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Controller reduction via stable factorization and balancing

YI LIU† and BRIAN D. O. ANDERSON†

A controller reduction procedure based on a representation of a controller as a matrix function defined using stable proper transfer functions and employing a balancing technique is studied in this paper. For a certain right coprime factorization of an LQG designed controller $K(s) = N(s)D^{-1}(s)$, we approximate using a balancing technique the pair $[D(s), N(s)]^{\mathrm{T}}$ by a low-order pair $[D_1(s), N_1(s)]^{\mathrm{T}}$ defining a factorization of the reduced-order controller $K_1(s) = N_1(s)D_1^{-1}(s)$. We show that reducing the controller order in this way is motivated in a natural way, which leads to the expectation of both good stability properties and good accuracy of approximation of closed-loop behaviour. This is also demonstrated in some examples.

1. Introduction

For physical systems described by high-order linear time-invariant state space models, controller design procedures will often lead to high-order compensators. For various reasons, reduction of the order of the compensator may be desirable. For example, in order to use fewer components or computing resources in an implementation, or to obtain higher reliability of an implementation, we need a lower-order controller.

There are two very important compensator design techniques which lead to high-order compensators for high-order plants. One is linear quadratic gaussian (LQG) design, which usually leads to a compensator with order roughly the same as the order of the original system to be controlled (Anderson and Moore 1971). Another is the H_∞ optimization technique, which may lead to a compensator with order as high as six times the order of the original system (Doyle, private communication, 1985). We focus in this paper on a controller reduction technique for controllers obtained via LQG design.

There are two general points which should be made concerning controller reduction. First, a choice presents itself between either approximating the original plant by one of lower order, doing a normal controller design for this lower-order plant and trying the result out on the original system, or designing the controller for the original plant and then approximating the controller. The second approach has the theoretical advantage that the approximation step is delayed and there is less propagation of the effects of approximation through the design procedure. Second, because controller and plant are connected in closed loop, the question of what constitutes a good approximation to a high-order controller is one that should not be answered without consideration of the presence of the plant being controlled. Generally speaking, one needs better accuracy of approximation of the full-order controller by the reduced-order controller at frequencies near the unity gain crossover frequencies of the loop transfer function or its singular values.

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† Department of Systems Engineering, Research School of Physical Sciences, Australian National University, P.O. Box 4, Canberra, A.C.T. 2600, Australia.

The new method of controller reduction presented in this paper follows the path of first designing a high-order controller and then approximating it. Also, the method reflects the presence of the plant being controlled, at least indirectly.

Tools used in the method include representation of the controller using coprime matrix fractions of stable proper matrices (Vidyasagar 1985b), and approximation using balanced realizations (Moore 1981); these tools allow the determination of an error bound between the full-order controller (designed by the LQG procedure) and the reduced-order controller, and the bound allows consideration of stability and the 'closeness' of the closed-loop performances. In the next section, we discuss as a basis for comparison various alternative approaches to controller reduction. In § 3 we present a justification for the new reduction method and its algorithm, and in § 4, some examples which compare our method with others. Conclusions and remarks will be found in § 5.

Before proceeding further, we introduce some notation and definitions.

The system to be controlled is linear, time-invariant and minimal and given in state space form:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^l$, $y \in \mathbb{R}^m$; there is also given a quadratic index

$$J = \frac{1}{2}\int_0^\infty (x^\mathrm{T}Qx + u^\mathrm{T}Ru)\,dt \tag{1.2}$$

where $Q \geqslant 0$, $R > 0$ and $[A, Q^{1/2}]$ is observable.

To minimize the index J, we use the control law $u = -Fx$, with control gain F taken as $R^{-1}B^\mathrm{T}P_\mathrm{c}$, where P_c is a positive definite matrix which satisfies the Riccati equation

$$P_\mathrm{c}A + A^\mathrm{T}P_\mathrm{c} - P_\mathrm{c}BR^{-1}B^\mathrm{T}P_\mathrm{c} + Q = 0 \tag{1.3}$$

If we cannot measure the states directly, and in the presence of input noise $w(t)$ and measurement noise $v(t)$, eqn. (1.1) becomes

$$\begin{aligned} \dot{x} &= Ax + Bu + w(t) \\ y &= Cx + v(t) \end{aligned} \tag{1.4}$$

where $v(t)$ and $w(t)$ are white noise processes and independent, with the covariances $V\delta(t-\tau) > 0$ and $W\delta(t-\tau) \geqslant 0$, respectively, and with $[A, W^{1/2}]$ controllable. Then, we use the Kalman filter to obtain an estimate of the state $\hat{x}$ in order to implement the state feedback control law as $u = -F\hat{x}$.

The estimation equation is

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(y - \hat{y}) \\ \hat{y} &= C\hat{x} \end{aligned} \tag{1.5}$$

where the estimator gain $L = P_\mathrm{f}C^\mathrm{T}V^{-1}$, and P_f is positive definite and satisfies another Riccati equation

$$AP_\mathrm{f} + P_\mathrm{f}A^\mathrm{T} - P_\mathrm{f}C^\mathrm{T}V^{-1}CP_\mathrm{f} + W = 0$$

Thus, we obtain an n^th-order dynamical compensator with the transfer function

$$K(s) = F(sI_n - A + BF + LC)^{-1}L$$

This is the standard LQG design procedure. In this paper, we focus on the reduction problem for the full-order controller designed by this procedure.

Now we summarize some definitions concerning stable factorization of a transfer function or transfer function matrix (Vidyasagar 1985b).

A rational function or matrix $G(s)$ is said to be *proper* if it is finite at $s=\infty$, *strictly proper* if it is 0 at $s=\infty$, *stable* if it is (or its entries are) analytic in $\mathrm{Re}(s)\geqslant 0$. Two rational, proper and stable matrices $N(s)$ and $D(s)$ with the same number of columns are *right-coprime* iff there exist rational, proper and stable matrices $X(s)$ and $Y(s)$ satisfying $X(s)N(s)+Y(s)D(s)=I$. Every rational matrix $G(s)$ has a right-coprime factorization, say, $G(s)=N(s)D^{-1}(s)$, where $N(s)$ and $D(s)$ are right coprime, rational, proper and stable matrices.

Consider the standard feedback system shown in Fig. 1; we say that the compensator $K(s)$ stabilizes the plant $G(s)$ (the arrangement is internally stable), if and only if the closed-loop transfer function

$$H(G,K)=\begin{bmatrix}(I_m+GK)^{-1} & -G(I_l+KG)^{-1}\\ (I_m+GK)^{-1}G & (I_l+KG)^{-1}\end{bmatrix} \tag{1.7}$$

is stable.

In this paper, the L^∞ norm is defined as

$$\|G(j\omega)\|_{L^\infty}=\sup_{\omega\in R}\bar{\sigma}[G(j\omega)]=\sup_{\omega\in R}\lambda^{\frac{1}{2}}\max\{G^*G\}$$

where $G^*(s)=G^{\mathrm{T}}(-s)$.

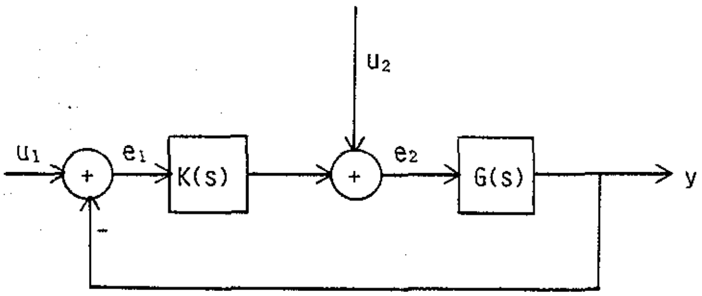

Figure 1. Standard feedback system.

2. Approaches to controller reduction involving balanced realization and Hankel norm optimal approximation

In this section, we shall discuss briefly some controller reduction methods based on approximation via balanced realization and the closely related Hankel norm optimal approximation procedure. Moore (1981) first introduced the internally balanced realization and showed its application to the model reduction problem. Subsequently, various applications of balanced realizations were made to the controller reduction problem, e.g. Jonckheere and Silverman (1981), Verriest (1981a, b), Yousuff and Skelton (1984), and Davis and Skelton (1985).

Enns (1984) developed a frequency-weighted balanced realization technique for use in either model or controller reduction problems. Hankel norm optimal approximation for controller reduction has been considered in Adamjan *et al.* (1971), Kung and Lin (1981), Glover and Limebeer (1983), and Glover (1984) and its extension to allow frequency weighting for controller reduction is discussed in Latham and Anderson (1985), and Anderson (1985). To understand these ideas in more detail, we first recall the definition of a balanced realization.

Definition 2.1

Given an nth-order, proper, asymptotically stable system $G(s)$, a realization of $G(s)=C(sI-A)^{-1}B+E$ is (internally) balanced if (A, B, C) satisfy the following Lyapunov equations:

$$A\bar{\Sigma}+\bar{\Sigma}A^{\mathrm{T}}+BB^{\mathrm{T}}=0 \tag{2.1}$$

$$A^{\mathrm{T}}\tilde{\Sigma}+\tilde{\Sigma}A+C^{\mathrm{T}}C=0 \tag{2.2}$$

and $\bar{\Sigma}=\tilde{\Sigma}=\Sigma=\operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$, where

$$\sigma_i \geqslant \sigma_{i+1} \geqslant 0, \quad i=1,2,\ldots,n-1 \tag{2.3}$$

In eqn. (2.1), $\bar{\Sigma}$ is the controllability gramian, and in equation (2.2), $\tilde{\Sigma}$ is the observability gramian, whether or not $\bar{\Sigma}=\tilde{\Sigma}$, i.e. for any stable system (A, B, C). Thus, a system is balanced just when its controllability gramian and observability gramian are equal and have a diagonal form. The balanced realization is for practical purposes unique (Moore 1981, Glover 1984). Notice that the matrix E plays no role in these considerations.

