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Exponential convergence of a model reference adaptive controller for plants with known high frequency gain

Soura DASGUPTA
Department of Electrical and Computer Engineering, University of Iowa, Iowa City, IA 52242, USA

Brian D.O. ANDERSON
Department of Systems Engineering, Australian National University, G.P.O. Box 4, Canberra, A.C.T. 2601, Australia

Ah-Chung TSOI
Department of Electrical & Electronic Engineering, Faculty of Military Studies, Royal Military College, Duntroon, A.C.T. 2600, Australia

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Abstract: A model reference adaptive controller proposed by Morse is studied. It is shown to be exponentially convergent when the high frequency gain is known and when the reference inputs satisfy a persistence of excitation assumption, which allows the inputs to be other than a linear combination of sinusoids.

Keywords: Model reference adaptive control, Adaptive control, Exponential convergence.

1. Introduction

In this paper we consider the exponential convergence of parameter estimates generated by a model reference adaptive control (MRAC) algorithm [1] for plants with known high frequency gain. Exponential convergence is important for robustness. Adaptive algorithms without such convergence have been known to display unacceptable behaviour in the presence of modelling inadequacies and noise [2–4]. Exponentially convergent algorithms on the other hand are totally stable [5, pp. 107–108], a property which allows them to retain stability even in the face of modest departures from the idealizing assumptions.

In [1], the algorithm in question has been shown to be globally stable in the sense that irrespective of the initial parameter estimates all system signals retain boundedness. This result does not require the knowledge of the high frequency gain parameter. Nor does it place any restriction on the reference input, except that it be bounded and piecewise continuous. However, even though the tracking error converges to zero, the parameter error may fail to do so. In [6], we have shown that when the high frequency gain is unknown, the algorithm will not be exponentially convergent and thus may not be robust, or even practical. Here, we demonstrate exponential convergence, whenever the reference input is persistently exciting (p.e.) and the high frequency gain is known. The persistence of excitation condition we shall propose will depend on the reference input only. Since the completion of an earlier version of this paper [7a], in [8,9] yet another MRAC scheme discussed in [10] has been analysed with similar conclusions. The persistence of excitation condition used in [8,9] is however less general than that used here in that the external input must have a certain number of spectral lines. The main contribution of this paper is in fact the work with a general condition.

In Section 2 we briefly recapitulate the essentials of the schemes in [1]. Section 3 states the p.e. conditions. Our proof is given in the appendix and appeals to simple tools developed in [7a,7b] and an important inequality from [12].

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In the sequel we shall abuse notation by referring to $h(s)$ as the Laplace transform of $h(t)$. Quantities like

$$c(t) = \frac{a(s)}{b(s)} d(t)$$

will refer to the solution of differential equation

$$b(p)c(t) = a(p)d(t)$$

where $p \triangleq \mathrm{d}/\mathrm{d}t$, with arbitrary finite initial conditions. Vectors like

$$W_{\nu+1}(s) \triangleq \left[u, \frac{u}{s+\beta}, \ldots, \frac{u}{(s+\beta)^{\nu}}\right]^{\mathrm{T}}$$

will have zero initial conditions and the subscript $\nu + 1$ refers to the dimension of $W(\cdot)$.

2. The control algorithm

First we briefly outline the philosophy and nature of Morse's algorithm, adhering closely to the terminology employed in [1], but omitting details irrelevant to the course of our development.

Consider a single-input single-output plant, modelled by a strictly proper transfer function

$$T(s) = \frac{g_{\mathrm{p}} \alpha_{\mathrm{p}}(s)}{\beta_{\mathrm{p}}(s)} \tag{2.1}$$

having degree and relative degree of n and n^* respectively; we assume g_{p} is a known nonzero constant (assumed positive without loss of generality) and α_{p} and β_{p} are monic, coprime polynomials, with unknown coefficients but with $\alpha_{\mathrm{p}}(s)$ known to be strictly stable. It is assumed that the plant output $y(t)$ is required to track a reference trajectory $y_{\mathrm{r}}(t)$, itself the output of a reference model with known, stable transfer function T_{r}, having relative degree no smaller than n^* (as otherwise explicit differentiation would be required). The reference model input is $r(t)$ and all of $y_{\mathrm{r}}(t)$, $y(t)$ and $r(t)$ are assumed measurable.

