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Nonlinear averaging theorems, and the determination of parameter convergence rates in adaptive control

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Abstract: The paper presents nonlinear averaging theorems for two-time scale systems, where the dynamics of the fast system are allowed to vary with the slow system. The results are applied to the Narendra–Valavani adaptive control algorithm, and estimates of the parameter convergence rates are obtained which do not rely on a linearization of the system around the equilibrium, and therefore are valid in a larger region in the parameter space.

Keywords: Adaptive control, Model reference, Parameter convergence rates, Averaging, Stability analysis, Nonlinear systems.

1. Introduction

Averaging methods were recently introduced for the analysis of deterministic adaptive systems [1–6]. The interest has been focused mostly on the assessment of stability of adaptive systems in the presence of unmodeled dynamics, and on the understanding of instability mechanisms. However, averaging is not only useful in stability problems, but in general as an *approximation* method, allowing one to replace a system of nonautonomous differential equations by an autonomous system. This aspect was emphasized in a recent paper [6], and new theorems were derived for one-time scale, and two-time scale systems such as those considered in [7], but without an almost periodicity assumption on the dynamics. In particular, convergence rate estimates were obtained for some identification and linearized adaptive control schemes. Such estimates are useful both for optimal input design, and for robustness considerations.

In this paper, we extend the results of [6], and establish theorems of averaging that are applicable to more general two-time scale systems, which are not considered in [7]. Our theorems are general in their formulation, and as such they constitute new results in the theory of averaging. The application is not limited to linearized adaptive control problems, but allow for the analysis of the full nonlinear equations. A recent research paper [3] considers such equations, and establishes the mathematical foundation to apply the integral manifold theory of Pliss, and reduce the nonlinear two-time scale system to a one-time scale system. This allows for the use of classical one-time scale averaging results. In this paper, we use the results of [6] to derive averaging theorems that are *directly* applicable to the two-time scale system of interest. These do not assume periodicity, but a more general averaging assumption. The results also include bounds on the approximation error, and justify the use of averaging for the determination of parameter convergence rates. Averaging is applied to the Narendra–Valavani algorithm [9], and it is shown that, despite the nonlinearity of the equations, a frequency-domain approach can still be used for the averaged system. A bound on the rate of convergence can be obtained, and is valid for a larger region in the parameter space than the region where the linearization is accurate. Simulations confirm that the original

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adaptive system is closely approximated by the averaged system, even for fairly large values of the parameter error and of the adaptation gain.

2. Two-time scale averaging with varying dynamics

2.1. Separated time scales

We first consider the system of differential equations

$$\dot{x} = \varepsilon f(t, x, y), \tag{2.1}$$

$$\dot{y} = A(x)y + \varepsilon g(t, x, y), \tag{2.2}$$

where $x(0) = x_0$, $y(0) = y_0$, $x \in R^n$, and $y \in R^m$.

The state vector is divided in a fast state vector y, and a slow state vector x, whose dynamics are of the order of ε with respect to the fast dynamics. The dominant term in (2.2) is linear in y, but is itself allowed to vary as a function of the slow state vector. This is a more general situation than the one considered in [5–7], where the matrix A is independent of x.

The following definitions will be useful in the sequel. Let B_h be the closed ball of radius h in R^n or R^m.

Definition 2.1. *Average value of a function, convergence function.* The function $f(t, x, 0)$ is said to have *average value* $f_{av}(x)$, if there exists a continuous, strictly decreasing function $\gamma(T): R_+ \to R_+$ such that $\gamma(T) \to 0$ as $T \to \infty$, and

$$\left\| \frac{1}{T} \int_t^{t+T} f(\tau, x, 0)\, d\tau - f_{av}(x) \right\| \leqslant \gamma(T) \tag{2.3}$$

for all t, $T \geqslant 0$, $x \in B_h$.

The function $\gamma(T)$ is called the *convergence function*, and the system

$$\dot{x}_{av} = f_{av}(x_{av}), \quad x_{av}(0) = x_0, \tag{2.4}$$

is called the *averaged system* corresponding to (2.1)–(2.2).

Definition 2.2. *Uniform exponential stability.* The family of matrices $A(x) \in R^{m \times m}$ is *uniformly exponentially stable for all* $x \in B_h$, if there exist m, λ, m', $\lambda' > 0$, such that, for all $x \in B_h$ and $t \geqslant 0$,

$$m' e^{-\lambda' t} \leqslant \| e^{A(x)t} \| \leqslant m\, e^{-\lambda t}. \tag{2.5}$$

Comments. This definition is equivalent to require that the solutions of the system $\dot{y} = A(x)y$ are bounded above and below by decaying exponentials, independently of the parameter x.