To use balanced realization in the model reduction problem, Moore's approach (1981) was to partition a balanced system (A, B, C) and Σ as

$$A=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B=\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C=[C_1, C_2], \; \Sigma=\begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \tag{2.4}$$

where $A_{11}, \Sigma_1 \in \mathbb{R}^{r\times r}$, $B_1 \in \mathbb{R}^{r\times l}$, $C_1 \in \mathbb{R}^{m\times r}$, $r<n$. When $\sigma_r \gg \sigma_{r+1}$, the subsystem (A_{11}, B_1, C_1) is internally dominant, i.e., subsystem (A_{11}, B_1, C_1) is the most controllable and observable part of the original system, and this subsystem can be used as a low-order approximation of the model.

For a reduced-order system obtained by truncating a balanced system, there are two important properties to note.

Lemma 2.1 (Pernebo and Silverman 1982)

For a balanced asymptotically stable system (A, B, C) satisfying eqns (2.1) and (2.2), and with Σ in the form (2.3) and partitioned as in (2.4), if $\sigma_r > \sigma_{r+1}$, then both subsystems (A_{ii}, B_i, C_i), $i=1, 2$, are asymptotically stable.

A natural corollary can be obtained.

Corollary 2.1 (Pernebo and Silverman 1982)

If Σ has distinct singular values, then every subsystem is asymptotically stable.

The second property concerns the frequency error bound between the reduced-order system and the original system.

Lemma 2.2 (Enns 1984, Glover 1984)

Given a balanced asymptotically stable system (A, B, C) and Σ, and an asymptotically stable reduced-order subsystem (A_{11}, B_1, C_1) obtained as above, there holds a frequency error bound.

$$\| C(j\omega I_n - A)^{-1}B - C_1(j\omega I_r - A_{11})^{-1}B_1 \|_{L^\infty} \leqslant 2\mathrm{tr}\,(\Sigma_2) = 2(\sigma_{r+1} + \dots + \sigma_n) \quad (2.5)$$

One approach to controller design is to approximate the original system using balanced approximation, and design a controller for the reduced-order model of the original system, connecting this to the original system. One objection to this approach has already been offered: approximation is not left as late as it might be. Two other possible objections are that the original system may not be stable, and that the procedure does not take into account frequency shaping in the loop caused by the simultaneous presence of controller and system. All these objections can in part be countered. First, one can design a high-order controller for the original system and then approximate the controller; see Yousuff and Skelton (1984). Second, one can additively decompose the system transfer function matrix into a stable and unstable part, and approximate just the stable part, and third, one can introduce appropriate frequency weighting into the approximation procedure, as in Enns (1984).

When one is approximating a high-order controller by a low-order controller, there also arises the possibility of approximating an unstable controller by approximating only its stable part in an additive approximation, and there arises the possibility of introducing frequency weighting, discussed further below. Yousuff and Skelton (1984) also suggest a rather *ad hoc* modification to cover the case of unstable controllers, and Davis and Skelton (1984) suggest a different modification which would deal with stable or unstable controllers (but without significantly more justification). *A priori* frequency domain bounds on the error associated with controller approximation are not known when the approximation procedure uses these modifications.

Enns (1984) developed a reduction method in terms of a consideration of the stability margin. Given a linear time-invariant system $G(s)$ and a full-order controller $K(s)$ stabilizing $G(s)$, suppose we replace the controller $K(s)$ by a low-order one $K_1(s)$. The closed-loop system with the reduced-order controller $K_1(s)$ can be described as in Fig. 2.

We know that if $K_1(s)$ has the same number of unstable poles as $K(s)$, and if

$$\| [K_1(j\omega) - K(j\omega)]G(j\omega)[I + K(j\omega)G(j\omega)]^{-1} \|_{L^\infty} < 1 \quad (2.6)$$

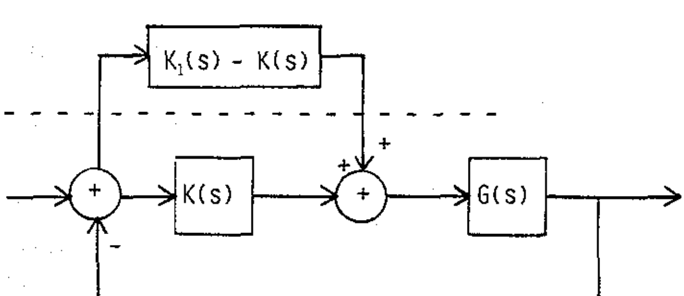

Figure 2. Closed-loop system with reduced-order controller.

then the closed-loop system with the reduced-order controller will be stable also. Also, the smaller the L^∞ norm on the left-hand side of (2.6), in one sense the better the approximation. This suggests a technique of frequency-weighted balancing which balances a system $K(s)$ with some input frequency weighting. An approximation $K_1(s)$ can be found by neglecting those states which are least observed, and least controlled (modulo the input frequency weighting). Enns also postulates a frequency error bound for this method of the form

$$\|[K(j\omega)-K_1(j\omega)]G(j\omega)(I+K(j\omega)G(j\omega)]^{-1}\|_{L^\infty} \leqslant 2(1+\alpha)\mathrm{tr}(\Sigma_2) \tag{2.7}$$

Here, Σ_2 is a modification of the earlier Σ_2, which reflects the frequency weighting; unfortunately a method of accurately evaluating the constant α is unknown so that the condition (2.6) is not easy to check using simply knowledge of Σ_2.

Hankel norm optimal approximation is another very important approach to model reduction problems. Glover (1984) characterized all stable approximations of a linear time-invariant stable system $G(s)$ of McMillan degree n by $\hat{G}(s)$ of McMillan degree r, $(r<n)$, which minimizes the Hankel norm $\|G(s)-\hat{G}(s)\|_H$. An L^∞-norm error bound has been also derived for the method proposed by Glover (1984),

$$\|G(j\omega)-\hat{G}(j\omega)\|_{L^\infty} \leqslant (\sigma_{r+1}+\dots+\sigma_n) \tag{2.8}$$

In principle, whenever a transfer function matrix can be approximated using balanced approximation, it can be approximated using Hankel norm ideas. It follows that many of the above controller reduction methods (and comments concerning these methods) which depend on using balanced approximation. This includes the frequency-weighted methods, since frequency-weighted Hankel norm approximation is possible (Latham and Anderson 1985). Indeed, frequency-domain error bounds are available for frequency-weighted Hankel norm approximations in contrast to frequency-weighted balanced approximation (Anderson 1985).

On the face of it, the lower error bound of (2.8) as compared with (2.5) suggests that Hankel norm approximation may be superior. This conclusion needs to be qualified, however, by noting that (2.8) in general requires $G(\infty)-\hat{G}(\infty)\neq 0$, which may not be desirable or acceptable. We return to this point later.

Jonckheere and Silverman (1981) and Verriest (1981a, b) have proposed yet another controller reduction method which also involves the balancing technique. They suggested balancing of the observability gramian for the pair of $A-BF$ and $\begin{bmatrix} Q^{1/2} \\ R^{1/2}F \end{bmatrix}$ and the controllability gramian for the pair of $A-LC$ and $[W^{1/2}, LV^{1/2}]$ i.e., to find a non-singular transformation T, such that the pair of $\bar{A}_F = T^{-1}(A-BF)T$ and $\bar{F} = \begin{bmatrix} Q^{1/2} \\ R^{1/2}F \end{bmatrix} T$ satisfy eqn. (2.2), and the pair of $\bar{A}_L = T^{-1}(A-LC)T$ and $\bar{L} = T^{-1}[W^{1/2}, LV^{1/2}]$ satisfy eqn. (2.1), with $\bar{\Sigma}=\tilde{\Sigma}=$ a diagonal matrix. Unfortunately, this procedure will not even eliminate unobservable or uncontrollable modes in the compensator. Furthermore, frequency-domain error bounds for the controller approximation have not yet been obtained.

We know that the purpose of controller reduction is not only to find a stabilizing reduced-order controller but also to ensure as far as possible that the performance of the closed-loop system with the reduced-order controller is as close as possible to that of the closed-loop system with the full-order controller. We have noted that some controller reduction methods mentioned above lead to an error bound

$$\|[K(s) - K_1(s)]W_i(s)\| \leqslant \varepsilon(K), \quad \varepsilon(K) > 0$$

where W_i may be an identity matrix, and $\|\cdot\|$ can be L^∞ norm or Hankel norm. Then we can ask a natural question: does a small value of $\varepsilon(K)$ necessarily imply a small value of $\delta \triangleq \|H(G, K) - H(G, K_1)\|$, where $H(G, K)$ is defined by (1.7)? It also raises the question, can we find an alternative way to approximate $K(s)$ by $K_1(s)$ which more directly seeks to ensure that δ is small? This motivating question is the starting point for our discussion of a new controller reduction method.

3. New approach to controller reduction

Instead of approximating $K(s)$ directly by $K_1(s)$, we shall now approximate the component parts of a fractional representation of $K(s)$. From Vidyasagar (1985 b), every rational matrix $K(s)$ has a right coprime factorization $K(s) = N(s)D^{-1}(s)$, where $N(s)$ and $D(s)$ are rational, proper, stable and right coprime, $D(s)$ is non-singular. The basic idea is to find an rth-order controller $(r < n)K_1(s) = N_1(s)D_1^{-1}(s)$, where $N_1(s)$ and $D_1(s)$ are rational, proper, stable and right coprime with $D_1(s)$ non-singular, such that the error

$$\left\|\begin{bmatrix} D(j\omega) - D_1(j\omega) \\ N(j\omega) - N_1(j\omega) \end{bmatrix}\right\|_{L^\infty}$$

is small.

