defined by (2.13), (2.12) matches the first q-Markov parameters for *any* choice of G.

Proof

From (2.7) we require

$$\hat{\mathbf{M}}_{1q}\begin{bmatrix} I_{n_w} \\ 0 \end{bmatrix} = \hat{\Theta}_{2q}\hat{D}_1 = [\hat{\Theta}_{2q} \quad 0]\begin{bmatrix} D_R \\ D_T \end{bmatrix} = \hat{\Theta}_{2q}D_R = \hat{\Theta}_{2q}\hat{D}_2 = \hat{\mathbf{M}}_{2q}\begin{bmatrix} I_{n_w} \\ 0 \end{bmatrix} \qquad (2.16)$$

where property (i) of Lemma 1 is used. □

Thus, the matching of q-Markov parameters is automatic (for all G) by the *structure* of (2.12). This now leaves only the task of matching covariances by choice of G.

Problem restatement

Find G such that $(A_R + BG, D_R)$ has state covariance $\hat{X}_2$ satisfying

$$\hat{\Theta}_{1q}\hat{X}_1\hat{C}_1^* = \hat{\Theta}_{2q}\hat{X}_2\hat{C}_2^* \qquad (2.17)$$

But from (2.14), (2.12) the left-hand-side becomes

$$\hat{\Theta}_{1q}\hat{X}_1\hat{C}_1^* = [\hat{\Theta}_{2q} \quad 0]\begin{bmatrix} X_a & X_b \\ X_c & X_d \end{bmatrix}\begin{bmatrix} C_{11}^* \\ 0 \end{bmatrix} = \hat{\Theta}_{2q}X_aC_{11}^* \tag{2.18}$$

where X_a is the upper left $r \times n_y$ block of matrix $\hat{X}_1$. Now since the columns of $\hat{\Theta}_{2q}$ are linearly independent and the rows of C_{11}^* are linearly independent, we have $\hat{\Theta}_{1q}\hat{X}_1\hat{C}_1^* = \hat{\Theta}_{2q}\hat{X}_2\hat{C}_2^*$ or

$$\hat{\Theta}_{2q}X_aC_{11}^* = \hat{\Theta}_{2q}\hat{X}_2\hat{C}_2^* = \hat{\Theta}_{2q}[\hat{X}_{R1} \quad \hat{X}_{R2}]\begin{bmatrix} C_{11}^* \\ 0 \end{bmatrix} = \hat{\Theta}_{2q}\hat{X}_{R1}C_{11}^* \tag{2.19}$$

iff

$$\hat{X}_{R1} = X_a \tag{2.20}$$

This proves the following.

Lemma 3

The projection (2.13) from coordinates (2.12) yields a q-Markov COVER of $(\hat{A}_1, \hat{D}_1, \hat{C}_1)$ iff there exists a G such that $(A_R + BG, D_R)$ has state covariance $\hat{X}_2$ which takes on the preassigned value for the first n_y columns (and hence also rows, by symmetry),

$$\hat{X}_{R1} = X_a$$

3. A covariance assignment problem

We now seek to solve the problem suggested by Lemma 3. This is recognized as a mathematical problem in covariance assignment using state feedback control. Such problems were introduced in Hotz and Skelton (1985) and Collins and Skelton (1985) for *control* design purposes, and we shall need one of those results here for our *model reduction* problem.

Define E_B such that

$$E_B^*BB^+E_B = \begin{bmatrix} I_{r_b} & 0 \\ 0 & 0 \end{bmatrix}, \quad E_B^*E_B = I_r, \quad \text{rank } B = r_b \tag{3.1}$$

where B^+ is the Moore–Penrose inverse of B, and I_r denotes the identity matrix of size rr. Now the needed result from Hotz and Skelton (1985) is as follows.

