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The Testing for Optimality of Linear Systems†

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Abstract

Procedures exist for designing optimal linear controllers for linear systems subjected to additive noise at the input and output. The paper considers the inverse problem of deciding whether a given controller, specified by its transfer function, is optimal for a prescribed linear plant. A closed form test for optimality is derived for single input, single output plants, while for all other situations the testing for optimality is reduced to the derivation of a non-singular solution of a prescribed quadratic matrix equation.

§ 1. Introduction

Of recent years there have appeared a number of results which unify parts of classical and modern control theory. While certainly of inherent interest, the results are also useful in that they assist the overall development of the subject, and provide greater insight into the already established theory.

One such result is concerned with showing that a linear, time-invarient, optimal control system is usually also a system in which feedback reduces the sensitivity of the system to plant parameter variations (Kalman 1964, Anderson 1966). A feature of a result such as this is the need to shuttle between the frequency-domain description of a system (the usual classical description) and the time-domain description of a system (the usual modern description); the modern question which asks, when is the linear system optimal?, is answered in a classical way by writing down a frequency-domain criterion. This criterion could naturally be expressed in time-domain terms, but only awkwardly and without the intuitive appeal of the frequency-domain format.

With such a result available, it is possible to examine a design carried out via classical methods, and to decide whether or not the design is also an optimal one in the sense of the modern theory.

Interest in control systems has spread rapidly of late to encompass *stochastic control* problems, or problems of control in a noisy environment. One of the most substantial advances has been due to Kalman and Bucy (1961). Their theory in essence shows how to construct a 'best' estimate of the state of a linear, finite-dimensional dynamic system, subjected at

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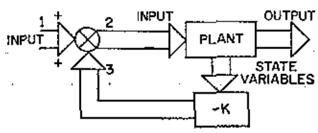
the input and output to additive white gaussian noise. This best estimate appears at the output of a state estimator, which is itself a linear, finite-dimensional dynamic system driven by the input and output of the plant.

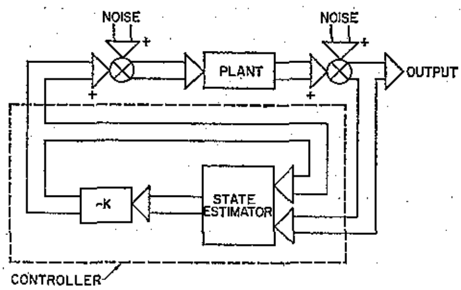
Deterministic linear optimal systems, or linear regulators, use for their input linear functions of the instantaneous state, as shown in fig. 1. In the figure K denotes a constant matrix, which multiplies the state vector to produce a plant input vector, the entries of which are the linear functions referred to. The thick flow lines are to symbolize the multi-variable nature of the plant. Gunckel (1961) and other workers have shown that in the stochastic case, optimality is achieved by feeding back to the plant input linear functions of the state estimate, derived from a Kalman–Bucy filter, and that these linear functions are the same as those used in the deterministic case, see fig. 2. If external inputs are applied to the plant in addition to the fedback component of the input, then these external inputs must be applied also to the input of the state estimator. The diagrams of fig. 3 (*a*) or fig. 3 (*b*) then apply; the difference between the latter two figures lies in the way the plant input, which is the sum of the external input and the fedback input, is applied to the input of the state estimator.

Fig. 1



Linear feedback system.

Fig. 2

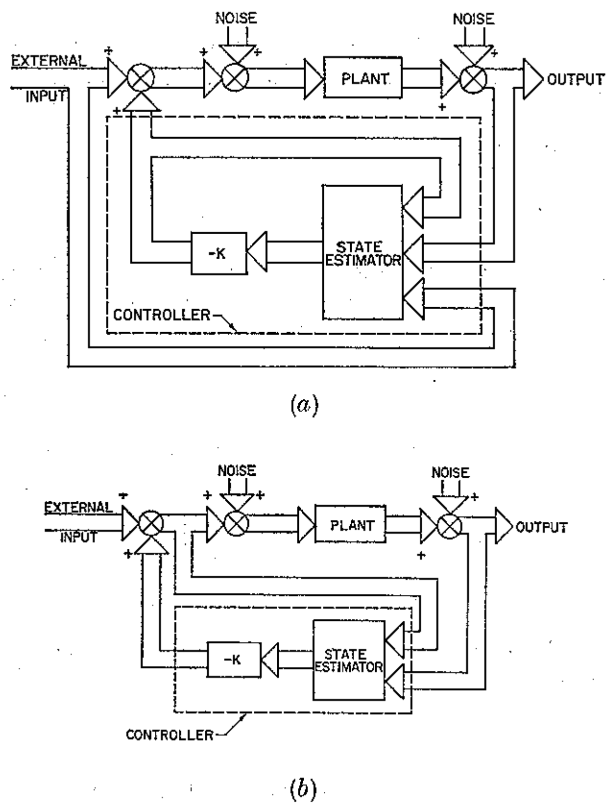


Linear regulator system.