It is also possible to show that the definition is equivalent to requiring that there exist p_1, p_2, q_1, $q_2 > 0$, such that for all $x \in B_h$, there exists $P(x)$ satisfying $p_1 I \leqslant P(x) \leqslant p_2 I$, and $-q_2 I \leqslant A^T(x)P(x) + P(x)A(x) \leqslant -q_1 I$.

We will make the following assumptions.

(C1) The functions f and g are piecewise continuous functions of time, and continuous functions of x and y. Moreover, $f(t, 0, 0) = 0$, $g(t, 0, 0) = 0$ for all $t \geqslant 0$, and for some l_1, l_2, l_3, $l_4 \geqslant 0$,

$$\| f(t, x_1, y_1) - f(t, x_2, y_2) \| \leqslant l_1 \| x_1 - x_2 \| + l_2 \| y_1 - y_2 \|, \tag{2.6}$$

$$\| g(t, x_1, y_1) - g(t, x_2, y_2) \| \leqslant l_3 \| x_1 - x_2 \| + l_4 \| y_1 - y_2 \|, \tag{2.7}$$

for all $t \geq 0$, $x_1, x_2 \in B_h$, $y_1, y_2 \in B_h$. Also assume that $f(t, x, 0)$ has continuous and bounded partial derivatives with respect to x, for all $t \geq 0$, and $x \in B_h$.

(C2) The function $f(t, x, 0)$ has average value $f_{av}(x)$. Moreover, $f_{av}(0) = 0$, and $f_{av}(x)$ has continuous and bounded partial derivatives with respect to x, for all $x \in B_h$, so that for some $l_{av} \geq 0$,

$$\| f_{av}(x_1) - f_{av}(x_2) \| \leq l_{av} \| x_1 - x_2 \| \tag{2.8}$$

for all $x_1, x_2 \in B_h$.

(C3) Let $d(t, x) = f(t, x, 0) - f_{av}(x)$, so that $d(t, x)$ has zero average value. Assume that the convergence function can be written as $\gamma(T)\|x\|$, and that $\partial d(t, x)/\partial x$ has zero average value, with convergence function $\gamma(T)$.

(C4) $A(x)$ is uniformly exponentially stable for all $x \in B_h$ and, for some $k_a \geq 0$,

$$\left\| \frac{\partial A(x)}{\partial x} \right\| \leq k_a \quad \text{for all } x \in B_h. \tag{2.9}$$

(C5) For some $h' < h$, $\| x_{av}(t) \| \in B_{h'}$ on the time intervals considered, and for some h_0, $y_0 \in B_{h_0}$ (where h', h_0 are constants to be defined later).

We are now ready to state the first averaging theorem concerning the differential system (2.1)–(2.2). Theorem 2.1 is an approximation theorem, guaranteeing that the trajectories of the original and averaged system are arbitrarily close on compact intervals, when ε tends to zero. Recall that a function $\psi(\varepsilon) \in$ class K, if it is continuous, strictly increasing, and $\psi(0) = 0$.

Theorem 2.1. Basic averaging theorem. *If the original system* (2.1)–(2.2) *and the averaged system* (2.4) *satisfy assumptions* (C1)–(C5), *then there exists* $\psi(\varepsilon) \in$ class K, *such that, given* $T \geq 0$,

$$\| x(t) - x_{av}(t) \| \leq \psi(\varepsilon) b_T \tag{2.10}$$

for some b_T, $\varepsilon_T > 0$, *and for all* $t \in [0, T/\varepsilon]$, $\varepsilon \leq \varepsilon_T$.

Proof. The proof assumes that for all $t \in [0, T/\varepsilon]$, the solutions $x(t)$, $y(t)$, and $z(t)$ (to be defined) remain in B_h. Since this is not guaranteed a priori, the steps of the proof are only valid for as long as the condition is verified. We will show that, under the conditions of the theorem, the time interval over which the condition is satisfied includes the interval $[0, T/\varepsilon]$, so that all steps will be valid.

It was shown in [6], that under assumptions (C1)–(C5), there exists a change of coordinates

$$x = z + \varepsilon w_\varepsilon(t, z) \tag{2.11}$$

such that

$$\| \varepsilon w_\varepsilon(t, z) \| \leq \xi(\varepsilon) \| z \| \quad \text{and} \quad \left\| \varepsilon \frac{\partial w_\varepsilon(t, z)}{\partial z} \right\| \leq \xi(\varepsilon) \tag{2.12}$$

for some $\xi(\varepsilon) \in$ class K. The change of coordinates is a homeomorphism in B_h for all $\varepsilon \leq \varepsilon_1$ (where ε_1 is a constant such that $\xi(\varepsilon_1) < 1$). Under the change of coordinates, it was also shown that z satisfies the differential equation

$$\dot{z} = \varepsilon f_{av}(z) = \varepsilon p_1(t, z, \varepsilon) + \varepsilon p_2(t, z, y, \varepsilon), \quad z(0) = x_0, \tag{2.13}$$

where

$$\| p_1(t, z, \varepsilon) \| \leq \xi(\varepsilon) k_1 \| z \| \quad \text{and} \quad \| p_2(t, z, y, \varepsilon) \| \leq k_2 \| y \| \tag{2.14}$$

and k_1, k_2 are constants depending on l_1, l_2, l_{av}, and $\xi(\varepsilon_1)$.