One might ask, why do controller approximation in this way? One reason is that we have the following statement (Vidyasagar 1985 b): let

$$\varepsilon \triangleq \left\|\begin{bmatrix} D(j\omega) - D_1(j\omega) \\ N(j\omega) - N_1(j\omega) \end{bmatrix}\right\|_{L^\infty} \tag{3.1}$$

If $\varepsilon \to 0$ as we change the pair $[N_1(s), D_1(s)]$, then

$$\delta \triangleq \|H(G(j\omega), K(j\omega)) - H(G(j\omega), K_1(j\omega))\|_{L^\infty} \to 0 \tag{3.2}$$

where $H(G, K)$ is defined by (1.7).

Below, we shall return to this point, and also give justification for the particular factorization of $K(s)$ to be used in the approximation. This particular factorization is obtained in the following way.

For an LQG designed controller $K(s)$, we can have a factorization

$$\begin{aligned} K(s) &= F(sI_n - A + BF + LC)^{-1}L \\ &= F(sI_n - A + BF)^{-1}L[I_m + C(sI_n - A + BF)^{-1}L]^{-1} \end{aligned}$$

Let

$$\begin{aligned} N(s) &= F(sI_n - A + BF)^{-1}L \\ D(s) &= I_m + C(sI_n - A + BF)^{-1}L \end{aligned}$$

Then

$$K(s) = N(s)D^{-1}(s)$$

It is easy to see that $N(s)$ is strictly proper and $D(s)$ is non-strictly-proper and non-singular. From LQG design theory we know that if $[A, B]$ is completely controllable, then $A - BF$ has eigenvalues with negative real parts, so that $N(s)$ and $D(s)$ are stable. Further, if $[A, C]$ is completely observable, then $A - LC$ has eigenvalues with negative real parts, so that

$$\tilde{X}(s) \triangleq C(sI_n - A + LC)^{-1}B \tag{3.3}$$

and

$$\tilde{Y}(s) \triangleq I_m - C(sI_n - A + LC)^{-1}L \tag{3.4}$$

are both stable.

It has been proved (Nett *et al.* 1984) that

$$G(s) = C(sI_n - A)^{-1}B = \tilde{Y}^{-1}(s)\tilde{X}(s)$$

and

$$\tilde{X}(s)N(s) + \tilde{Y}(s)D(s) = I_m$$

This Bezout identity shows that $N(s)$ and $D(s)$ are right coprime, or, $N(s)D^{-1}(s)$ is a right coprime factorization of $K(s)$. Incidentally, the only way in which the LQG origins of the controller are exploited above is when we appeal to the eigenvalue properties of $A - BF$ and $A - LC$. One could conceive of F and L being obtained by non-LQG methods and yielding the same eigenvalue properties.

Suppose we have determined a reduced-order controller $K_1(s) = N_1(s)D_1^{-1}(s)$ which satisfies (3.1); then we can derive a relation between ε and δ for the $N(s)$, $D(s)$ and $H(G, K)$ defined as before.

Lemma 3.1

Let $\eta \triangleq \|[\tilde{Y}(j\omega), \tilde{X}(j\omega)]\|_{L^\infty}$; if $\varepsilon\eta < 1$, then

$$\delta \leqslant \xi\varepsilon\eta + O(\varepsilon^2) \tag{3.5}$$

where ε is defined by (3.1), δ is defined by (3.2), $\tilde{X}(s)$ is defined by (3.3), $\tilde{Y}(s)$ is defined by (3.4) and

$$\xi = \left\| I_{m+l} - \begin{bmatrix} D(j\omega) \\ N(j\omega) \end{bmatrix} [\tilde{Y}(j\omega), \tilde{X}(j\omega)] \right\|_{L^\infty}$$

The proof of this lemma is in the Appendix.

The meaning of (3.5) is very clear. When we do the controller reduction, if the error bound ε for the factorizations of the controllers is small enough, then the error bound of the closed-loop system δ will be small also, and if we neglect the high-order ε term for sufficiently small ε, then the multiplicative factor in the bound relating δ and ε is just a constant $\xi\eta$, determinable from the given original system and full-order controller.

Thus, we can say that approximating a controller by approximating the component parts of its factorization is a natural method of tackling the controller reduction problem, in that we can expect some good results in the 'closeness' of the closed-loop performance for this controller reduction method.

One significant question, however, remains. There is no single fractional representation of the controller, and no guarantee that if approximations are made using two different fractional representations, then the same approximate controller will result. So we need to consider why the particular fractional representation defined above is

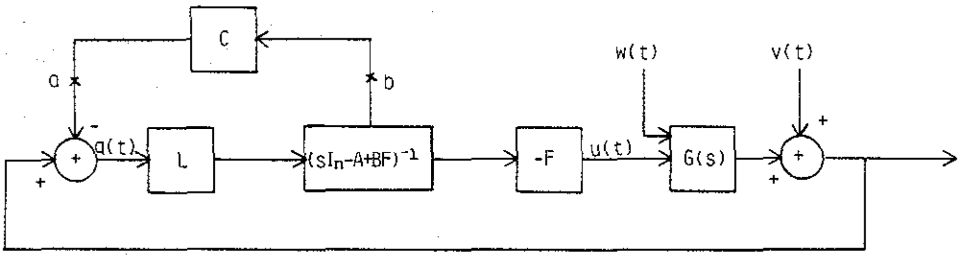

Figure 3.

to be preferred to any other one. To answer this question, let us consider the closed-loop system in Fig. 3 under the assumptions of the LQG problem. It is well known that the innovation process $q(t)$ is white and with covariance $V\delta(t-\tau)$, the same as the covariance of the measurement noise $v(t)$ (Kwakernaak and Sivan 1972).

Think of the controller as being formed by two open-loop transfer functions, that from $q(t)$ to $u(t)$, and that from $q(t)$ to point a, followed by subsequent connection of the negative feedback junction to the left of $q(t)$. Thus the controller prior to connection of the feedback is defined by

$$\begin{bmatrix} -F(sI-A+BF)^{-1}L \\ C(sI-A+BF)^{-1}L \end{bmatrix} = \begin{bmatrix} -N(s) \\ D(s)-I \end{bmatrix}$$

Because the innovations noise process $q(t)$ is white, every frequency is equally important when we approximate $\begin{bmatrix} D(s)-I_m \\ -N(s) \end{bmatrix}$ by some $\begin{bmatrix} D_1(s)-I_m \\ -N_1(s) \end{bmatrix}$, or, approximate $\begin{bmatrix} D(s) \\ N(s) \end{bmatrix}$ by $\begin{bmatrix} D_1(s) \\ N_1(s) \end{bmatrix}$. Thus we should not put any frequency weighting here but just a constant weighting M, where $MM^T = V$, covariance of $v(t)$. Equivalently, it is *not* more appropriate to approximate

$$\begin{bmatrix} \bar{D}(s) \\ \bar{N}(s) \end{bmatrix} = \begin{bmatrix} D(s)P(s) \\ N(s)P(s) \end{bmatrix}$$

by some $[\bar{D}_1^T(s)\ \bar{N}_1^T(s)]$, where $P(s)$ and $P^{-1}(s)$ are stable; note that $P(s)$ both defines a different fractional representation $\bar{N}\bar{D}^{-1}$ (and all other coprime representations can be found in this way) and can be thought of as a way of introducing frequency weighting to the original representation by N, D. Now we state the algorithm for controller approximation.

Algorithm

Given: a system (A, B, C), completely controllable and observable, and control gain F and estimator gain L (obtained by LQC design), and the weighting M, where $MM^T = V$, the covariance of the measurement noise $v(t)$.

Step 1. For the system $(A-BF, LM, [C^T, F^T]^T)$, find a balanced realization, i.e., find a non-singular transformation matrix T, such that with

$$\bar{A} = T^{-1}(A-BF)T, \quad \bar{L} = T^{-1}L, \quad \bar{C} = CT, \quad \bar{F} = FT$$

the equations

$$\bar{A}\Sigma + \Sigma\bar{A}^{\mathrm{T}} + \bar{L}MM^{\mathrm{T}}\bar{L}^{\mathrm{T}} = 0$$

$$\bar{A}^{\mathrm{T}}\Sigma + \Sigma\bar{A} + [\bar{C}^{\mathrm{T}}, \bar{F}^{\mathrm{T}}]\begin{bmatrix}\bar{C}\\ \bar{F}\end{bmatrix} = 0$$

hold, where $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$, $\sigma_i \geqslant \sigma_{i+1} \geqslant 0$, $i = 1, 2, \ldots, n-1$.

Step 2. Partition $\bar{A}$, $\bar{L}$, $\bar{C}$, $\bar{F}$ and Σ as

$$\bar{A} = \begin{bmatrix}\bar{A}_{11} & \bar{A}_{12}\\ \bar{A}_{21} & \bar{A}_{22}\end{bmatrix}, \quad \bar{L} = \begin{bmatrix}\bar{L}_1\\ \bar{L}_2\end{bmatrix}, \quad \bar{C} = [\bar{C}_1, \bar{C}_2]$$

$$\bar{F} = [\bar{F}_1, \bar{F}_2], \Sigma = \begin{bmatrix}\Sigma_1 & \\ & \Sigma_2\end{bmatrix}$$

where $\bar{A}_{11}, \Sigma_1 \in \mathbb{R}^{r\times r}$, $\bar{C}_1 \in \mathbb{R}^{m\times r}$, $\bar{F}_1 \in \mathbb{R}^{l\times r}$, $\Sigma_2 = \mathrm{diag}(\sigma_{r+1}, \ldots, \sigma_n)$, and $1 \leqslant r < n$.