The combination of the state estimator and the block that has been labelled with the matrix K constitutes the controller for the plant. The controller of fig. 2 has one vector input only, whilst those of fig. 3 have two. Since the state estimators under consideration are linear, time-invariant, finite-dimensional, dynamical systems, they, and thus also the controllers, can be described from the input–output point of view by a transfer function matrix. In the case of the controllers of fig. 3, this matrix is naturally sub-divided into two sub-matrices, in accordance with the sub-division of the controller input into two (vector) inputs.

It is now possible to formulate the following naturally arising problem: given the transfer function matrix of a linear, time-invariant, finite-dimensional plant and a similar matrix for a controller, determine whether or not the controller may be represented as a cascade of an optimal state estimator, and an optimal linear feedback law K. In other words, given a frequency-domain description of a plant and controller, decide whether the system is (stochastically) optimal.

Fig. 3



Optimal linear system-two viewpoints.

If such a system is found to be stochastically optimal, then the somewhat qualitative conclusions can be drawn that the system will behave better in the presence of plant parameter variations and external noise than one which is not optimal. The knowledge of a criterion for optimality thus allows the checking of a design, perhaps derived via classical design methods, in respect of its sensitivity and its noise immunity.

This paper presents a detailed consideration of the problem of establishing optimality. It would appear that for some of the situations available, the problem is of extreme difficulty, while for other situtations an explicit solution is available. In particular, when the plant is single-input, single-output and the controller is of the type shown in fig. 3 (*a*) or fig. 3 (*b*), it is relatively simple to check optimality.

The paper is structured in the following way. In § 2, brief reviews of the control and filtering problems are given, and frequency-domain criteria for the separate optimality of the control law and the state estimator are given. Section 3 discusses the question of establishing controller optimality given a controller transfer function matrix for a controller of the form shown in fig. 2. Section 4 discusses the same question for controllers of the form shown in fig. 3.

§ 2. Review of Control and Filtering Problems

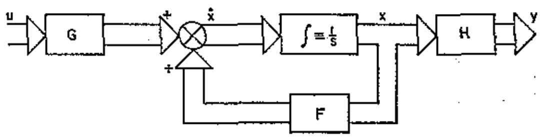
The plants under consideration are linear, time-invariant, finite-dimensional, dynamic systems which are assumed to be described by deterministic equations of the form:

$$\dot{x} = Fx + Gu, \tag{1a}$$

$$y = Hx. \tag{1b}$$

Here u is an m-vector, the input; y is a p-vector, the output; and x is an n-vector, the state; the matrices F, G and H are constant and of dimension $n \times n$, $n \times m$, $p \times n$ respectively. Diagrammatic representation is given in fig. 4.

Fig. 4



State variable view of a linear system.

The design of optimal linear regulators proceeds by examining the following performance index:

$$V(x_0, t_0, u) = \int_{t_0}^{\infty} (x'Qx + u'Ru)\, dt. \tag{2}$$

(The prime denotes matrix transposition.) This performance index is a function of the initial state of the plant, x_0, the initial time, t_0, and the control u employed over the interval (t_0, ∞). Usually the matrices Q and R are non-negative and positive definite respectively. By a suitable normalization of the input vector, it is then possible to take $R = I$, the unit matrix; thus (2) becomes:

$$V(x_0, t_0, u) = \int_{t_0}^{\infty} (x'Qx + u'u)\, dt. \tag{3}$$

In the optimal linear regulator problem, the object is to return a non-zero state x_0 to the zero state by a control u which minimizes the integral (2). It can be shown, see, for example, Athans and Falb (1966), that the minimization can be achieved by selecting

$$u(t) = -Kx(t), \tag{4}$$

where K is a constant matrix depending on F, G and Q. The optimal system of fig. 1 then results.