A bound on the error $\| z(t) - x_{av}(t) \|$ can be obtained by integrating (2.4)–(2.13), and using (2.14):

$$\| z(t) - x_{av}(t) \| \leqslant \varepsilon l_{av} \int_0^t \| z(\tau) - x_{av}(\tau) \| d\tau + \varepsilon \xi(\varepsilon) k_1 \int_0^t \| z(\tau) \| d\tau + \varepsilon k_2 \int_0^t \| y(\tau) \| d\tau. \qquad (2.15)$$

To obtain a bound on $\| y(t) \|$, we divide the interval $[0, T/\varepsilon]$ in intervals $[t_i, t_{i+1}]$ of length ΔT (the last interval may be of smaller length and ΔT will be defined later). The differential equation for y is

$$\dot{y} = A(x) y + \varepsilon g(t, x, y) \qquad (2.16)$$

and is rewritten on the time interval $[t_i, t_{i+1}]$ as follows:

$$\dot{y} = A_{x_i} y + \varepsilon g(t, x, y) + (A_{x_t} - A_{x_i}) y \qquad (2.17)$$

where $A_{x_t} = A(x(t))$, and $A_{x_i} = A(x(t_i))$, so that the solution $y(t)$, for $t \in [t_i, t_{i+1}]$ is given by

$$y(t) = e^{A_{x_i}(t-t_i)} y_i + \varepsilon \int_{t_i}^t e^{A_{x_i}(t-\tau)} g(\tau, x, y)\, d\tau + \int_{t_i}^t e^{A_{x_i}(t-\tau)} (A_{x_\tau} - A_{x_i}) y(\tau)\, d\tau \qquad (2.18)$$

where $y_i = y(t_i)$. From the assumptions, it follows that

$$\| A_{x_\tau} - A_{x_i} \| \leqslant k_a \| \dot{x} \| (\tau - t_i) \leqslant \varepsilon (l_1 + l_2) h k_a\, \Delta T \qquad (2.19)$$

and, using the uniform exponential stability assumption on $A(x)$,

$$\| y(t) \| \leqslant m \| y_i \|\, e^{-\lambda(t-t_i)} + \varepsilon \frac{m}{\lambda} h\big((l_3 + l_4) + (l_1 + l_2) k_a\, \Delta T\big). \qquad (2.20)$$

Let the last term in (2.20) be denoted by εk_b, and use (2.20) as a recursion formula for y_i, so that

$$\| y_i \| \leqslant (m\, e^{-\lambda \Delta T})^i \| y_0 \| + \varepsilon k_b \sum_{j=0}^{i-1} (m\, e^{-\lambda \Delta T})^j. \qquad (2.21)$$

Choose ΔT sufficiently large that

$$m\, e^{-\lambda \Delta T} \leqslant e^{-\lambda \Delta T/2}, \quad \text{i.e} \quad \Delta T \geqslant \frac{2}{\lambda} \ln m. \qquad (2.22)$$

It follows that

$$\sum_{j=0}^{i-1} (m\, e^{-\lambda \Delta T})^j \leqslant \sum_{j=0}^{\infty} (e^{-\lambda \Delta T/2})^j = \frac{1}{1 - e^{-\lambda \Delta T/2}}. \qquad (2.23)$$

Combining (2.21)–(2.23), and using the assumption $y_0 \in B_{h_0}$,

$$\| y_i \| \leqslant e^{-\lambda \Delta T i/2} h_0 + \frac{\varepsilon k_b}{1 - e^{-\lambda \Delta T/2}} := e^{-\lambda t_i/2} h_0 + \varepsilon k_c. \qquad (2.24)$$

Using this result in (2.20), it follows that for all $t \in [t_i, t_{i+1}]$,

$$\| y(t) \| \leqslant m\, e^{-\lambda t_i/2} h_0\, e^{-\lambda(t-t_i)} + m \varepsilon k_c\, e^{-\lambda(t-t_i)} + \varepsilon k_b \leqslant m h_0\, e^{-\lambda t/2} + \varepsilon(m k_c + k_b). \qquad (2.25)$$

Since the last inequality does not depend on i, it gives a bound on $\| y(t) \|$ for all $t \in [0, T/\varepsilon]$.