Step 3. Form the reduced-order controller

$$K_1(s) = \bar{F}_1(sI_r - \bar{A}_{11} + \bar{L}_1\bar{C}_1)^{-1}\bar{L}_1$$

Now we are going to check whether $K_1(s)$ is what we wanted.
Let

$$N_1(s) = \bar{F}_1(sI_r - \bar{A}_{11})^{-1}\bar{L}_1$$

$$D_1(s) = I_m + \bar{C}_1(sI_r - \bar{A}_{11})^{-1}\bar{L}_1$$

It is trivial to prove that $K_1(s) = N_1(s)D_1^{-1}(s)$, i.e., $N_1(s)D_1^{-1}(s)$ is a factorization of $K_1(s)$. It is easy to see that $N_1(s)$ is strictly proper, and $D_1(s)$ is non-strictly proper and non-singular. From Lemma 2.2, we have a frequency error bound

$$\left\|\left[\begin{bmatrix}\bar{C}\\ \bar{F}\end{bmatrix}(j\omega I_n - \bar{A})^{-1}\bar{L} - \begin{bmatrix}\bar{C}_1\\ \bar{F}_1\end{bmatrix}(j\omega I_r - \bar{A}_{11})^{-1}\bar{L}_1\right]M\right\|_{L^\infty} \leqslant 2\mathrm{tr}(\Sigma_2)$$

that is

$$\left\|\begin{bmatrix}C(j\omega I_n - A + BF)^{-1}L - \bar{C}_1(j\omega I_r - \bar{A}_{11})^{-1}\bar{L}_1\\ F(j\omega I_n - A + BF)^{-1}L - \bar{F}_1(j\omega I_r - \bar{A}_{11})^{-1}\bar{L}_1\end{bmatrix}M\right\|_{L^\infty} \leqslant 2\mathrm{tr}(\Sigma_2)$$

thus,

$$\left\|\begin{bmatrix}(D(j\omega) - I_m) - (D_1(j\omega) - I_m)\\ N(j\omega) - N_1(j\omega)\end{bmatrix}M\right\|_{L^\infty} = \left\|\begin{bmatrix}D(j\omega) - D_1(j\omega)\\ N(j\omega) - N_1(j\omega)\end{bmatrix}M\right\|_{L^\infty} \leqslant 2\mathrm{tr}(\Sigma_2) \tag{3.6}$$

or

$$\left\|\begin{bmatrix}D(j\omega) - D_1(j\omega)\\ N(j\omega) - N_1(j\omega)\end{bmatrix}\right\|_{L^\infty} \leqslant 2\mathrm{tr}(\Sigma_2)/|\lambda_{\min}(M)| = 2\mathrm{tr}(\Sigma_2)|\lambda_{\max}(M^{-1})|$$

Now we need only check whether $N_1(s)D_1^{-1}(s)$ is a right coprime factorization of $K_1(s)$. From the LQG design procedure we believe that it is easy to ensure, through slight modifications of the performance weighting matrices Q and R in (1.2) or the filter weighting matrices W and V if necessary, that $\sigma_r > \sigma_{r+1}$ in the balanced realization of $(A-BK,\ L,\ [C^T,\ K^T]^T)$. Then by Lemma 2.1, we know that $\bar{A}_{11}$ is asymptotically stable because $A-BF$ is asymptotically stable, and $N_1(s)$ and $D_1(s)$ are also asymptotically stable. Again, from Vidyasahar (1985a), if $2\mathrm{tr}(\Sigma_2)$ is small enough, then $N_1(s)$ and $D_1(s)$ are right coprime. This means that $N_1(s)D_1^{-1}(s)$ is a right coprime factorization of $K_1(s)$. Hence, we can say that $\begin{bmatrix} D_1(s) \\ N_1(s) \end{bmatrix}$ is a factorization approximation of $\begin{bmatrix} D(s) \\ N(s) \end{bmatrix}$ with weighting M and bounded by $2\mathrm{tr}(\Sigma_2)$.

The stability of the closed-loop system with the reduced-order controller is also guaranteed by a sufficiently small value of the error bound $2\mathrm{tr}(\Sigma_2)$ (Vidyasagar 1985 b). Just how small should $\mathrm{tr}(\Sigma_2)$ be for stability? This question is in part addressed as follows.

Lemma 3.2

For a LQG designed controller $K(s)$, with controller reduction performed as above, if $\mathrm{tr}(\Sigma_2)$ in (3.6) satisfies

$$\mathrm{tr}(\Sigma_2) < \frac{1}{(1 + \|[I_m - D\tilde{Y} - D\tilde{X}]\|_{L^\infty})\|D^{-1}\|_{L^\infty}}, \tag{3.7}$$

then the closed-loop system with the reduced-order controller $K_1(s)$ will be stable provided that $K(s)$ and $K_1(s)$ have the same number of unstable poles.

The proof of this lemma is also in the Appendix.

The condition that $K(s)$ and $K_1(s)$ have the same number of unstable poles will be true for a sufficiently small ε (Vidyasagar 1985 b), but how small ε should be is not *a priori* clear. This may make (3.7) meaningless, but in every example where we have used our method, the condition that $K(s)$ and $K_1(s)$ have the same number of unstable poles is always true. So it seems that this is a very weak condition; thus, (3.7) may still have some meaning in the estimation of the upper bound of $\mathrm{tr}(\Sigma_2)$ for the stability.

4. Examples

We have used the example in Enns' paper (1984) to compare our method to Enns' method (1984), Glover's method (1984) applied to the controller, Davis and Skelton's method (1984), and Yousuff and Skelton's method (1984).

The plant to be controlled is a four-disk system and is linear, time-invariant, SISO, unstable and non-minimum phase, and of eighth order. The transfer function is

$$G(s) = \frac{0{\cdot}01(0{\cdot}64s^5 + 0{\cdot}235s^4 + 7{\cdot}13s^3 + 100{\cdot}02s^2 + 10{\cdot}45s + 99{\cdot}55)}{s^2(s^6 + 0{\cdot}161s^5 + 6{\cdot}004s^4 + 0{\cdot}5822s^3 + 9{\cdot}9835s^2 + 0{\cdot}4073s + 3{\cdot}982)}$$

The performance and stability robustness requirements for this system result in a loop shape constraint shown in Fig. 4.

To do LQG design, we use the observable canonical form realization of $G(s)$, say (A, B, C), and choose the control performance weightings as

$$Q = q_1 H^T H, \quad R = 1$$

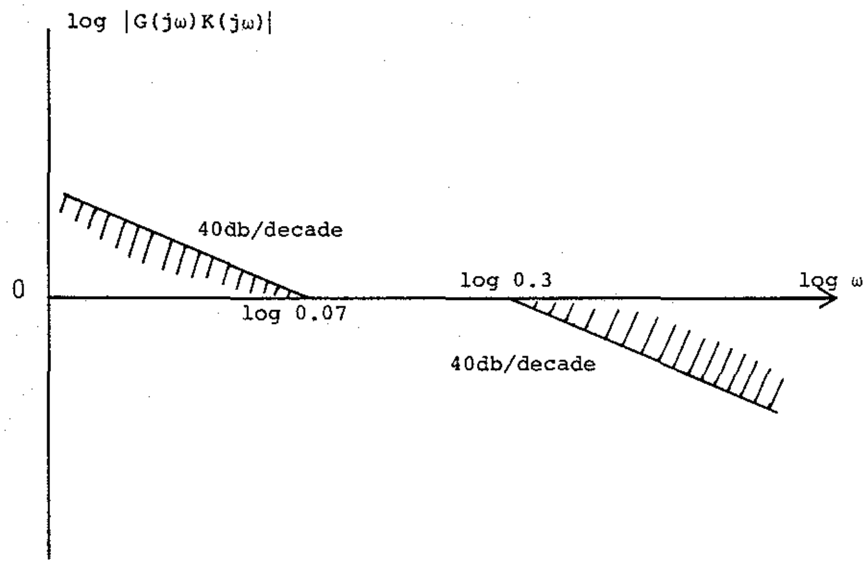

Figure 4. Loop shape constraints (shaded region to be avoided).

where

$$H = [0, 0, 0, 0, 0{\cdot}55, 11, 1{\cdot}32, 18]$$
$$q_1 = 10^{-6}$$

and the filter weightings as

$$W = q_2 BB^{\mathrm{T}}, \quad V = 1, \text{(hence } M = 1)$$

where q_2 is a design parameter.

Then, when $0{\cdot}01 \leqslant q_2 \leqslant 10^6$, every LQG designed full-order controller $K(s)$ satisfies the loop shape constraints in Fig. 4. The phase margins and gain margins of the closed-loop system with full order controller for different values of q_2 are listed in Table 1.

q_2	10^{-4}	10^{-2}	1	10^2	10^4	10^6
Gain margins	−58·99	−68·02	−71·754	−72·83	−73·41	−73·84
Phase margins	32·79	35·98	47·795	52·40	55·19	57·54

Table 1. Gain and phase margins of full-order systems.

For the different controllers obtained by LQG designs for the different q_2 ($q_2 = 0{\cdot}01$, 0·1, 1, 10, 100, 1000, 2000), we have carried out the controller reduction by the four methods metioned before and our method, for different reduced orders (from second to seventh order). Then we have checked the stability of each closed-loop system. The results are shown in Table 2.