The inverse problem of optimality can be explained as follows. A plant is specified with state-variable feedback; determine whether or not there exists a performance index of the form of (3) for which the feedback law K is optimal.

The problem is solved by Kalman (1964) for the single-loop case and Anderson (1966) for the multi-loop case; it is shown that an optimal design implies:

$$[I + G'(-j\omega I - F')^{-1}K'][I + K(j\omega I - F)^{-1}G] \geqslant I \tag{5}$$

for all ω, or equivalently:

$$I - [I - G'(-j\omega I - F' + K'G')^{-1}K'][I - K(j\omega I - F + GK)^{-1}G] \geqslant 0 \tag{6}$$

for all ω.

The notation $\geqslant 0$ is shorthand for 'is non-negative definite.'

Conversely, if (5) and (6) are satisfied with strict inequality signs, then the closed-loop system is optimal for some performance index which may be calculated by the methods of Anderson (1966), under the condition that it is stable, and that $[F, K]$ is completely observable (Kalman 1964).

The matrix $I + K(j\omega I - F)^{-1}G$ appearing in (5) is the *return difference* matrix: its inverse is the transfer function matrix relating the external input at point 1 of fig. 1 to the plant input at point 2, and is given by $I - K(j\omega I - F + GK)^{-1}G$. Condition (5) or (6) with the strict inequality sign is the condition for improvement of the system in its sensitivity to plant parameter variations, see Cruz and Perkins (1964) and Anderson and Newcomb (1966).

Let us now consider the optimal state estimator or filtering problems. Instead of eqns. (1), we have:

$$\dot{x} = Fx + Gu + Gv, \tag{7 a}$$

$$z = Hx + w, \tag{7 b}$$

where v and w are assumed to be independent white gaussian noise, of mean zero, with

$$\text{cov}\,[Gv(t), Gv(\tau)] = \hat{Q}\delta(t - \tau), \tag{8}$$

and

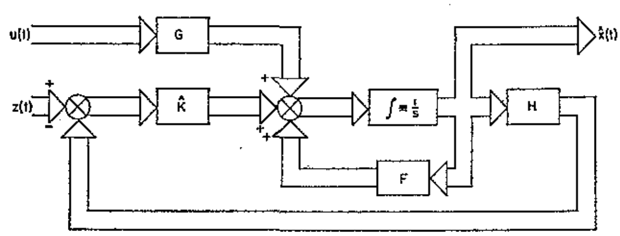
$$\text{cov}[w(t), w(\tau)] = \hat{R}\delta(t - \tau). \tag{9}$$

It is moreover assumed that $\hat{R}$ is non-singular. Both $\hat{Q}$ and $\hat{R}$ are by their nature symmetric and non-negative definite.

The optimal state estimator has the plant input u and noisy plant output z for its inputs, and produces at its output a state estimate $\hat{x}$; this estimate is a best or optimal estimate in the sense that it minimizes the expected error in estimating any linear functional of the state (Kalman and Bucy 1961). The form of the state est mator is shown in fig. 5. As may be seen from the diagram, the estimator consists of a model of the plant together with a feedback loop which feeds back linear functions of the difference between the plant output and that variable of the estimator which simulates the plant output. The matrix $\hat{K}$ which defines these linear functions is determined from F, H, $\hat{Q}$ and $\hat{R}$.

As with the regulator problem, by normalization, this time of the outputs, we can assume $\hat{R}=I$. It is then pertinent to ask: given an estimator of the form of (4), with an arbitrary $\hat{K}$, is it optimal for some $\hat{Q}$?

Fig. 5



Detail of state estimator.

The very close connection between estimation and control, brought out by Kalman and Bucy (1961), makes this question easy to answer. A detailed analysis of the estimation and control problems shows that

$$K=G'P, \tag{10}$$

where P is the positive definite solution of

$$F'P+PF-PGG'P+Q=0. \tag{11}$$

(P is unique for a completely controllable plant which is also completely observable with respect to $Q^{1/2}$, see Kalman 1964.) On the other hand,

$$\hat{K}=\hat{P}H', \tag{12}$$

where P is the positive definite solution of

$$F\hat{P}+\hat{P}F'-\hat{P}H'H\hat{P}+\hat{Q}=0, \tag{13}$$

(again with suitable conditions, $\hat{P}$ is unique). Evidently, by making the interchanges $F\leftrightarrow F'$, $G\leftrightarrow H'$, $K\leftrightarrow\hat{K}'$ and $Q\leftrightarrow\hat{Q}$, eqns. (12) and (13) follow from (10) and (11). By making similar interchanges in (5), we observe therefore that optimal estimation implies that

$$[I+H(-j\omega I-F)\hat{K}][I+\hat{K}'(j\omega I-F')^{-1}H']\geqslant I, \tag{14}$$

for all ω; conversely if the strict inequality holds for some $\hat{R}$, then the estimation scheme is optimal for some $\hat{Q}$.