Consider next the scheme depicted in Figure 1. Here $1/\beta_{\mathrm{r}}$ is an arbitrary, known, stable, all pole transfer function, with β_{r} monic and of degree n^*, $W(s) = (sI - A_0)^{-1} b_0$ for some $n \times n$ A_0 with all eigenvalues in $\mathrm{Re}[s] < 0$ and with $[A_0, b_0]$ completely controllable. The vector $\theta^{\mathrm{T}} = (\theta_u^{\mathrm{T}}(t), \theta_y^{\mathrm{T}}(t))$ is an auxiliary signal, and $\hat{k}(t) = (\hat{k}_u^{\mathrm{T}}(t), \hat{k}_y^{\mathrm{T}}(t))$ is a parameter estimate vector. There exists a k_{p} such that with $\hat{k}(t) = k_{\mathrm{p}}$ the transfer function relating θ_{r} to y_{p} equals $1/\beta_{\mathrm{r}}$, with $k_{u\mathrm{p}}$, $k_{y\mathrm{p}}$ serving to assign the zeros, poles of the plant respectively. With $\hat{k}(t) \equiv k_{\mathrm{p}}$, it follows that the transfer function from $r(\cdot)$ to $y_{\mathrm{p}}(\cdot)$ will be $T_{\mathrm{r}}(\cdot)$, as required. Thus we must be interested in the question of convergence of $\hat{k}(t)$ to k_{p}.

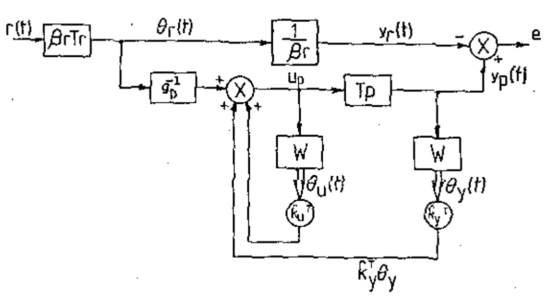

Fig. 1. A block diagram representation of Morse's algorithm.

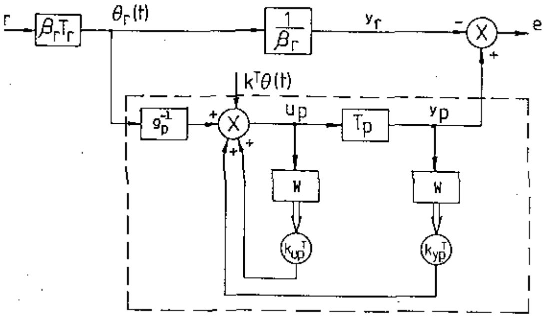

Fig. 2. An equivalent representation of Morse's algorithm.

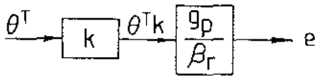

Fig. 3. Error model.

Defining $k(t) = \hat{k}(t) - k_p$ we can redraw Figure 1 as Figure 2 with the transfer function representing the system within the dotted box equalling $1/\beta_r(s)$. Accordingly we have the error model of Figure 3 as

$$e = y_p - y_r = \frac{1}{\beta_r}\left(\theta_r + g_p k^T\theta\right) - \frac{1}{\beta_r}\theta_r = \frac{g_p}{\beta_r}k^T\theta. \tag{2.2}$$

Now if we were to identify k by performing a steepest descent on e^2, existing results tell us [11] that $1/\beta_r$ will need to be strictly positive real, a condition clearly unattainable for $n^* > 1$.

Were the error model of Figure 4 replaced by

$$e' = k^T\left(\frac{g_p}{\beta_r}I\theta\right) = g_p k^T\phi \tag{2.3}$$

where $\phi = (\beta_r I)^{-1}\theta$, see Figure 4, it would be possible to identify k be a steepest descent procedure on $(e')^2$, assuming too that e' could be measured. Such an error model can in fact be obtained by adding an auxiliary error signal to e. The auxiliary signal is obtainable from measurements and is depicted in Figure 5. It is $g_p\psi$ where

$$\psi = -\frac{1}{\beta_r}\left[\theta^T\hat{k}(t)\right] + \phi^T\hat{k}(t) = -\frac{1}{\beta_r}\left[\theta^T(k_p + k)\right] + \phi^T(k_p + k)$$
$$= \phi^T k - \frac{1}{\beta_r}(\theta_k^T). \tag{2.4}$$

Observe that

$$e' = e + g_p\psi = g_p k^T\phi$$

as required.