Theorem 2 (Hotz and Skelton 1985)

Let $(E_B^*QE_B)_{22}$ denote the lower right $(r - r_b) \times (r - r_b)$ block of matrix $E_B^*QE_B$, and define

$$Q \triangleq A_R\bar{X}_2 + \bar{X}_2A_R^* + D_RD_R^* \tag{3.2}$$

There exists a G such that $(A_R + BG,\ D_R)$ has a specified state covariance $\hat{X}_2 = \bar{X}_2 > 0$ if and only if $(A_R + BG,\ D_R)$ is controllable and

$$(E_B^*QE_B)_{22} = 0 \tag{3.3}$$

The class of all non-trivial solutions (define trivial ones such that $BG = 0$) which exist are given by

$$G=-\tfrac{1}{2}B^{+}(Q+E_BSE_B^*)\bar{X}_2^{-1},\quad S=\begin{array}{c}\left[\begin{array}{cc} S_{11} & S_{12} \\ -S_{12}^* & 0 \end{array}\right] \\ \begin{array}{cc} \quad r_b & \quad r-r_b \end{array}\end{array} \tag{3.4}$$

subject to (A_R+BG, D_R) controllable, where

$$S_{12}=-(E_B^*QE_B)_{12},\quad E_B^*QE_B=\begin{array}{c}\left[\begin{array}{cc} ()_{11} & ()_{12} \\ ()_{12}^* & ()_{22} \end{array}\right] \\ \begin{array}{cc} \quad r_b & \quad r-r_b \end{array}\end{array} \tag{3.5}$$

and where S_{11} is an arbitrary skew symmetric matrix. Note that (A_R+BG) will be asymptotically stable if (A_R+BG, D_R) is controllable since $\bar{X}_2>0$.

Theorem 2 holds for any (A_R, B, D_R), but we wish now to specialize Theorem 2 for our problem. This is accomplished by the use of Lemma 3, which requires the choice of $\bar{X}_2$ in our use of Theorem 2 to be

$$\bar{X}_2=\left[\begin{array}{cc} I_{n_1} & 0 \\ 0 & X_f \end{array}\right],\quad X_f=X_f^*>0 \tag{3.6}$$

where X_f is arbitrary. In other words, in the special coordnates (2.12), X_a in (2.20) is

$$X_a=\left[\begin{array}{c} I_{n_1} \\ 0 \end{array}\right] \tag{3.7}$$

Next, to simplify our presentation, assume rank $B=r_b=n_q$. (This assumption is not necessary for application of the concepts.) Then

$$BB^{+}=\left[\begin{array}{cc} 0 & 0 \\ 0 & B_qB_q^{+} \end{array}\right]=\left[\begin{array}{cc} 0 & 0 \\ 0 & I_{n_q} \end{array}\right] \tag{3.8}$$

$$E_B=\left[\begin{array}{cc} 0 & I_{r-n_q} \\ I_{n_q} & 0 \end{array}\right] \tag{3.9}$$

and (3.3) becomes

$$Q_{11}\triangleq(A_R\bar{X}_2+\bar{X}_2A_R^*+D_RD_R^*)_{11}=0 \tag{3.10 a}$$

where $Q_{11}\in\mathbf{R}^{(r-n_q)\times(r-n_q)}$. Now in order to exploit the freedom in the choice of X_f, we proceed to use the partitioned notation of (2.12 e) to write

$$A_R\bar{X}_2=\left[\begin{array}{ccc} A_{11} & \Gamma & 0 \\ \Psi & \Lambda & \Phi \\ A_{q1} & \Xi & A_{qq} \end{array}\right]\left[\begin{array}{ccc} I_{n_1} & 0 & 0 \\ 0 & X_{f11} & X_{f12} \\ 0 & X_{f12}^* & X_{f22} \end{array}\right]\begin{array}{l} \\ \\ \}n_q \end{array} \tag{3.11}$$

where $\Phi^*=[0 \quad A_{q-1,q}^*]$.

Hence (3.10) becomes the following three partitioned equations:

$$0=A_{11}+A_{11}^*+D_aD_a^* \tag{3.10 b}$$

$$0=\Gamma X_{f11}+\Psi^*+D_aD_b^* \tag{3.10 c}$$

$$0=\Lambda X_{f11}+X_{f11}\Lambda^*+\Phi X_{f12}^*+X_{f12}\Phi^*+D_bD_b^* \tag{3.10 d}$$

But from (2.12 *g*), (3.10 *b*) is automatically satisfied and (2.12 *h*) reduces (3.10 *c*) to

$$0=\Gamma[X_{f11}-I] \tag{3.12 a}$$

and (2.12 *j*) reduces (3.10 *d*) to

$$0=\Lambda[X_{f11}-I]+[X_{f11}-I]\Lambda^*+\Phi X_{f12}^*+X_{f12}\Phi^* \tag{3.12 b}$$