In this section we have shown how optimality of a control law and a state estimator can be examined separately. When the feedback law and state estimator are incorporated in one unit, with no distinct separation of the two, the examination of optimality becomes much harder. This is the subject of the next two sections.

§ 3. Optimality with no External Inputs

In this section we consider the somewhat artificial situation of fig. 2. It is clear that very few plants will be designed so that there can be no external input, while if there is an external input then the state estimator in fig. 2 is not correctly depicted. There is, however, some sense in considering the situation of fig. 2 as being devoid of external input for two reasons. First, an optimal linear regulator, when it is actually regulating, requires only fed-back signals. If we are prepared to accept non-optimal operations when some other activity than regulating is taking place, i.e. when an activity for which the system has not been designed is taking place, then the situation of fig. 2 is valid. Second, the typical classically designed controller will not incorporate the plant external input as one of its inputs, and should we be interested in checking the optimality of a number of classical designs, again fig. 2 provides the more appropriate viewpoint.

As far as the first point is concerned, it is fair to point out that an optimally designed linear servomechanism requires a feedback law just like that of the regulator, *and* an external input. The overall system of fig. 2 in conjunction with an external input will then function not just as a regulator, but as a follow-up system where the output can be made to track some prescribed path.

Suppose now that the plant is described by a transfer function matrix $W(s)$ and the controller by a transfer function matrix $W_c(s)$. Then $W(s)$ does not determine the triple $\{F, G, H\}$ uniquely in the state variable description (1). It is however true that if a description of the form of (1) is given, and this description is completely controllable and completely observable (Kalman 1963), then with the replacements $F \leftrightarrow TFT^{-1}$, $G \leftrightarrow TG$, $H \leftrightarrow HT^{-1}$ for any non-singular matrix T, the resulting state variable equations define a plant with the same transfer function matrix. Moreover, if T ranges through all non-singular matrices, all state-variable, completely controllable, completely observable descriptions result.

A triple which defines such a description is termed a minimal realization of $W(s)$ (Kalman 1963, 1965) and the dimension of the F matrix, is termed the dimension of the realization.

By taking the Laplace transform of (1) and eliminating $X(s)$ it is found that

$$Y(s) = H(sI - F)^{-1}GU(s), \tag{15}$$

and thus

$$W(s) = H(sI - F)^{-1}G. \tag{16}$$

Methods are available, see, e.g. Kalman (1963), for passing from $W(s)$ to a minimal realization $\{F, G, H\}$ (for which (16) holds). All other minimal realizations may be written then as $\{TFT^{-1}, TG, HT^{-1}\}$ for non-singular T.

Consider now fig. 2 and fig. 5 together. Explicit calculation readily shows that

$$W_c(s) = K(sI - F + GK + \hat{K}H)^{-1}\hat{K}. \tag{17}$$

Because of the way this form for $W(s)$ results, it will generally be true that $\{F - GK - \hat{K}H, \hat{K}, K\}$ is a minimal realization; thus minimal realizations of $W_c(s)$ and $W(s)$ will *generally* have the same dimension.

Now suppose that we *pick* on a fixed triple $\{F, G, H\}$ so that (16) holds. Then in (17) there remain two unknowns, K and $\hat{K}$. If it is possible to extract K and $\hat{K}$ from (17), then optimality can readily be checked using the results of § 2. This extraction, which simply amounts to solving (17) for K and $\hat{K}$ when everything else is known, is however, easier said than done. The best we can do is summed up in the following theorem:

Theorem 1. If $W(s)$, $W_c(s)$ are the transfer function matrices of a plant and optimal controller respectively, if $\{F, G, H\}$ is a minimal realization for $W(s)$ and $\{F_1, \hat{K}_1, K_1\}$ a realization for $W_c(s)$, and if F and F_1 have the same dimension, then there exists a non-singular matrix T such that

$$F_1T - TF + TGK_1T + \hat{K}_1H = 0, \tag{18}$$

and moreover

$$\hat{K} = T^{-1}\hat{K}_1, \tag{19}$$

$$K = K_1T, \tag{20}$$

from which optimality can be verified.