We now return to (2.15), and to the approximation error, using the bound on $\| y(t) \|$,

$$\| z(t) - x_{av}(t) \| \leqslant \varepsilon l_{av} \int_0^t \| z(\tau) - x_{av}(\tau) \|\, d\tau + \varepsilon \xi(\varepsilon) k_1 \int_0^t h\, d\tau + \varepsilon k_2 \int_0^t \big(m h_0\, e^{-\lambda \tau/2} + \varepsilon(m k_c + k_b)\big)\, d\tau, \qquad (2.26)$$

so that, using the *Generalized Bellman–Gronwall Lemma* (see the appendix),

$$\|z(t)-x_{av}(t)\| \leqslant \int_0^t \left(\xi(\varepsilon)k_1 h + k_2 m h_0\, e^{-\lambda\tau/2} + k_2\varepsilon(mk_c+k_b)\right)\varepsilon\, e^{\varepsilon l_{av}(t-\tau)}\, d\tau$$

$$\leqslant (\varepsilon+\xi(\varepsilon))\left(k_1 h + \frac{k_2 m h_0 l_{av}}{\lambda/2+\varepsilon l_{av}} + k_2(mk_c+k_b)\right)\frac{e^{l_{av}T}}{l_{av}}$$

$$:= \psi(\varepsilon)a_T \tag{2.27}$$

and, using (2.12),

$$\|x(t)-x_{av}(t)\| \leqslant \psi(\varepsilon)b_T \tag{2.28}$$

for some b_T.

We assumed in the proof that all signals remained in B_h. By assumption, $x_{av}(t)\in B_{h'}$, for some $h'<h$. Let h_0, and ε_T be sufficiently small so that, for all $\varepsilon\leqslant\varepsilon_T\leqslant\varepsilon_1$, we have that $mh_0+\varepsilon(mk_c+k_b)\leqslant h$ (cf. eqn. (2.25)), and that $\psi(\varepsilon)b_T\leqslant h-h'$. It follows, from a simple contradiction argument, that the solutions $x(t)$, $y(t)$, and $z(t)$ remain in B_h for all $t\in[0, T/\varepsilon]$, so that all steps of the proof are valid, and (2.28) is in fact satisfied over the whole time interval.

Theorem 2.2. Exponential stability theorem. *If the original system* (2.1)–(2.2), *and the averaged system* (2.4) *satisfy assumptions* (C1)–(C5), *and if the averaged system is exponentially stable*, ***then*** *the original system is exponentially stable for* ε *sufficiently small.*

Proof. The proof relies on a converse theorem of Lyapunov for exponentially stable systems (see [8], p. 273). Under the hypotheses, there exists a function $v(x_{av}): R^n\to R_+$, and strictly positive constants α_1, α_2, α_3, α_4 and $h'<h$ such that, for all $x_{av}\in B_{h'}$

$$\alpha_1\|x_{av}\|^2 \leqslant v(x_{av}) \leqslant \alpha_2\|x_{av}\|^2, \tag{2.29}$$

$$\dot v(x_{av})\,|_{(2.4)} \leqslant -\varepsilon\alpha_3\|x_{av}\|^2, \tag{2.30}$$

$$\left\|\frac{\partial v}{\partial x_{av}}\right\| \leqslant \alpha_4\|x_{av}\|. \tag{2.31}$$

The derivative in (2.30) is taken along the trajectories of the averaged system (2.4). We now study the stability of the original system (2.1)–(2.2), through the transformed system (2.13)–(2.2), where $x(z)$ is defined in (2.11). Consider the following Lyapunov function

$$v_1(z, y) = v(z) + \frac{\alpha_2}{p_2}y^{\mathrm T}P(x(z))y \tag{2.32}$$

where $P(x)$, p_2 are defined in the comments after the definition of uniform exponential stability of $A(x)$. Defining $\alpha_1' = \min(\alpha_1, (\alpha_2/p_2)p_1)$, it follows that

$$\alpha_1'(\|z\|^2+\|y\|^2) \leqslant v_1(z, y) \leqslant \alpha_2(\|z\|^2+\|y\|^2). \tag{2.33}$$

The derivative of v_1 along the trajectories of (2.13)–(2.2) can be bounded, using the previous results,

$$\dot v_1(z, y) \leqslant -\varepsilon\alpha_3\|z\|^2 + \varepsilon\xi(\varepsilon)k_1\alpha_4\|z\|^2 + \varepsilon k_2\alpha_4\|z\|\,\|y\|$$
$$+\frac{\alpha_2}{p_2}\left\|\frac{\partial P(x)}{\partial x}\right\|\left\|\frac{\partial x}{\partial z}\right\|\|\dot z\|\,\|y\|^2 - \frac{\alpha_2}{p_2}q_1\|y\|^2 + 4\varepsilon l_3\alpha_2\|z\|\,\|y\| + 2\varepsilon l_4\alpha_2\|y\|^2 \tag{2.34}$$

for $\varepsilon\leqslant\varepsilon_1$ (so that, in particular, $\|x\|\leqslant 2\|z\|$). We now calculate bounds on the terms in (2.34). Note that $P(x)$ can be defined by

$$P(x) = \int_0^\infty e^{A^{\mathrm T}(x)t}Q\,e^{A(x)t}\,dt \tag{2.35}$$

so that

$$\frac{\partial P(x)}{\partial x_i}=\int_0^{\infty}\left\{\left(\frac{\partial}{\partial x_i}e^{A^{T}(x)t}\right)Q\,e^{A(x)t}+e^{A^{T}(x)t}Q\left(\frac{\partial}{\partial x_i}e^{A(x)t}\right)\right\}\mathrm{d}t. \tag{2.36}$$