Hence (3.10 *a*) holds if and only if (3.12) holds. Using (3.9) and (3.11), (3.5) becomes

$$S_{12}=-Q_{12}^* \tag{3.13}$$

Finally, (3.4) reduces (using (3.9), (3.11)) to

$$G=-\tfrac{1}{2}B_q^+[Q_{22}+S_{11}][X_{f22}-X_{f12}^*X_{f11}^{-1}X_{f12}]^{-1}[O_{n_1} \quad -X_{f12}^*X_{11}^{-1} \quad I_{n_q}] \tag{3.14 a}$$

$$Q_{22}=\Xi X_{f12}+X_{f12}^*\Xi^*+A_{qq}[X_{f22}-I]+[X_{f22}-I]A_{qq}^* \tag{3.14 b}$$

These results may now be summarized. In the following theorem we consider the system $(\hat{A}_1, \hat{D}_1, \hat{C}_1, \hat{X}_1)$ described by (2.12).

Definition

An 'admissible' state covariance of a q-Markov COVER is defined by (3.6) and (3.12) described in terms of coordinates (2.12).

Theorem 3

A q-Markov COVER $(\hat{A}_2, \hat{D}_2, \hat{C}_2, \hat{X}_2)$ of $(\hat{A}_1, \hat{D}_1, \hat{C}_1, \hat{X}_1)$ exists if $\hat{X}_2$ is 'admissible'. If $\hat{X}_2=\bar{X}_2$ is admissible, then many q-Markov COVERs may exist satisfying

$$\hat{A}_2=A_R+BG, \quad \hat{D}_2=D_R, \quad \hat{C}_2=C_R \tag{3.15 a}$$

$$\hat{X}_2=\bar{X}_2=\begin{bmatrix} I_{n_1} & 0 \\ 0 & X_f \end{bmatrix}, \quad X_f=\begin{bmatrix} X_{f11} & X_{f12} \\ X_{f12}^* & X_{f22} \end{bmatrix}\begin{matrix} \}r-n_1-n_q \\ \}n_q \end{matrix} \tag{3.15 b}$$

where G is given by (3.14), for arbitrary values of the skew-symmetric matrix $S_{11}=-S_{11}^*$, and for any admissible values of $X_{f11}, X_{f12}, X_{f22}$.

Corollary 1

The 1-Markov COVER is unique within a similarity transformation P, and is given by (3.15 *a*) with $\hat{X}_2=\bar{X}_2=I_n$, $G=0$, and the reduced model is stable.

Corollary 2

For stable systems, q-Markov COVERs always exist and the choice $\hat{X}_2=\bar{X}_2=I_r$ is always admissible.

Corollary 3

If $\lambda_i[A_{qq}]+\lambda_j[A_{qq}]\neq 0$ for all i, j, then q-Markov COVERs with the admissible choice $\hat{X}_2=\bar{X}_2=I_r$, are given by (3.15 *a, b*) with

$$G=-\tfrac{1}{2}B_q^+S_{11}[0 \quad 0 \quad I_{n_q}] \tag{3.16}$$

and the reduced model is stable by choosing S_{11} such that $(A_R + BG, D_R)$ is controllable. This is always possible.

Proofs

Corollary 1 follows by setting $r = n_1 = n_y$ in Theorem 3. Corollary 2 follows by noting that $X_f = I$ satisfies the admissible conditions (3.12). Corollary 3 is verified by substituting $X_f = I$ in (3.14). Since (A_R, D_R) is controllable by the structure of (2.12) there is at least one choice $(S_{11} = 0)$ which makes $(A_R + BG, D_R)$ controllable. □

Theorem 3 provides the *explicit* structure of the q-Markov COVER problem and allows the designer much freedom in their construction. Specifically, the free parameters (S_{11}, X_f, P) are available to accomplish other objectives. We shall need this freedom to obtain a q-Markov COVER with desirable robustness properties. This is the subject of a future paper.