Proof. Since $W_c(s)$ can be written in the form of (17) for some K and $\hat{K}$, it follows that $\{F_1, \hat{K}_1, K_1\}$, and $\{F - GK - \hat{K}H, \hat{K}, K\}$ are two minimal realizations for W_c. Thus there exists a non-singular matrix T such that

$$T^{-1}F_1T = F - GK - \hat{K}H, \tag{21}$$

$$T^{-1}\hat{K}_1 = \hat{K}, \tag{19}$$

and

$$K_1T = K. \tag{20}$$

Using (19) and (20) in (21) leads to:

$$F_1T - TF + TGK_1T + \hat{K}_1H = 0. \tag{18}$$

This completes the proof.

Note that in (18), all matrices other than T are known. *Thus if* (19) *can be solved for* T, *optimality can be checked* (when $\hat{K}$ and K have been found from (20) and (21)) *by using the results of* § 2.

The situation however is still unsatisfactory. We have required $W(s)$ and $W_c(s)$ to have minimal realizations of the same dimension, and though this will in general be the case, it may not always be so. We note also that the solution of (19) for T poses extremely formidable computational problems, and raises many uniqueness questions.* Even in the case where F and F_1 degenerate to scalars, and thus T is a scalar also, the possibilities arise of there being no solution, two solutions, or one degenerate solution. Kalman (1966) has commented that in the two-solution case, there seems to be in some sense interchangeability of noise performance and reduction in sensitivity to plant parameter variation.

§ 4. Optimality with External Inputs

In this section we consider controllers of the form shown in fig. 3 (*a*) and 3 (*b*). As before, we suppose the plant is described by a transfer function matrix $W(s)$ with a minimal realization $\{F, G, H\}$. We shall suppose further that the controller of fig. 3 (*a*) is specified by two transfer function matrices: $W_c(s)$, the matrix relating the controller output to that controller input derived from the plant output and $W_{c1}(s)$, the matrix relating the controller output to that controller input derived from the external input to the plant. Then figs. 3 (*a*) and 5 can be used to show that

$$W_{c1}(s) = K(sI - F + GK + \hat{K}H)^{-1}G, \tag{22}$$

while the relation (17), repeated here for convenience,

$$W_c(s) = K(sI - F + GK + \hat{K}H)^{-1}\hat{K}, \tag{23}$$

still holds.

For the controller of fig. 3 (*b*) we define $W_c(s)$ as before, and $W_{c2}(s)$ to be the transfer function matrix relating the controller output to that controller input derived from the plant input. Explicit calculation yields:

$$W_{c2}(s) = K(sI - F + \hat{K}H)^{-1}G. \tag{24}$$

It is not difficult to verify algebraically the following relations between $W_{c1}(s)$ and $W_{c2}(s)$:

$$W_{c1}(s) = [I + W_{c2}(s)]^{-1}W_{c2}(s), \tag{25}$$

and

$$W_{c2}(s) = [I - W_{c1}(s)]^{-1}W_{c1}(s). \tag{26}$$

Because of (25) and (26), questions relating to the optimality of the controller of fig. 3 (*a*) can then be settled by considering the optimality of the controller of fig. 3 (*b*) and (vice versa).

The checking of optimality for the controller of fig. 3 (*b*) can now be viewed as a problem of finding K and $\hat{K}$ such that both (23) and (24) hold (when the triple $\{F, G, H\}$ has been specified), and then using the tests of § 2 to examine the resulting K and $\hat{K}$ for optimality. Note that if, for example, (24) is used to derive K and $\hat{K}$, then these K and $\hat{K}$ must also satisfy (23) *as well as* the optimality conditions of § 2.

The procedure suggested in the last section (using (23)) is still naturally a valid one for determining K and $\hat{K}$, but in the case where the plant is single-input, single-output, an alternative and much simpler procedure exists to identify K and $\hat{K}$, which are then unique.

* Techniques for solving this equation are discussed in a note by the author "Solution of Quadratic Matrix Equations" to appear in *Electronics Letters*.