From Table 2, it is easy to see that only in four cases does our method yield an unstable closed-loop system with the reduced-order controller when Enns' method (and Davis and Skelton's method) do yield a stabilizing reduced-order controller. But, generally speaking, our method gives a stable closed-loop system in the same number of cases as Enns' method for this example, and this is more than any other method.

Order	Method, q_2 =	0·01	0·1	1	10	100	1000	2000
	Enns (E)	S	S	S	S	S	S	S
	Glover (G)	S	S	S	S	S	U	S
7	Davis and Skelton (D)	S	U	U	S	S	S	S
	Yousuff and Skelton (Y)	S	S	S	S	U	U	U
	New method (N)	S	S	S	S	S	S	U
	E	S	S	S	S	S	S	S
	G	S	S	S	S	U	U	U
6	D	S	S	S	S	S	S	S
	Y	S	S	S	S	U	U	U
	N	S	S	S	S	S	S	U
	E	S	S	S	S	S	S	S
	G	S	S	S	S	U	U	U
5	D	S	U	S	S	S	U	U
	Y	S	S	S	U	U	U	U
	N	S	S	S	S	S	S	S
	E	S	S	S	S	S	S	U
	G	S	S	S	S	U	U	U
4	D	S	S	U	S	S	U	U
	Y	S	S	S	U	U	U	U
	N	S	S	S	S	S	S	S
	E	S	S	S	S	S	S	S
	G	S	S	U	S	U	U	U
3	D	U	U	U	U	U	U	U
	Y	S	U	U	U	U	U	U
	E	S	S	S	S	S	U	U
	E	S	S	S	S	U	U	U
	G	S	U	S	U	S	U	U
2	D	S	U	S	U	U	U	U
	Y	S	S	S	U	U	U	U
	N	S	S	S	S	S	S	S

S—the closed loop system is stable; U—unstable.

Table 2. Stability of the reduced-order controller by different methods.

We have also compared the performances of the closed-loop systems with different reduced-order controllers. We have checked the step responses, impulse responses, and the responses to an approximate white noise input for all those closed-loop systems shown in Fig. 5.

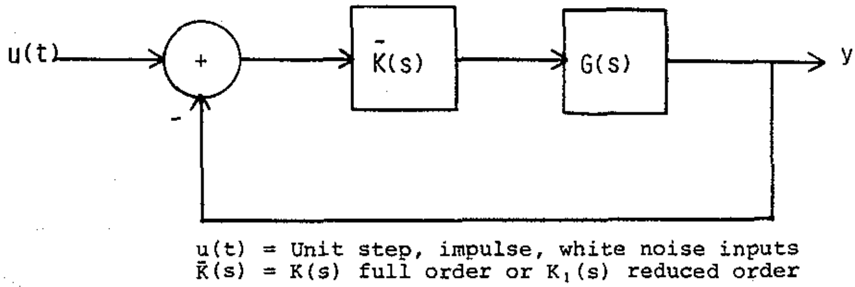

Figure 5. Comparison of performances of full- and reduced-order controllers.

We also have compared the frequency behaviour of the loop gain $G(s)K_1(s)$ with that of the full-order one, $G(s)K(s)$, where $K_1(s)$ is the reduced-order controller obtained by different methods. In general, we find that in most cases the closed-loop system with a reduced-order controller obtained by our method has a much closer response to that of the full-order closed-loop system than for the other methods. To show this clearly, we choose two rather critical cases with different values of q_2 and

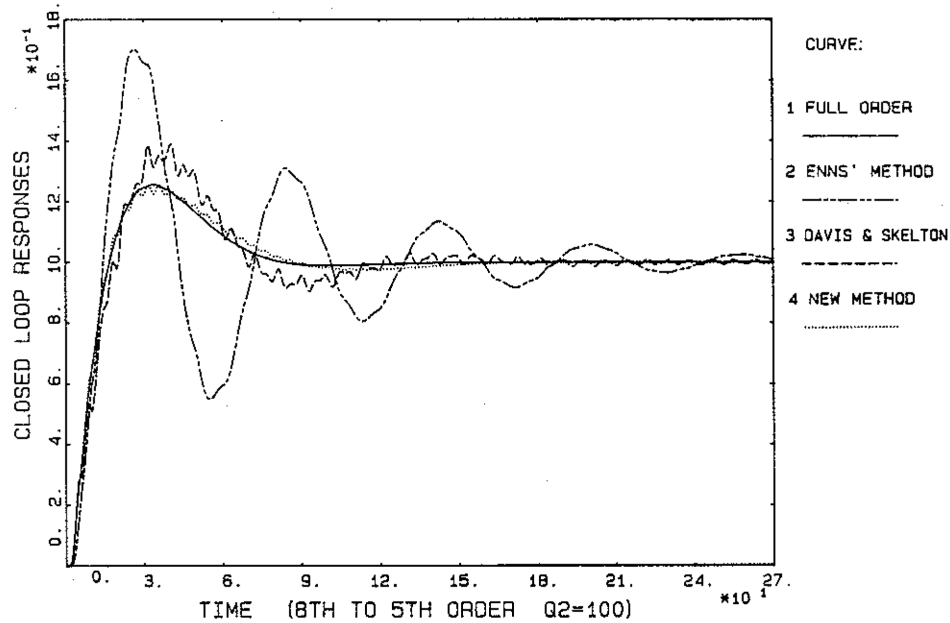

Figure 6. Step response comparison.

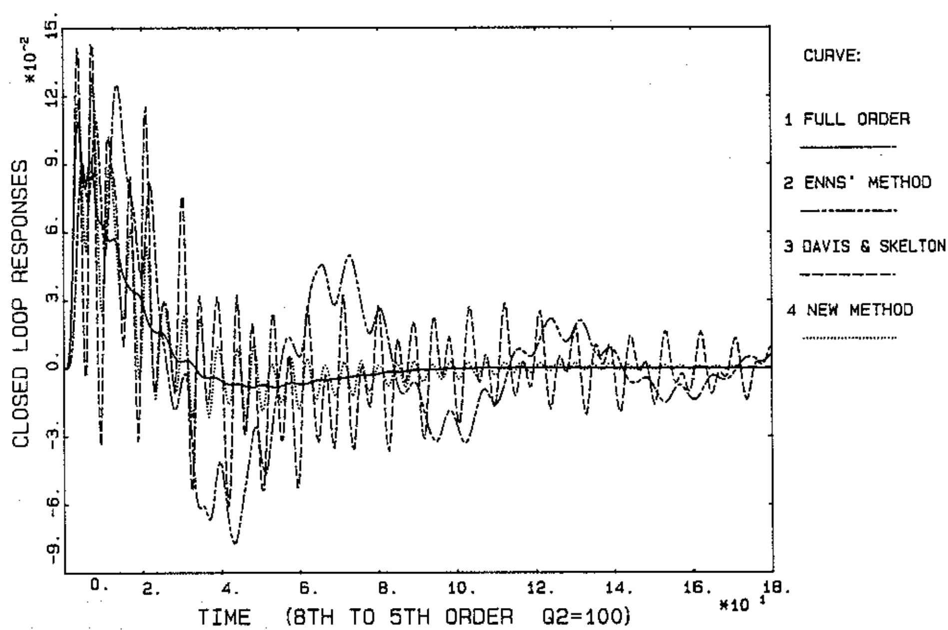

Figure 7. Impulse response comparison.

different reduced orders. When $q_2 = 100$, and reduced order of 5, two methods fail to yield a stabilizing fifth-order controller, viz. Yousuff and Skelton's and Glover's methods. Comparisons of the closed-loop responses due to different input signals for the different reduction methods are shown in Figs. 6–8. Note that the approximate white noise signal, shown in Fig. 12 and the closed-loop responses to these inputs shown in Figs. 8 and 15, are not plotted with the same vertical axis scaling. Figure 9

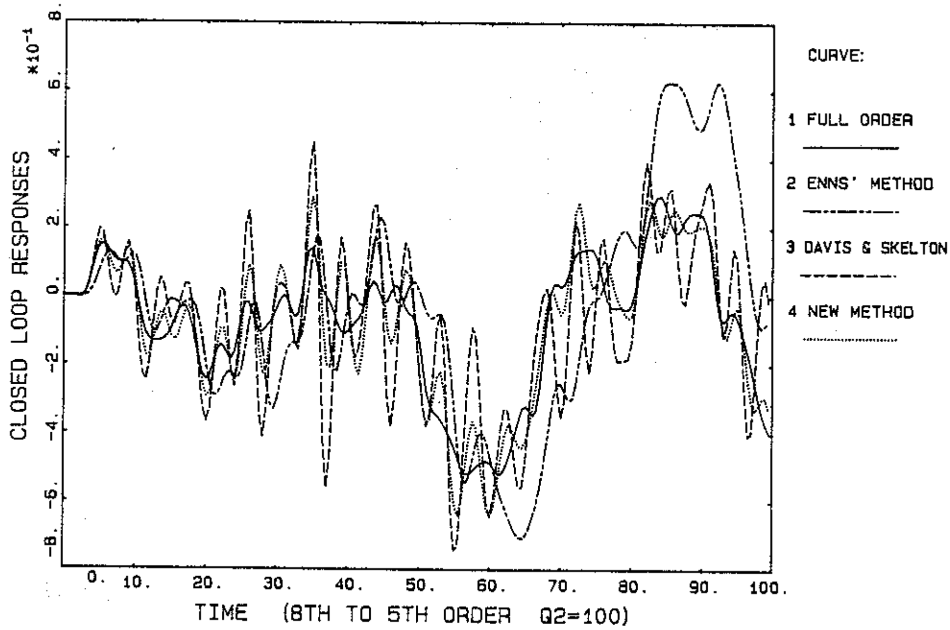

Figure 8. Responses to external white noise.