The partial derivatives in parentheses solve the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial}{\partial x_i}e^{A(x)t}\right)=A(x)\left(\frac{\partial}{\partial x_i}e^{A(x)t}\right)+\frac{\partial A(x)}{\partial x_i}e^{A(x)t} \tag{2.37}$$

with zero initial conditions, so that

$$\frac{\partial}{\partial x_i}e^{A(x)t}=\int_0^{t}e^{A(x)(t-\tau)}\frac{\partial A(x)}{\partial x_i}e^{A(x)\tau}\,\mathrm{d}\tau. \tag{2.38}$$

From the boundedness of $\partial A(x)/\partial x_i$, and from the exponential stability of $A(x)$, it follows that

$$\left\|\frac{\partial}{\partial x}e^{A(x)t}\right\|\leqslant m^2k_a t\,e^{-\lambda t}. \tag{2.39}$$

With (2.36), this implies that $\|\partial P(x)/\partial x\|$ is bounded by some $k_p\geqslant 0$. On the other hand, from (2.12)–(2.14),

$$\left\|\frac{\partial x}{\partial z}\right\|<1+\xi(\varepsilon)<2 \quad\text{and}\quad \|\dot{z}\|\leqslant \varepsilon h(l_{\mathrm{av}}+\xi(\varepsilon)k_1+k_2). \tag{2.40}$$

Using these results in (2.34), and noting the fact that

$$\varepsilon\|z\|\,\|y\|\leqslant\tfrac{1}{2}(\varepsilon^{4/3}\|z\|^2+\varepsilon^{2/3}\|y\|^2), \tag{2.41}$$

it follows that

$$\begin{aligned}\dot{v}_1(z,\,y)\leqslant{}&-\varepsilon\left(\alpha_3-\xi(\varepsilon)k_1\alpha_4-\varepsilon^{1/3}\frac{k_2\alpha_4}{2}-2\varepsilon^{1/3}l_3\alpha_2\right)\|z\|^2\\&-\left(\frac{\alpha_2}{P_2}q_1-2\varepsilon l_4\alpha_2-\varepsilon^{2/3}\frac{k_2\alpha_4}{2}-2\varepsilon^{2/3}l_3\alpha_2+2\varepsilon\frac{\alpha_2}{p_2}k_ph(l_{\mathrm{av}}+\xi(\varepsilon)k_1+k_2)\right)\|y\|^2\\:={}&-2\varepsilon\alpha_2\alpha(\varepsilon)\|z\|^2-q(\varepsilon)\|y\|^2.\end{aligned} \tag{2.42}$$

Note that, with this definition, $\alpha(\varepsilon)\to\frac{1}{2}\alpha_3/\alpha_2$ as $\varepsilon\to 0$, while $q(\varepsilon)\to(\alpha_2/p_2)q_1$. Let $\varepsilon\leqslant\varepsilon_1$ be sufficiently small that $\alpha(\varepsilon)>0$, and $2\varepsilon\alpha_2\alpha(\varepsilon)\leqslant q(\varepsilon)$. Then

$$\dot{v}_1(z,\,y)\leqslant -2\varepsilon\alpha(\varepsilon)v_1(z,\,y) \tag{2.43}$$

so that the z, y system is exponentially stable with rate of convergence $\varepsilon\alpha(\varepsilon)$ (v_1 being bounded above and below by the *square* of the norm of the state). The same conclusion holds for the x, y system, given the transformation (2.11)–(2.12). Also, for ε, h_0 sufficiently small, all signals are actually guaranteed to remain in B_h so that all assumptions are valid.

Comments. The proof of Theorem 2.2 gives a useful bound on the rate of convergence of the nonautonomous system. As $\varepsilon\to 0$, the rate tends to $\frac{1}{2}\varepsilon\alpha_3/\alpha_2$, which is the bound on the rate of convergence of the averaged system that one would obtain using the Lyapunov function $v(x_{\mathrm{av}})$. Since the averaged system is autonomous, such a Lyapunov function is usually easier to find than for the original nonautonomous system, and conclusions about its exponential convergence can be applied to the nonautonomous system for ε sufficiently small.