Example 1

Find all 1-Markov COVERs of

$$G(s) = \frac{\alpha s + 1}{(s+\beta)(s+\gamma)}, \quad \beta > 0, \quad \gamma > 0$$

Solution

Begin with the representation $(\hat{A}_1, \hat{D}_1, \hat{C}_1)$

$$\hat{A}_1 = \begin{bmatrix} -\dfrac{\alpha^2\beta\gamma(\beta+\gamma)}{1+\alpha^2\beta\gamma} & \dfrac{\sqrt{\beta\gamma}(1+\alpha(\alpha\beta\gamma-\beta-\gamma))}{1+\alpha^2\beta\gamma} \\ -\dfrac{\sqrt{\beta\gamma}(1+\alpha(\alpha\beta\gamma+\beta+\gamma))}{1+\alpha^2\beta\gamma} & \dfrac{-(\beta+\gamma)}{1+\alpha^2\beta\gamma} \end{bmatrix}$$

$$\hat{D}_1 = \begin{bmatrix} \alpha\sqrt{\dfrac{2\beta\gamma(\beta+\gamma)}{1+\alpha^2\beta\gamma}} \\ \sqrt{\dfrac{2(\beta+\gamma)}{1+\alpha^2\beta\gamma}} \end{bmatrix}$$

$$\hat{C}_1 = \begin{bmatrix} \sqrt{\dfrac{1+\alpha^2\beta\gamma}{2\beta\gamma(\beta+\gamma)}} & 0 \end{bmatrix}$$

since in this representation $\hat{X}_1 = I$ and (2.12) is satisfied. From Corollary 1, the 1-Markov COVER is unique to within a similarity transformation:

$$B = B_q = \frac{\sqrt{\beta\gamma}(1+\alpha^2\beta\gamma-\alpha(\beta+\gamma))}{1+\alpha^2\beta\gamma}, \quad G = 0 \quad \text{(Corollary 1)}$$

$$A_2 = \hat{A}_2 = -\frac{\alpha^2\beta\gamma(\beta+\gamma)}{1+\alpha^2\beta\gamma}, \quad D_2 = P^{-1}\hat{D}_2 = P^{-1}\alpha\sqrt{\left[\frac{2\beta\gamma(\beta+\gamma)}{1+\alpha^2\beta\gamma}\right]}$$

$$C_2 = \hat{C}_2 P = P\sqrt{\left[\frac{1+\alpha^2\beta\gamma}{2\beta\gamma(\beta+\gamma)}\right]}$$

$$\hat{X}_2 = \bar{X}_2 = 1, \quad X_2 = P^{-2} \quad \text{for any} \quad P \neq 0$$

For existence we require (3.12), which is automatically satisfied because $\bar{X}_2 = I_{n_1}$ and Γ, Λ and X_f are all void in this problem. The transfer function of the 1-Markov COVER is

$$G_R(s) = \left[\frac{\alpha}{s + \dfrac{\alpha^2\beta\gamma(\beta+\gamma)}{1+\alpha^2\beta\gamma}} \right]$$

Example 2

Find all 1-Markov COVERS of stable third-order systems.

Solution

All stable third-order systems may be transformed to coordinates (2.12), in which case the parameters are

$$\hat{A}_1 = \begin{bmatrix} -d_1^2 & -2d_1d_2 - d_4 & 0 \\ d_4 & -d_2^2 & -2d_2d_3 - d_5 \\ -2d_1d_3 & d_5 & -d_3^2 \end{bmatrix}, \quad \hat{D}_1 = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \sqrt{2}$$

$$\hat{C}_1 = [d_6 \quad 0 \quad 0], \quad \hat{X}_1 = I$$

From Corollary 1, all 1-Markov COVERs are similar to

$$\hat{A}_2 = -d_1^2, \quad \hat{D}_2 = d_1\sqrt{2}, \quad \hat{C}_2 = d_6, \quad \hat{X}_2 = 1$$

Example 3

Find all 2-Markov COVERs of stable third-order systems.

Solution

Existence is guaranteed by (3.12), since X_{f22}, X_{f12}, Ξ and Φ are void and the admissible $\bar{X}_2 = \begin{bmatrix} 1 & 0 \\ 0 & X_{f11} \end{bmatrix}$, where $X_{f11} = 1$ from (3.12 *b*). From (3.14),

$$Q_{22} = 2A_{qq}(X_{f11} - 1) = 0, \quad S_{11} = 0 \text{ (since } S_{11} \text{ is scalar)}$$