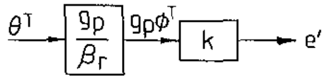

Fig. 4. Error model for which a strict positive real condition is not required.

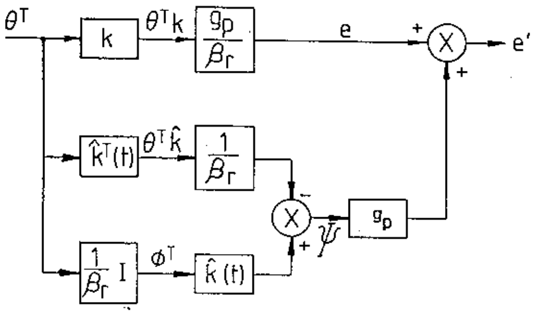

Fig. 5. Augmented error model.

The update equation for the parameter estimate vector $\hat{k}$ is given by

$$\dot{\hat{k}} = -Q\phi\bar{e} \tag{2.5}$$

where Q is a positive definite matrix and

$$\bar{e} = \frac{1}{\lambda_0 + \phi^T Q \phi} e', \quad \lambda_0 > 0. \tag{2.6}$$

The above discussion is a minor specialisation of [1], in that we have assumed g_p to be known. We have also suppressed terms which are known to be exponentially decaying, being associated for example with initial conditions in a $\beta_r^{-1}(s)$ blocks.

3. Stability analysis

The stability results of [1], applicable for unknown g_p, may in this case be trivially modified to yield the proposition below. Following the proposition, we indicate in Theorem 3.1 a development of this stability result to reflect exponential stability under a persistence of excitation condition.

Proposition 3.1. *For any time $t > 0$ and bounded, piecewise continuous input $r(t)$, the state response of the adaptive control system defined above is bounded and the quantities $\bar{e}$ and $\dot{\hat{k}}$ decay asymptotically to zero.*

Remark. The phrase 'state response' is meant to include the state variables of the plant, of the state-variable filters, of the filters in the auxiliary error generator and of the adaptive loop [i.e. $\hat{k}(t)$]. Observe then that Proposition 3.1 assures the boundedness of all signals of interest without a persistency of excitation condition. As matters stand, however, only the tracking error converges to zero. Although $\dot{\hat{k}}$ decays to zero, $\hat{k}$ need not converge to the correct parameter values, let alone do so exponentially. Theorem 3.1 presents a condition on the reference input which guarantees exponential convergence.

To restrict the input space appropriately, define the set $\Omega_\Delta[0, \infty)$ as follows. Let $C = \{t_1, t_2, \dots\}$ be an ordered set $0 < t_1 < t_2 < \cdots$ with $t_{i+1} - t_i > \Delta$ for all i. Then $r(\cdot) \in \Omega_\Delta[0, \infty)$ if r and $\dot{r}$ are continuous and bounded on $R_+ - C_\Delta$, and finite upper and lower limits as $t \to t_i$ exist for r and $\dot{r}$ for all i.

Theorem 3.1. *For a reference input $r(t) \in \Omega_\Delta[0, \infty)$ and the adaptive control system described above with known g_p, the quantities, k, $\hat{k}$ and $\bar{e}$ approach zero exponentially fast, provided that for some α_1 and $\delta' > 0$ and all $\sigma \in R^+$ the following relation holds:*

$$\alpha_1 I < \int_\sigma^{\sigma+\delta'} R(t) R^T(t)\, dt \tag{3.1}$$

where

$$R = \left[r, \frac{r}{s+\bar{\beta}}, \dots, \frac{r}{(s+\bar{\beta})^{2n+n_r-1}} \right]^{T}, \tag{3.2}$$

$\bar{\beta}$ is any positive number, and n_r is the number of imaginary axis zeros of $T_r(s)$.