2.2. Mixed time scales

We now discuss a more general class of two-time scale systems, arising in adaptive control:

$$\dot{x} = \varepsilon f'(t, x, y'), \tag{2.44}$$

$$\dot{y}' = A(x)y' + h(t, x) + \varepsilon g'(t, x, y'). \tag{2.45}$$

We will show that system (2.44)–(2.45) can be transformed into the system (2.1)–(2.2). In this case, x is a slow variable, but y' has both a fast, and a slow component.

The averaged system corresponding to (2.44), (2.45) is obtained as follows. Define the function

$$v(t, x) = \int_0^t e^{A(x)(t-\tau)} h(\tau, x)\, d\tau \tag{2.46}$$

and assume that the following limit exists uniformly in t and x:

$$f_{av}(x) = \lim_{T\to\infty} \frac{1}{T}\int_t^{t+T} f'(\tau, x, v(\tau, x))\, d\tau. \tag{2.47}$$

Intuitively, $v(t, x)$ represents the steady-state value of the variable y with x frozen and $\varepsilon = 0$ in (2.45).

Consider now the transformation

$$y = y' - v(t, x). \tag{2.48}$$

Since $v(t, x)$ satisfies

$$\frac{\partial}{\partial t} v(t, x) = A(x)v(t, x) + h(t, x), \quad v(t, 0) = 0, \tag{2.49}$$

we have that

$$\dot{y} = A(x)y + \varepsilon\left(-\frac{\partial v(t, x)}{\partial x} f'(t, x, y + v(t, x)) + g'(t, x, y + v(t, x))\right) \tag{2.50}$$

which is of the form studied previously when

$$f(t, x, y) = f'(t, x, y + v(t, x)), \tag{2.51}$$

$$g(t, x, y) = -\frac{\partial v(t, x)}{\partial x} f'(t, x, y + v(t, x)) + g'(t, x, y + v(t, x)). \tag{2.52}$$

The averaged system is obtained by averaging the right-hand side of (2.51) with $y = 0$, so that the definitions (2.47), and (2.3) agree.

We require assumptions (C1)–(C5) to be satisfied. In particular, we assume similar Lipschitz conditions on f', g', and the following assumption on $h(t, x)$:

(C6) $h(t, 0) = 0$ for all $t \geq 0$, and

$$\left\| \frac{\partial h(t, x)}{\partial x} \right\| \leq k \tag{2.53}$$

for all $t \geq 0$, $x \in B_h$.

This new assumption implies that $v(t, 0) = 0$. It also implies that $\| \partial v(t, x)/\partial x \|$ is bounded for all $t \geq 0$, $x \in B_h$, since

$$\frac{\partial v(t, x)}{\partial x_i} = \int_0^t \left(e^{A(x)(t-\tau)} \frac{\partial h(\tau, x)}{\partial x_i} + \frac{\partial}{\partial x_i}(e^{A(x)(t-\tau)})h(\tau, x)\right) d\tau \tag{2.54}$$

and using the fact that $e^{A(x)(t-\tau)}$ and $(\partial/\partial x)\, e^{A(x)(t-\tau)}$ are bounded by exponentials ((2.5 and (2.39)).

3. Two-time scale averaging applied to model reference adaptive control

3.1. Application to the Narendra–Valavani algorithm

We apply the averaging results to the model reference adaptive control system of Narendra and Valavani [9] for the relative degree 1 case. It was shown in [6] that the equations describing the system can be written as

$$\dot{e} = Ae + bw_m(t)^{\mathrm{T}}\phi + b\phi^{\mathrm{T}}Qe, \tag{3.1}$$

$$\dot{\phi} = -\varepsilon w_m(t)c^{\mathrm{T}}e - \varepsilon Qec^{\mathrm{T}}e. \tag{3.2}$$

The last terms, quadratic in e, ϕ, were neglected in [6]. This restricts attention to the linearized adaptive control scheme, i.e. to its behavior around the equilibrium. We consider here the complete set of differential equations, and extend the analysis to the nonlinear case. As in Section 2, we define

$$v(t, \phi) = \int_0^t \mathrm{e}^{(A+b\phi^{\mathrm{T}}Q)(t-\tau)}bw_m^{\mathrm{T}}(\tau)\phi \,\mathrm{d}\tau \tag{3.3}$$

so that the averaged system

$$\dot{\phi}_{\mathrm{av}} = -\varepsilon f_{\mathrm{av}}(\phi_{\mathrm{av}}) \tag{3.4}$$

is defined by the limit

$$f_{\mathrm{av}}(\phi_{\mathrm{av}}) = \lim_{T\to\infty} \frac{1}{T}\int_s^{s+T}\left(w_m(t)c^{\mathrm{T}}v(t, \phi_{\mathrm{av}}) + Qv(t, \phi_{\mathrm{av}})c^{\mathrm{T}}v(t, \phi_{\mathrm{av}})\right)\mathrm{d}t. \tag{3.5}$$

The assumptions of the theorems will be satisfied if the limit in (3.5) is uniform in the sense of (C3), and provided that the matrix $A + b\phi^{\mathrm{T}}Q$ is uniformly exponentially stable for $\phi \in B_h$.In effect, this means that if the controller parameters are frozen at any point of the trajectory — the adaptation then being switched off — the resulting time-invariant system must be closed-loop stable. Naturally, this precludes consideration of adaptation from initial parameter values which define an unstable closed-loop system.