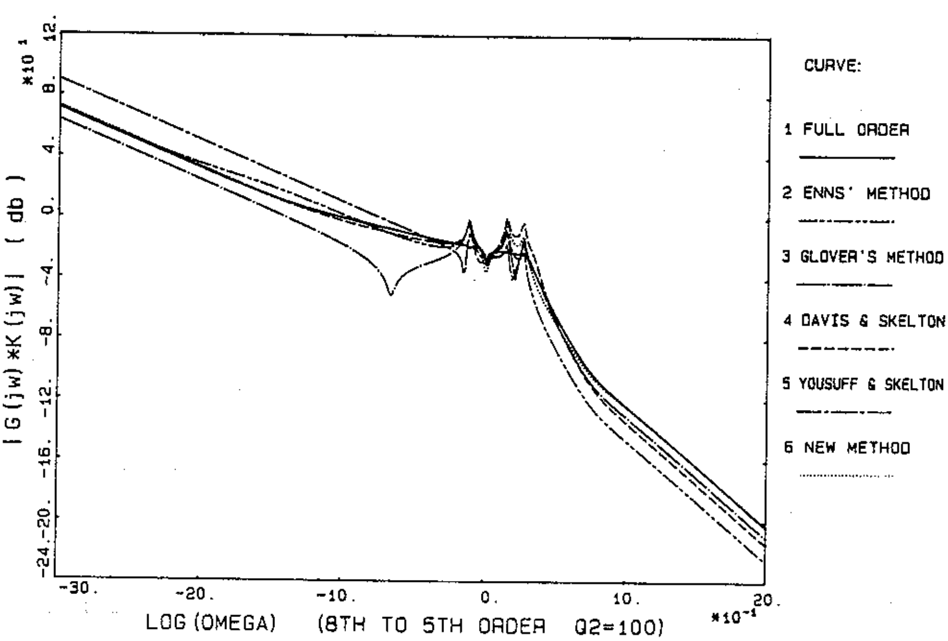

Figure 9. Bode plot comparison.

yield a stabilizing second-order controller. The comparison results are shown in Figs. 13–18. We can also draw the same conclusions from these comparisons as before. These figures do not of course suggest that all methods other than the one of this paper are so poor as to be almost unacceptable. In fact, we can see that when q_2 is very small, all methods (including our new method) may yield a very good result. For

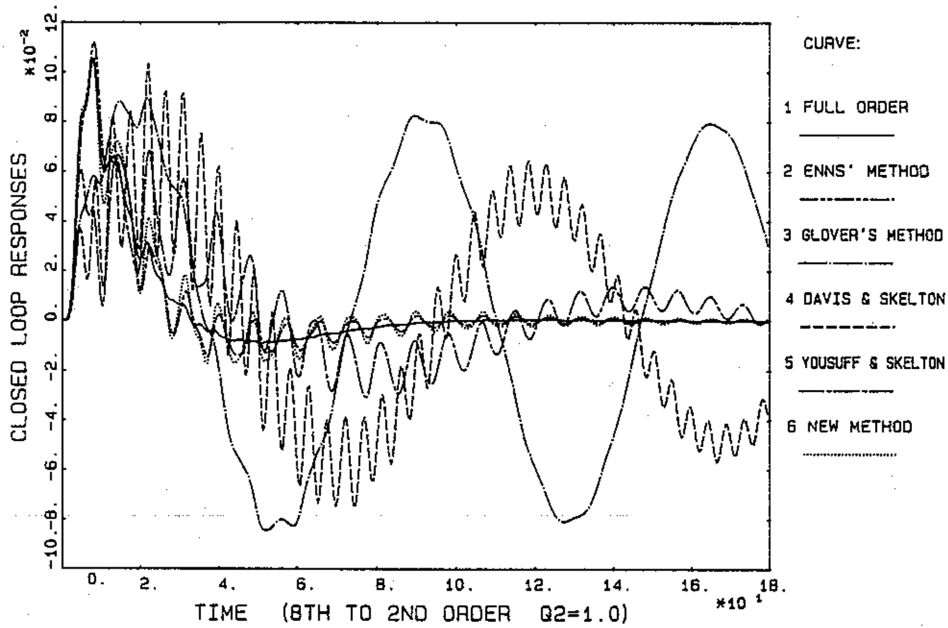

Figure 14. Impulse response comparison.

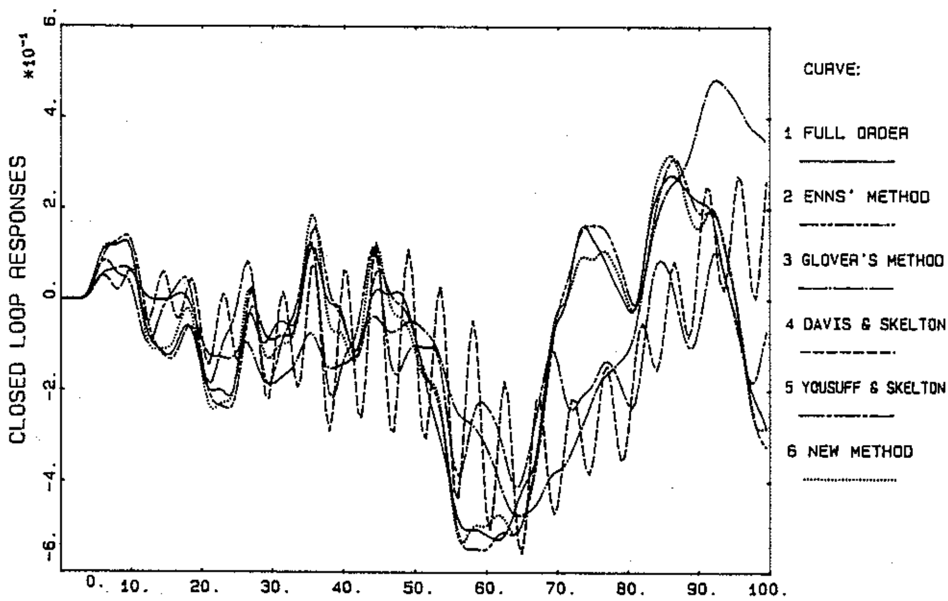

Figure 15. Response to external white noise.

example, when $q_2 = 0{\cdot}01$ and the reduced order is 6, all methods yield responses almost the same as the response of the full-order controller. This means that every method here may sometimes work well, and sometimes not. But comparatively speaking, our method seems to do well in ensuring 'closeness' of the closed-loop responses.

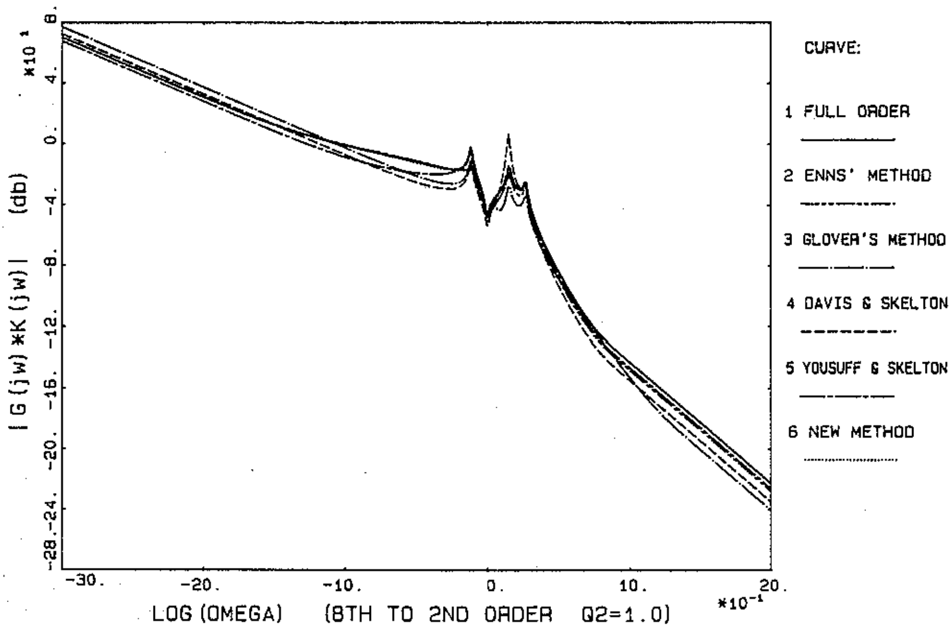

Figure 16. Bode plot comparison.

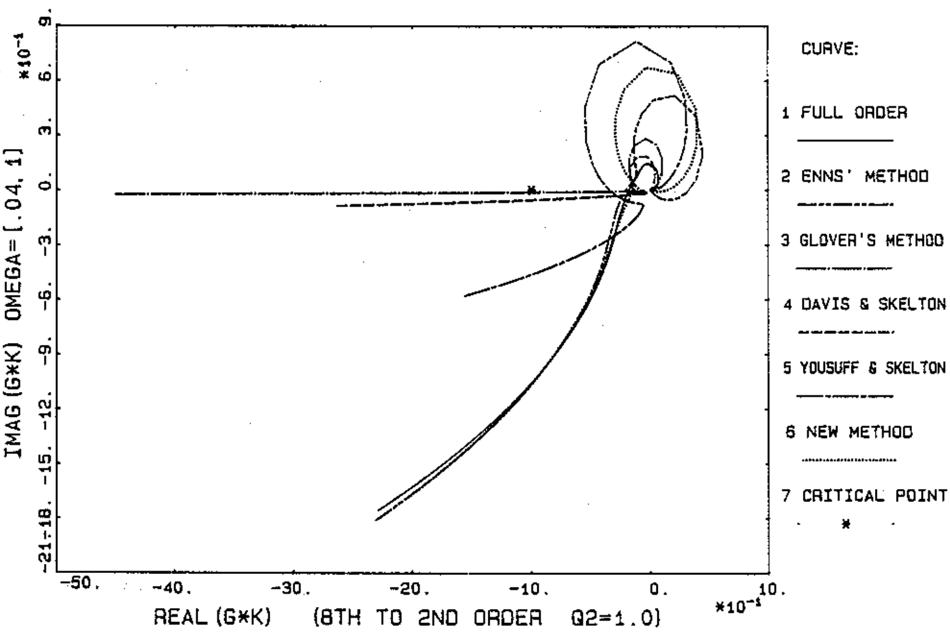

Figure 17. Nyquist plot comparison.