The identification proceeds from (24) rather than (23). Let us suppose

$$W_{c2}(s)=\frac{a_n s^{n-1}+\ldots+a_1}{s^n+b_n s^{n-1}+\ldots+b_1}, \tag{27}$$

while

$$W(s)=\frac{c_n s^{n-1}+\ldots+c_1}{s^n+d_n s^{n-1}+\ldots+d_1}. \tag{28}$$

Because $\{F, G, H\}$ is a minimal realization for the plant, $\{F, H\}$ is completely observable. A canonical form for a completely observable plant is available in the single-output case, see Kalman (1963), which allows us to assume that

$$F=\begin{bmatrix} 0 & 0 & 0\ldots 0 & -d_1 \\ 1 & 0 & 0\ldots 0 & -d_2 \\ 0 & 1 & 0\ldots 0 & -d_3 \\ \cdot & \cdot & \cdot\quad\ \ \cdot & \cdot \\ \cdot & \cdot & \cdot\quad\ \ \cdot & \cdot \\ \cdot & \cdot & \cdot\quad\ \ \cdot & \cdot \\ 0 & 0 & 0\ldots 1 & -d_n \end{bmatrix} \tag{29}$$

with

$$H=[\ 0\quad 0\quad 0\ldots 1\ ] \tag{30}$$

and

$$G=\begin{bmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_n \end{bmatrix} \tag{31}$$

Now from (24) and (27) we observe that

$$\det(sI-F+\hat{K}H)=s^n+b_n s^{n-1}+\ldots+b_1, \tag{32}$$

while use of (29) and (30) shows that if

$$\hat{K}=\begin{bmatrix} \hat{k}_1 \\ \hat{k}_2 \\ \cdot \\ \cdot \\ \cdot \\ \hat{k}_n \end{bmatrix} \tag{33}$$

then

$$\det(sI-F+\hat{K}H)=s^n+(d_n+\hat{k}_n)s^{n-1}+\ldots+(d_1+\hat{k}_1). \tag{34}$$

Equations (32) and (34) now identify $\hat{K}$ via

$$\hat{k}_i = b_i - d_i \quad (i = 1, 2, \dots, n). \tag{35}$$

To find K, we must solve the equation:

$$K(sI - F + \hat{K}H)^{-1}G = \frac{a_n s^{n-1} + \dots + a_1}{s^n + b_n s^{n-1} + \dots + b_1}, \tag{36}$$

for K, where all other matrices on the left-hand side are known. There are various ways of doing this, and we note here one, due to Ho (1966), which is suited to digital computation.

An arbitrary minimal realization of $W_{c2}(s)$ is chosen, call it $(\tilde{F}, \tilde{G}, \tilde{K})$. From $F - \hat{K}H$ and G there is formed the matrix:

$$W = [G \vdots (F - \hat{K}H)G \vdots \dots \vdots (F - \hat{K}H)^{n-1}G] \tag{37}$$

and from $\tilde{F}$ and $\tilde{G}$ the matrix:

$$\tilde{W} = [\tilde{G} \vdots \tilde{F}\tilde{G} \dots \vdots \tilde{F}^{n-1}\tilde{G}], \tag{38}$$

then K is given by:

$$K = \tilde{K}\tilde{W}W(WW')^{-1}. \tag{39}$$

(The existence of the inverse in (39) is actually guaranteed by complete controllability.)

The following theorem summarizes the content of the section:

Theorem 2. If $W(s)$ is the transfer function of a (single-input, single-output) plant, and $W_c(s)$ and $W_{c2}(s)$ the transfer functions assocoiated with a controller as shown in fig. 3 (*b*), there is an explicit procedure for checking the optimality of the controller. The checking proceeds by identifying plant and estimator feedback laws, K and $\hat{K}$, from $W_{c2}(s)$, observing that $W_c(s)$ has the correct form as determined by K and $\hat{K}$, and checking the optimality relations of § 2.

§ 5. Conclusions

It is clear that the optimality of a controller is considerably more difficult to establish than the optimality of a linear feedback law. For the single-input, single-output plant case, treated in § 4, an explicit method for checking optimality (via a finite number of steps) is given. Other cases however require the solution of a non-linear matrix equation (see § 3) for the verification of optimality. In view of the non-uniqueness of solutions to this equation, its investigation presents many difficulties.

Acknowledgments

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