3.2. Frequency domain analysis

The expression of f_{av} in (3.5) can be translated into the frequency domain, using Parseval's equality, and noting that w_m is related to r through some vector transfer function $\hat{n}(s)$ (cf. [6]):

$$f_{\mathrm{av}}(\phi_{\mathrm{av}}) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\hat{n}(\mathrm{j}\omega) + Q\left(\mathrm{j}\omega I - A - b\phi_{\mathrm{av}}^{\mathrm{T}}Q\right)^{-1}b\phi_{\mathrm{av}}^{\mathrm{T}}\hat{n}(\mathrm{j}\omega)\right)$$
$$\cdot\left(c^{\mathrm{T}}\left(-\mathrm{j}\omega I - A - b\phi_{\mathrm{av}}^{\mathrm{T}}Q\right)^{-1}b\phi_{\mathrm{av}}^{\mathrm{T}}\hat{n}(-\mathrm{j}\omega)\right)s_r(\mathrm{d}\omega) \tag{3.6}$$

where $s_r(\mathrm{d}\omega)$ is the spectral measure of r (cf. [10]). Although there is apparently no direct way to simplify this expression (as is possible in the linearized case), we claim the following.

Claim. (1) *$f_{\mathrm{av}}(\phi_{\mathrm{av}})$ can be factored as $f_{\mathrm{av}}(\phi_{\mathrm{av}}) = A_{\mathrm{av}}(\phi_{\mathrm{av}})\phi_{\mathrm{av}}$.*

(2) *$\phi_{\mathrm{av}}^{\mathrm{T}}A_{\mathrm{av}}(\phi_{\mathrm{av}})\phi_{\mathrm{av}} = \phi_{\mathrm{av}}^{\mathrm{T}}R(\phi_{\mathrm{av}})\phi_{\mathrm{av}}$, where $R(\phi_{\mathrm{av}})$ is a positive semidefinite matrix (not necessarily symmetric) for all ϕ_{av}. Whenever $w_m(t)$ is persistently exciting, $R(\phi_{\mathrm{av}}) \geqslant \alpha > 0$ for all $\phi_{\mathrm{av}} \in B_h$, and for some constant α.*

Proof. Part (1) is trivial, while part (2) requires some manipulations. We will use the fact that the zeroes of single-input *multiple*-output (SIMO) transfer functions remain constant under output feedback. For convenience, we will also drop the 'av' subscripts.

We have that

$$\phi^{T}A_{av}(\phi)\phi = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\phi^{T}\hat{n}(j\omega)\right)\left(1+\phi^{T}Q(j\omega I - A - b\phi^{T}Q)^{-1}b\right)$$
$$\cdot\left(c^{T}(-j\omega I - A - b\phi^{T}Q)^{-1}b\right)\left(\phi^{T}\hat{n}(-j\omega)\right)s_{r}(d\omega) \tag{3.7}$$

where the four main terms in parenthesis are *scalars*. The two terms in the middle represent SISO transfer functions which will be calculated now. By definition, the transfer function

$$c^{T}(sI-A)^{-1}b = k_{p}\frac{\hat{n}_{m}(s)}{\hat{d}_{m}(s)} = \frac{1}{c_0^*}\hat{m}(s) \tag{3.8}$$

where $\hat{m}(s)$ is strictly positive real, $\hat{n}_m(s)$, $\hat{d}_m(s)$ are monic polynomials in s, and $c_0^* = k_m/k_p$ is a positive constant. The $(A,\ b,\ c^T)$ representation in (3.8) is not minimal since dim $A = 3n-2$, and $\hat{d}_m(s)$ is an n-th order polynomial. Let $\hat{D}_m(s) = \det(sI-A) = \hat{d}_m(s)\hat{l}(s)$, where $\hat{l}(s)$ is a monic polynomial containing the unobservable modes of A.

The SISO transfer function $\phi^T Q(sI-A)^{-1}b$ can be written

$$\phi^{T}Q(sI-A)^{-1}b = \frac{\hat{n}_{\phi}(s)}{\hat{D}_{m}(s)} \tag{3.9}$$

where $\hat{n}_\phi(s)$ is a polynomial in s, not necessarily monic.