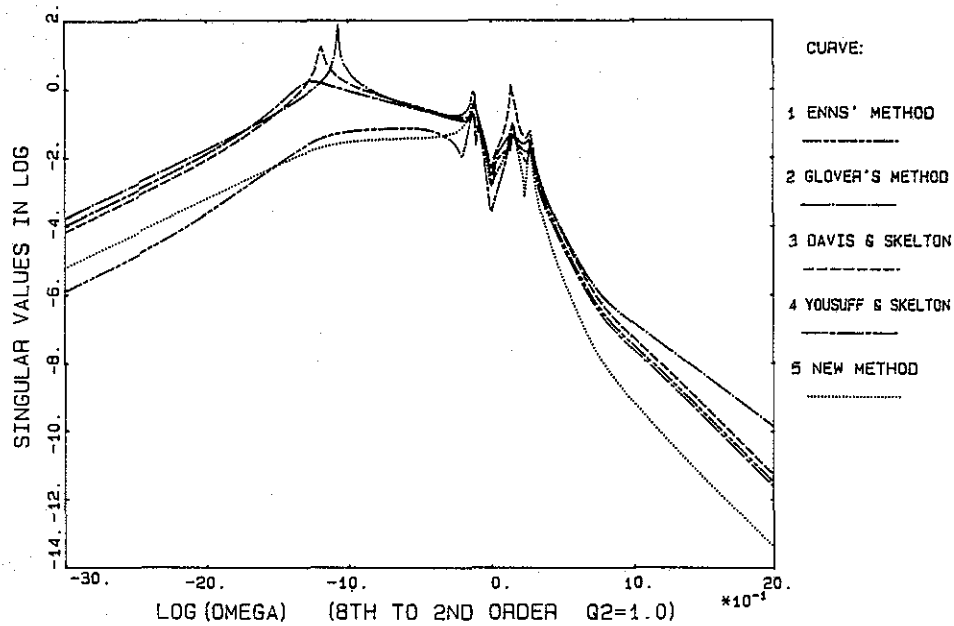

Figure 18. Error of closed-loop systems.

5. Conclusions and remarks

(*a*) The simulation results in § 4 have shown that our new method is very attractive in terms both of the stability properties and of the accuracy of closed-loop approximation. The algorithm of this method is also very simple to implement. The procedure is also naturally motivated. For the non-minimum-phase system, the results show that it is much harder to stabilize the original system by a reduced-order controller (no matter what method is used!) when the filter design parameter q_2 becomes larger. Another interesting observation is that for the same method and the same q_2, in many cases, a second-order controller is better than a third-order one, a fourth-order controller is better than a fifth-order one, and a sixth-order controller is better than a seventh-order one. This suggests that in general, it may be better to reduce an odd-number-order controller to an odd-order one, and an even-number-order controller to an even-order one for better closed-loop performance and stability.

(*b*) An advantage of this new method is that provided that the original system is completely controllable and observable, then even if the LQG designed controller is unstable, this method still works without changing anything. Specifically, there is no need to decompose the unstable controller $K(s)$ into an asymptotically stable part $K_s(s)$ and an unstable part $K_u(s)$, and then to reduce the stable part $K_s(s)$, as suggested in Enns (1984) and Glover (1984). Nor is any other *ad hoc* modification needed. Thus our method also involves the information contained in the unstable part of the controller in the approximation process. So, from this point of view, our method may yield a better result than any other method when we deal with unstable controller reduction. This has also been verified in some simulation results (not presented here).

(*c*) It is in principle possible to start with a left coprime factorization of the controller $K(s)$.

$$K(s) = [I_l + F(sI_n - A + LC)^{-1}B]^{-1}F(sI_n - A + LC)^{-1}L = \tilde{D}^{-1}(s)\tilde{N}(s)$$

where

$$\tilde{D}(s) = I_l + F(sI_n - A + LC)^{-1}B$$
$$\tilde{N}(s) = F(sI_n - A + LC)^{-1}L$$

We can also use essentially the same method to approximate $K(s)$ by balancing the system $(A - LC,\ [B, L], F)$, approximating, and then forming the reduced-order controller in a similar way. The results do not seem as good as those obtained with the right coprime factorization approximation, perhaps because the justification we gave for the use of the particular right coprime factorization in terms of the whiteness of the innovation process can no longer be advanced.

(*d*) A natural question about this method is, why not use a Hankel norm optimal approximation of $\begin{bmatrix} D(s) \\ N(s) \end{bmatrix}$, since, especially, the approximation error would seem less than when balanced approximation is used? One answer to this question is that we may prefer a strictly proper reduced-order controller to a non-strictly-proper one, since, in some sense, stabilization by a non-strictly-proper controller is never robust against singular perturbations while stabilization by any strictly proper controller is robust against singular perturbations (Vidyasagar 1985 a). The reduced-order controller obtained by the Hankel norm approximation method in Glover (1984) is generally a non-strictly-proper one. One might then consider working with the strictly proper part only of the optimal Hankel approximation; let us therefore consider the L^∞ norm bound of the Hankel norm optimal approximation to a strictly proper system $G(s)$, when we also require the reduced-order system $\hat{G}(s)$ strictly proper. For an rth-order Hankel norm optimal approximation of $\hat{G}(s)$ which is constrained to be strictly proper we know, see Glover (1984), that there exists a constant matrix E_0, such that

$$\| G(j\omega) - \hat{G}(j\omega) - E_0 \|_{L^\infty} \leqslant \mathrm{tr}(\Sigma_2) = (\sigma_{r+1} + \dots + \sigma_n) \tag{5.1}$$

Letting $\omega \to \infty$, (5.1) becomes

$$\| E_0 \|_{L^\infty} \leqslant \mathrm{tr}(\Sigma_2)$$

Thus,

$$\| G(j\omega) - \hat{G}(j\omega) \|_{L^\infty} \leqslant \| G(j\omega) - \hat{G}(j\omega) - E_0 \|_{L^\infty} + \| E_0 \|_{L^\infty} \leqslant 2\mathrm{tr}(\Sigma_2) \tag{5.2}$$

This means that when we require the Hankel norm optimal approximation $\hat{G}(s)$ to be strictly proper, the L^∞ norm error bound will be the same as the error bound of the balanced realization truncated approximation. Since the algorithm of the Hankel norm optimal approximation is more complex than the algorithm of balanced realization, it may therefore offer no advantage at all.

(*e*) There would seem to be no difficulty associated with carrying the ideas over to discrete-time systems.

Appendix

1. *Proof of Lemma* 3.1

Let

$$G(s) = C(sI_n - A)^{-1}B = \tilde{Y}^{-1}(s)\tilde{X}(s) \tag{A 1}$$

and

$$K(s) = F(sI_n - A + BF + LC)^{-1}L = N(s)D^{-1}(s) \tag{A 2}$$

where

$$\tilde{X}(s) = C(sI_n - A + LC)^{-1}B$$

$$\tilde{Y}(s) = I_m - C(sI_n - A + LC)^{-1}L$$

$$N(s) = F(sI_n - A + BF)^{-1}L$$

$$D(s) = I_m + C(sI_n - A + BF)^{-1}L$$

Then we have (*Nett et al.* 1984)

$$\tilde{X}(s)N(s) + \tilde{Y}(s)D(s) = I_m \tag{A 3}$$

For $H(G, K)$ defined by (1.7), ε defined by (3.1) and

$$K_1(s) = N_1(s)D_1^{-1}(s)$$

let

$$\delta \triangleq \| H(G(j\omega), K(j\omega)) - H(G(j\omega), K_1(j\omega)) \|_{L^\infty}$$

and note that

$$G(I_l + KG)^{-1} = (I_m + GK)^{-1}G$$

We have

$$H(G, K) - \begin{bmatrix} 0 & 0 \\ 0 & I_l \end{bmatrix} = \begin{bmatrix} (I_m + GK)^{-1} & -G(I_l + KG)^{-1} \\ K(I_m + GK)^{-1} & (I_l + KG)^{-1} - I_l \end{bmatrix}$$
$$= \begin{bmatrix} (I_m + GK)^{-1} & -(I_m + GK)^{-1}G \\ K(I_m + GK)^{-1} & -K(I_m + GK)^{-1}G \end{bmatrix}$$
$$= \begin{bmatrix} (I_m + GK)^{-1} \\ K(I_m + GK)^{-1} \end{bmatrix} [I_m - G]$$

Similarly, we have

$$H(G, K_1) - \begin{bmatrix} 0 & 0 \\ 0 & I_l \end{bmatrix} = \begin{bmatrix} (I_m + GK_1)^{-1} \\ K_1(I_m + G + K_1)^{-1} \end{bmatrix} [I_m \quad -G]$$