For a proof, see the appendix.

Remark. To understand this result, observe that (3.1) is satisfied if $r(t)$ is a linear combination of at least $\frac{1}{2}(2n+n_r-1)$ independent sinusoids, a dc signal counting as half a sinusoid. Now the unknown parameters are the coefficients of $\alpha_p(s)$ and $\beta_p(s)$ and are $2n-1$ in number. Thus, as θ_r excites the closed loop system, one would expect that θ_r having $\frac{1}{2}(2n-1)$ frequencies should suffice for identification. The transfer function relating θ_r to r has n_r imaginary axis zeros. Thus as many as $\frac{1}{2}n_r$ frequencies in $r(t)$ may fail to be reflected in θ_r. Thus $r(t)$ must contain $\frac{1}{2}(2n+n_r-1)$ independent sinusoids.

4. Conclusions

A model reference adaptive control algorithm is shown to be exponentially convergent whenever the reference input is persistently exciting and the high frequency gain of the plant is known. This conclusion holds we believe for the other algorithm in [1], as well as that of for example [10]. When the high frequency gain is unknown [6] demonstrates the impossibility of attaining such convergence.

Appendix

We need the following lemmas and theorem, the first of which has been obtained from [12] and the rest from [7a] or [7b].

Lemma A.1. *If $f(\cdot)$ is an n times differentiable function on an interval I of length Δ and if $|f(x)| < M_0$ and $|f^{(n)}(x)| < M_n$, then for $x \in I$ and $0 < k < n$,*

$$|f^{(k)}(x)| < 4e^{2k}\{{}^{n}C_k\}^{k} M_0^{(1-k/n)} M_n^{1(k/n)}$$

where $M_n^1 = \max(M_n, M_0 n! \Delta^{-n})$ and ${}^{n}C_k = n!/\{(n-k)!k!\}$.

Lemma A.2. *For any stable system with a proper transfer function $T(s)$, if the input $v(t)$ is such that there exist M and $\varepsilon > 0$ for which $|v(t)| < M$ on $[0, T]$ and $|v(t)| < \varepsilon$ $\forall t > T$, and if the initial state lies in a fixed ball B of radius R, then there exists a $\nu(\varepsilon, m, R)$ independent of T, such that $\forall t > \nu + T$, $|y(t)| < O(\varepsilon)$.*

Lemma A.3. *If $u(t) \in \Omega_\Delta[0, \infty)$, then under the assumption of arbitrary finite initial conditions, for any Hurwitz polynomial $D(s)$ and polynomials $N_1(s)$ and $N_2(s)$, such that $\delta[N_1(s)] \leqslant \delta[D(s)]$ and $\delta[N_2(s)] = 1 + \delta[D(s)]$, the following properties hold:* (i) *$\{N_1(s)/D(s)\}u(t) \in \Omega_\Delta[0, \infty)$, and* (ii) *$\{N_2(s)/D(s)\}u(t)$ is continuous and bounded on $\{[0, \infty) - C_\Delta\}$ and has finite limits as $t \downarrow t_i$ and $t \uparrow t_i$, $t_i \in C_\Delta$.*

Theorem A.1. *Consider the proper time invariant SISO system described by*

$$y(t) = \frac{B(s)}{A(s)} u(t)$$

with $A(s)$ and $B(s)$ polynomials of degree n and m respectively with $n \geqslant m$. Assume $u(t) \in \Omega_\Delta[0, \infty)$ and let z be the number of zeros of $B(s)$ with zero real part. Define

$$Y_\nu = \left[y, \frac{y}{s+\beta}, \dots, \frac{y}{(s+\beta)^\nu} \right]^{\mathrm{T}} \quad \textit{and} \quad W_{\nu+z} = \left[u, \frac{u}{s+\bar{\beta}}, \dots, \frac{u}{(s+\bar{\beta})^{\nu+z}} \right]^{\mathrm{T}}.$$

Let β, $\bar{\beta} \in R_+$. Suppose there exist positive α_1 and δ' such that $\forall \sigma \in R_+$,

$$\int_\sigma^{\sigma+\delta'} W_{\nu+z}(t) W_{\nu+z}^{\mathrm{T}}(t)\, \mathrm{d}t \geqslant \alpha_1 I.$$

Then there exist $\delta'' > \delta'$ and $\alpha_2 > 0$ such that $\forall \sigma \in R_+$,

$$\int_\sigma^{\sigma+\delta''} Y_\nu(t) Y_\nu^{\mathrm{T}}(t)\, \mathrm{d}t \geqslant \alpha_2 I.$$

Proof. See [7b].