Hence $G = 0$, and

$$\hat{A}_2 = A_R + BG = \begin{bmatrix} -d_1^2 & -2d_1d_2 - d_4 \\ d_4 & -d_2^2 \end{bmatrix}$$

$$\hat{D}_2 = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \sqrt{2}, \quad \hat{C}_2 = (d_6 \quad 0), \quad \hat{X}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The transfer function of the 2-Markov COVER is

$$G_R(s) = \hat{C}_2(sI - \hat{A}_2)^{-1}\hat{D}_2 = \frac{\sqrt{2}[(s+d_2^2)d_1 + d_2(2d_1d_2 + d_4)]d_6}{(s+d_1^2)(s+d_2^2) + (2d_1d_2 + d_4)d_4}$$

The transfer function of the original system is $G(s)=N(s)/D(s)$, where

$$N(s)\triangleq(\sqrt{2})\{[(s+d_2^2)(s+d_3^2)+d_5(2d_2d_3+d_5)]d_1-(s+d_3^2)d_2(2d_1d_2+d_4)\}d_6$$
$$+(\sqrt{2})d_3(2d_1d_2+d_4)(2d_2d_3+d_5)d_6$$

$$D(s)\triangleq(s+d_1^2)(s+d_2^2)(s+d_3^2)+(s+d_3^2)(d_4+2d_1d_2)d_4+(s+d_1^2)(d_5+2d_2d_3)d_5$$
$$+2d_1d_3(sd_1d_2+d_4)(2d_2d_3+d_5)$$

Finally, in this section we must show how the class of q-Markov COVERs defined in this paper relates to the specific q-Markov COVER derived earlier by Yousuff *et al.* (1985).

Theorem 4

The particular q-Markov COVER of $(\hat{A}_1, \hat{D}_1, \hat{C}_1)$ defined by the projection

$$\hat{A}_2=\mathbf{L}\hat{A}_1\mathbf{R},\quad \hat{D}_2=\mathbf{L}\hat{D}_1,\quad \hat{C}_2=\hat{C}_1\mathbf{R} \tag{3.17 a}$$

$$\mathbf{L}\triangleq U_1^*\hat{\Theta}_{1q},\quad \mathbf{R}\triangleq\hat{X}_1\mathbf{L}^*(\mathbf{L}\hat{X}_1\mathbf{L}^*)^{-1} \tag{3.17 b}$$

$$\hat{\Theta}_{1q}=[U_1\quad U_2]\begin{bmatrix}\Sigma & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}V_1^*\\ V_2^*\end{bmatrix} \tag{3.17 c}$$

is related by a similarity transformation P to that special q-Markov COVER in Theorem 3, which is associated with the values $X_f=I$, $G=0$. The similarity transformation that relates them is $P=[U_1^*\quad \hat{\Theta}_{2q}]^{-1}$.

Proof

It is proven in Yousuff *et al.* (1985) that (3.17) is related by a similarity transformation P to a *truncation* of the cost decoupled coordinates (2.12). A *truncation* of (2.12) is equivalent to (2.13) with $G=0$. To show the transformation, write the singular value decomposition of $\hat{\Theta}_{1q}$. From Lemma 1

$$\hat{\Theta}_{1q}=[\hat{\Theta}_{2q}\quad 0]=[U_1\quad U_2]\begin{bmatrix}\Sigma & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}V_1^*\\ V_2^*\end{bmatrix}$$

and hence by defining $P^{-1}\triangleq U_1^*\hat{\Theta}_{2q}$, (3.17 *b*) becomes

$$\mathbf{L}=U_1^*\hat{\Theta}_{1q}=\Sigma V_1^*=U_1^*[\hat{\Theta}_{2q}\quad 0]=P^{-1}[I\quad 0]$$

$$\mathbf{R}=\begin{bmatrix}I\\ 0\end{bmatrix}P^{-*}\left[P^{-1}[I\quad 0]\begin{bmatrix}I\\ 0\end{bmatrix}P^{-*}\right]^{-1}=\begin{bmatrix}I\\ 0\end{bmatrix}P$$

which is precisely the projection (2.2) used to develop the results of § 4 if $G=0$ and $T=I$ (since we started with coordinates (2.12)). □

Conclusions

A class of reduced-order models is described which preserve the first q-Markov parameters and the first q-output covariances. Results published earlier are shown to be special cases obtained by a similarity transformation of these more general results. The explicit formulae describing the freedom in the reduced models may be used to assign still other properties to the reduced-order model, such as the treatment of nonwhite noise inputs and robustness with respect to parameter variations. These topics will follow in future work.

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