Thus

$$\delta = \left\| \begin{bmatrix} (I_m + GK)^{-1} - (I_m + GK_1)^{-1} \\ K(I_m + GK)^{-1} - K_1(I_m + GK_1)^{-1} \end{bmatrix} [I_m \quad -G] \right\|_{L^\infty}$$
$$= \left\| \begin{bmatrix} (I_m + GK)^{-1} - I_m - (I_m + GK_1)^{-1} + I_m \\ K(I_m + GK)^{-1} - K_1(I_m + GK_1)^{-1} \end{bmatrix} [I_m \quad -G] \right\|_{L^\infty}$$
$$= \left\| \begin{bmatrix} -GK(I_m + GK)^{-1} + GK_1(I_m + GK_1)^{-1} \\ K(I_m + GK)^{-1} - K_1(I_m + GK_1)^{-1} \end{bmatrix} [I_m \quad -G] \right\|_{L^\infty}$$
$$= \left\| \begin{bmatrix} -G \\ I_l \end{bmatrix} [K(I_m + GK)^{-1} - K_1(I_m + GK_1)^{-1}] [I_m \quad -G] \right\|_{L^\infty}$$

From (A 1), (A 2) and (A 3), we have

$$\delta = \left\| \begin{bmatrix} -G \\ I_l \end{bmatrix} [N\tilde{Y} - N_1\Delta^{-1}\tilde{Y}][I_m \quad -G] \right\|_{L^\infty}$$
$$= \left\| \begin{bmatrix} -G \\ I_l \end{bmatrix} (N - N_1\Delta^{-1})[\tilde{Y} \quad -\tilde{X}] \right\|_{L^\infty}$$

where $\Delta = \tilde{X}N_1 + \tilde{Y}D_1$. Using (A 3), we have

$$\Delta^{-1} = (\tilde{X}N_1 + \tilde{Y}D_1)^{-1} = (I_m - \tilde{X}(N - N_1) - \tilde{Y}(D - D_1))^{-1}$$
$$= \left[I_m - [\tilde{Y} \quad \tilde{X}] \begin{bmatrix} D - D_1 \\ N - N_1 \end{bmatrix} \right]^{-1}$$

Let

$$\eta = \| [\tilde{Y}(j\omega), \tilde{X}(j\omega)] \|_{L^\infty}$$

Then, if

$$\varepsilon\eta < 1 \tag{A 4}$$

we have

$$\Delta^{-1} = I_m + [\tilde{Y} \quad \tilde{X}] \begin{bmatrix} D - D_1 \\ N - N_1 \end{bmatrix} + O_m(\varepsilon^2)$$

where $O_m(\varepsilon^2)$ refers to a matrix and $\| O_m(\varepsilon^2) \|_{L^\infty} / \varepsilon^2$ is bounded as $\varepsilon \to 0$. Hence

$$N - N_1\Delta^{-1} = N - N_1[I_m + \tilde{X}(N - N_1) + \tilde{Y}(D - D_1) + O_m(\varepsilon^2)]$$
$$= (N - N_1)[I_m + \tilde{X}(N - N_1) + \tilde{Y}(D - D_1)]$$
$$- N[\tilde{X}(N - N_1) + \tilde{Y}(D - D_1)] - N_1 O_m(\varepsilon^2)$$
$$= [-N\tilde{Y} \quad I_l - N\tilde{X}] \begin{bmatrix} D - D_1 \\ N - N_1 \end{bmatrix} + \bar{O}_m(\varepsilon^2)$$

where $\bar{O}_m(\varepsilon^2) = (N - N_1)\tilde{X}(N - N_1) + (N - N_1)\tilde{Y}(D - D_1) - N_1 O_m(\varepsilon^2)$.

Finally, we have Lemma 3.1.

If (A 4) holds, then

$$\delta = \| H(G, K) - H(G, K_1) \|_{L^\infty}$$
$$\leqslant \left\| \begin{bmatrix} -G \\ I_l \end{bmatrix} [-N\tilde{Y} \quad I_l - N\tilde{X}] \begin{bmatrix} D - D_1 \\ N - N_1 \end{bmatrix} [\tilde{Y} \quad -\tilde{X}] + \tilde{O}_m(\varepsilon^2) \right\|_{L^\infty}$$

where

$$\tilde{O}_m(\varepsilon^2) = \begin{bmatrix} -G \\ I_l \end{bmatrix} \bar{O}_m(\varepsilon^2) [\tilde{Y} \quad -\tilde{X}]$$

Let

$$\varepsilon = \left\| \begin{bmatrix} -G \\ I_l \end{bmatrix} [-N\tilde{Y} \quad I_l - N\tilde{X}] \right\|_{L^\infty} = \left\| I_{m+l} - \begin{bmatrix} D \\ N \end{bmatrix} [\tilde{Y} \quad \tilde{X}] \right\|_{L^\infty}$$

Then

$$\delta \leqslant \xi \varepsilon \eta + O(\varepsilon^2)$$

where

$$O(\varepsilon^2) = \| \tilde{O}_m(\varepsilon^2) \|_{L^\infty}$$

Proof of Lemma 3.2

Using notation as in § 3, we consider the closed-loop system with the reduced-order controller in Fig. 2.

We know that if $K(s)$ stabilizes $G(s)$, and $K(s)$ and $K_1(s)$ have the same number of unstable poles, then $K_1(s)$ will be stabilizing when either

$$\| (K_1 - K) G (I_l + KG)^{-1} \|_{L^\infty} < 1 \tag{A 5}$$

or

$$\| (I_m + GK)^{-1} G (K_1 - K) \|_{L^\infty} < 1 \tag{A 6}$$

From the factorization of K and K_1, we have

$$\begin{aligned} K_1 - K &= N_1 D_1^{-1} - N D^{-1} \\ &= N D^{-1} (D - D_1) D_1^{-1} - (N - N_1) D_1^{-1} \\ &= [N D^{-1} (D - D_1) - (N - N_1)] D^{-1} [I_m - (D - D_1) D^{-1}]^{-1} \end{aligned}$$

It follows that

$$(I_m + GK)^{-1} G (K_1 - K) = D \tilde{X} [N D^{-1} \quad -I_l] \begin{bmatrix} D - D_1 \\ N - N_1 \end{bmatrix} D^{-1} [I_m - (D - D_1) D^{-1}]^{-1}$$

Hence, if

$$1 - \operatorname{tr}(\Sigma_2) \| D^{-1} \|_{L^\infty} > 0 \tag{A 7}$$

then

$$\begin{aligned} &\| (I_m + GK)^{-1} G (K_1 - K) \|_{L^\infty} \\ &\qquad \leqslant \| [I_m - D\tilde{Y} \quad -D\tilde{X}] \|_{L^\infty} \operatorname{tr}(\Sigma_2) \| D^{-1} \|_{L^\infty} [1 - \operatorname{tr}(\Sigma_2) \| D^{-1} \|_{L^\infty}]^{-1} \end{aligned}$$

If we arrange for

$$\| [I_m - D\tilde{Y} \quad -D\tilde{X}] \|_{L^\infty} \| D^{-1} \|_{L^\infty} \frac{\operatorname{tr}(\Sigma_2)}{1 - \operatorname{tr}(\Sigma_2) \| D^{-1} \|_{L^\infty}} < 1 \tag{A 8}$$

then (A 6) holds. The condition

$$\operatorname{tr}(\Sigma_2) < \frac{1}{[1 + \| I_m - D\tilde{Y} \quad -D\tilde{X} \|_{L^\infty}] \| D^{-1} \|_{L^\infty}}$$

is equivalent to (A 8) and guarantees (A 7).

References

Adamjan, V. M., Arov, D. Z., and Krein, M. G., 1971, *Math. U.S.S.R. Sb.*, **15**.
Anderson, B. D. O., 1985, *Proc. IFAC Workshop on Model Error Concepts and Compensation.*
Anderson, B. D. O., and Moore, J. B., 1971, *Linear Optimal Control* (Englewood Cliffs, NJ: Prentice-Hall).

DAVIS, J. A., and SKELTON, R. E., 1984, *Syst. Control Lett.*, **4**, 79.
ENNS, D., 1984, Ph.D. thesis, Dept of Electrical Engineering, Stanford University.
GLOVER, K., 1984, *Int. J. Control*, **39**, 1115.
GLOVER, K., and LIMEBEER, D. J. N., 1983, *Proc. Amer. Contr. Conf.*, San Francisco, CA, 644.
JONCKHEERE, E. A., and SILVERMAN, L. M., 1981, *Internat. Symp. Theory Networks and Systems*, Santa Monica, CA.
KUNG, S.-Y., and LIN, D. W., 1981, *I.E.E.E. Trans. autom. Control*, **26**, 832.
KWAKERNAAK, H., and SIVAN, R., 1972, *Linear Optimal Control Systems* (New York: Wiley).
LATHAM, G. A., and ANDERSON, B. D. O., 1985, *Syst. Control Lett.*, **5**, 229.
MOORE, B. C., 1981, *I.E.E.E. Trans. autom. Control*, **26**, 17.
NETT, C. N., JACOBSON, C. A., and BALAS, M. J., 1984, *I.E.E.E. Trans. autom. Control*, **29**, 831.
PERNEBO, L., and SILVERMAN, L. M., 1982, *I.E.E.E. Trans. autom. Control*, **27**, 382.
VERRIEST, E. I., 1981 a, *Proc. 24th Symp. on Circuits and Systems*. 365; 1981 b, *Proc. 20th-I.E.E.E. Conf. on Decision and Control*, p. 686.
VIDYASAGAR, M., 1985 a, *Syst. Control Lett.*, **5**, 413; 1985 b, *Control Systems Synthesis: a factorization Approach* (Cambridge, Mass.: MIT Press), to be published.
YOUSUFF, A., and SKELTON, R. E., 1984, *I.E.E.E. Trans. autom. Control*, **29**, 254.