Proof of Theorem 3.1. We shall prove exponential convergence of k to zero. Equations (2.2) through (2.6) then yield exponential convergence of e and $\hat{k}$.

Equations (2.3), (2.5) and (2.6) imply

$$\dot{k} = -\frac{g_p Q \phi \phi^{\mathrm{T}} k}{\lambda_0 + \phi^{\mathrm{T}} Q \phi}. \tag{A.1}$$

By the boundedness of ϕ and a result in [11], k converges exponentially if for arbitrary σ_0 there exists a positive α_1 and δ such that for all $\sigma \geqslant \sigma_0$,

$$\alpha_1 I < \int_\sigma^{\sigma+\delta} \phi \phi^{\mathrm{T}}\, \mathrm{d}t. \tag{A.2}$$

Suppose the lower bound is violated irrespective of the choice of δ. Then for arbitrary $\varepsilon > 0$ and arbitrary $\delta > 0$, there exists a σ and a unit length vector γ such that

$$f(t) = \int_\sigma^t [\gamma^{\mathrm{T}} \phi]^2\, \mathrm{d}t < \varepsilon^4 \quad \forall t \in [\sigma, \sigma+\delta].$$

(Below, we shall impose lower bounds on the selection of σ_0 and δ.)

Now, $\ddot{f}$ is bounded, since $\phi = (\beta_r I)^{-1} \theta$ and θ is bounded. Hence by Lemma A.1 with $n = 2$,

$$|\gamma^{\mathrm{T}} \phi| < \mathrm{O}(\varepsilon) \quad \forall t \in [\sigma, \sigma+\delta].$$

From the relation between ϕ and θ and the definition of θ,

$$\left| \frac{1}{\beta_r(s)} \gamma^{\mathrm{T}} [\theta_u, \theta_y]^{\mathrm{T}} \right| < \mathrm{O}(\varepsilon) \quad \forall t \in [\sigma, \sigma+\delta]. \tag{A.3}$$

Now recognise that

$$\theta_y = W y_p = (sI - A_0)^{-1} b_0 y_p \quad \text{and} \quad \theta_u = W u_p + W T_p^{-1} y_p = \frac{\beta_p(s)}{g_p \alpha_p(s)} (sI - A_0)^{-1} b_0 y_p$$

and so (A.3) yields

$$\left| \left\{ \left[\frac{\beta_p(s)}{g_p \alpha_r(s) \beta_r(s)} \gamma_u^{\mathrm{T}} + \frac{1}{\beta_r(s)} \gamma_y^{\mathrm{T}} \right] (sI - A_0)^{-1} b_0 y_p + i(t) \right\} \right| < \mathrm{O}(\varepsilon) \tag{A.4}$$

where $i(t)$ denotes initial condition effects which decay to zero due to the assumed stability of $\alpha_p(s)$, $\beta_r(s)$ and $\det(sI - A_0)$. Suppose $\delta_1(\varepsilon)$ is the time required by $i(t)$ to decay to $\frac{1}{2}\varepsilon$; then by choosing $\delta > \delta_1$, we have

$$\left\{\left[\frac{g_p^{-1}\beta_p(s)}{\alpha_p(s)\beta_r(s)}\gamma_u^T + \frac{1}{\beta_r(s)}\gamma_y^T\right](sI - A_0)^{-1}b_0 y_p\right\} < O(\varepsilon)$$

on $[\sigma + \delta_1, \sigma + \delta]$. Note that $y_p - y_r \to 0$ as $t \to \infty$. Then for sufficiently large σ_0,

$$\left|\left\{\left[\frac{g_p^{-1}\beta_p(s)}{\alpha_p(s)\beta_r(s)}\gamma_u^T + \frac{1}{\beta_r(s)}\gamma_y^T\right](sI - A_0)^{-1}b_0\frac{1}{\beta_r(s)}\theta_r(t)\right\}\right| < O(\varepsilon). \tag{A.5}$$

Denote $\gamma_u^T(sI - A_0)^{-1}b_0$ and $\gamma_y^T(sI - A_0)^{-1}b_0$ by $\gamma_u(s)/\lambda(s)$ and $\gamma_y(s)/\lambda(s)$ respectively. Because α_p and β_p are coprime with deg $\beta_p = n$ and because $\gamma_u(s)$, $\gamma_y(s)$ have degree smaller than n,

$$\mu(s) \triangleq g_p^{-1}\beta_p(s)\gamma_u(s) - \alpha_p(s)\gamma_y(s) \not\equiv 0.$$

Rewrite (A.5) as

$$\left|\frac{\mu(s)}{\alpha_p\beta_r(s)\lambda(s)}\frac{1}{\beta_r(s)}\theta_r(t)\right| < O(\varepsilon) \quad \text{on } [\sigma + \delta_1, \sigma + \delta]. \tag{A.6}$$

Now use Lemma A.2 with the term in the modulus sign as $v(t)$ and $T(s) = \alpha_p(s)\beta_r(s)\lambda(s)/(s + \beta)^{2n}$. By choosing $\delta_2 - \delta_1$ to exceed the quantity ν of the lemma statement, and $\delta > \delta_2$, we obtain

$$\left|\frac{\sum_{i=0}^{2n-1}\eta_i(s+\beta)^i}{(s+\beta)^{2n}}\frac{1}{\beta_r(s)}\theta_r(t)\right| < O(\varepsilon)$$

on $[\sigma + \delta_2, \sigma + \delta]$. Now, $\theta_r(t) \in \Omega_\Delta(0, \infty)$, because $r(t) \in \Omega_\Delta[0, \infty)$, see Lemma A.3. Hence

$$\left|s(s+\beta)\beta_r(s)\frac{\sum_{i=1}^{2n-1}\eta_i(s+\beta)^i}{(s+\beta)^{2n}}\frac{1}{\beta_r(s)}\theta_r(t)\right|$$

is bounded on intervals of length Δ. Use of Lemma A.1 with $f(t)$ identified with

$$\frac{\sum_{i=0}^{2n-1}(s+\beta)^i}{(s+\beta)^{2n}}\frac{1}{\beta_r(s)}\theta_r(t)$$

and r, k in the lemma identified with $n^* + 2$, $n^* + 1$ leads to

$$\left|(s+\beta)\beta_r(s)\frac{\sum_{i=0}^{2n-1}(s+\beta)^i}{(s+\beta)^{2n}}\frac{1}{\beta_r(s)}\theta_r(t)\right| < O\left(\varepsilon^{1/(n^*+2)}\right),$$

i.e.

$$\left| \frac{\sum_{i=0}^{2n-1} \eta_i (s+\beta)^i}{(s+\beta)^{2n-1}} \theta_r(t) \right| < O\left(\varepsilon^{1/(n^*+2)}\right) \tag{A.7}$$

on $[\sigma + \delta_2,\ \sigma + \delta]$. Now (A.7) is a consequence of assuming that (A.2) does not hold for some $\alpha_1 > 0$, $\delta > 0$ and all $\sigma \geqslant \sigma_0$. We shall show this contradicts the persistency of excitation assumption in Theorem 3.1 summed up in (3.1) and (3.2). Suppose that (3.1) holds. Apply Theorem A.1 with u, y and $B(s)/A(s)$ in the theorem statement identified with r, θ_r and $\beta_r T_r$. Then there exists $\delta'' > \delta'$ and $\alpha_2 > 0$ such that

$$\int \left[\theta_r \frac{\theta_r}{s+\beta}, \ldots, \frac{\theta_r}{(s+\beta)^{2n-1}} \right]^T \left[\theta_r \frac{\theta_r}{s+\beta}, \ldots, \frac{\theta_r}{(s+\beta)^{2n-1}} \right] \mathrm{d}t > \alpha_2 I$$

for all σ. If $\delta + \delta_2 > \delta''$, we obtain the desired contradiction of (A.7); so (A.2) does hold and with it, the exponential convergence of $k(t)$ to zero.

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