We can also write the SISO transfer function

$$\phi^{T}Q(sI-A-b\phi^{T}Q)^{-1}b = \frac{\hat{n}_{\phi}(s)}{\hat{D}_{\phi}(s)} \tag{3.10}$$

where $\hat{D}_\phi(s) = \det(sI - A - b\phi^T Q)$ is a monic polynomial which must satisfy $\hat{D}_\phi(s) = \hat{D}_m(s) - \hat{n}_\phi(s)$. From this it follows that

$$1+\phi^{T}Q(sI-A-b\phi^{T}Q)^{-1}b = \frac{\hat{D}_{m}(s)}{\hat{D}_{\phi}(s)}. \tag{3.11}$$

Finally, we also have, using the above mentioned fact about SIMO transfer functions, that

$$c^{T}(sI-A-b\phi^{T}Q)^{-1}b = k_{p}\frac{\hat{n}_{m}(s)\hat{l}(s)}{\hat{D}_{\phi}(s)}. \tag{3.12}$$

Using the previous identities in (3.7), the result follows with

$$R(\phi) := \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{k_p}{k_m}\left|\frac{\hat{D}_{m}(j\omega)}{\hat{D}_{\phi}(j\omega)}\right|^{2}\hat{n}(j\omega)\hat{n}^{T}(-j\omega)\hat{m}(-j\omega)s_{r}(d\omega). \tag{3.13}$$

It is remarkable that the result differs from the expression obtained by linearization followed by averaging in [6] only by the *scalar* weighting factor $|\hat{D}_m/\hat{D}_\phi|^2$. This term is strictly positive, given any ϕ bounded, and it approaches unity continuously as ϕ approaches zero. Since $\hat{m}(s)$ is strictly positive real, $R(\phi)$ is at least positive semi-definite. As in the linearized case, it is positive definite if $w_m(t)$ is persistently exciting [6].

Comments. As can easily be shown, using the Lyapunov function $v_{av}(\phi_{av}) = \|\phi_{av}\|^2$, the claim itself constitutes a proof of exponential stability of the averaged system. By Theorem 2.2, the exponential stability of the original system is also guaranteed for ε sufficiently small. The persistency of excitation condition is not a condition on signals located inside the adaptive system, but on exogenous model signals, and can be directly translated into a sufficient richness condition on the input [10].

Rates of convergence can also be determined, using the Lyapunov function $v_{av}(\phi_{av}) = \phi_{av}^{T}\phi_{av}$, so that

$$-\dot{v}_{av} = \varepsilon\phi_{av}^{T}\left(R(\phi_{av}) + R^{T}(\phi_{av})\right)\phi_{av} \geq \varepsilon \inf_{\phi_{av}\in B_h}\left(\lambda_{\min}\left(R(\phi_{av}) + R^{T}(\phi_{av})\right)v_{av} := 2\varepsilon\alpha v_{av}\right. \tag{3.14}$$

and the guaranteed rate of parameter convergence of the averaged adaptive system is $\varepsilon\alpha$. The rate of convergence of the original system can be estimated by the same value, for ε sufficiently small.

It is interesting to note that, as $\|\phi_{av}\|$ increases, $\lambda_{\min}(R(\phi_{av}))$ tends to zero in some directions. This indicates that the adaptive control system is not globally exponentially stable (with uniform rate of convergence).

3.3. Example

We now consider a simple two parameter example, in fact the well-known 'Rohrs example' [11] when no unmodeled dynamics are present. The adaptive system is described by

$$\dot{e} = -a_m e + b_p(\phi_r r + \phi_y e + \phi_y y_m), \tag{3.15}$$

$$\dot{\phi}_r = -\varepsilon e r, \tag{3.16}$$

$$\dot{\phi}_y = -\varepsilon e^2 - \varepsilon e y_m. \tag{3.17}$$

Consider the case when $r = r_0 \sin(\omega_0 t)$. Using (3.3), (3.5), the averaged system can be computed. After lengthy but straightforward manipulations, we obtain, for the averaged system (dropping again the 'av' subscripts for simplicity),

$$\dot{\phi}_r = -\varepsilon b_p \frac{r_0^2}{2}\frac{1}{\omega_0^2 + (a_m - b_p\phi_y)^2}\left((a_m - b_p\phi_y)\phi_r + \left(\frac{2a_m^2 b_m}{\omega_0^2 + a_m^2} - b_m\right)\phi_y - \frac{b_p a_m b_m}{\omega_0^2 + a_m^2}\phi_y^2\right), \tag{3.18}$$

$$\dot{\phi}_y = -\varepsilon b_p \frac{r_0^2}{2}\frac{1}{\omega_0^2 + (a_m - b_p\phi_y)^2}\left(b_m\phi_r + \frac{a_m b_m^2}{\omega_0^2 + a_m^2}\phi_y + b_p\phi_r^2 + \frac{b_p a_m b_m}{\omega_0^2 + a_m^2}\phi_r\phi_y\right). \tag{3.19}$$

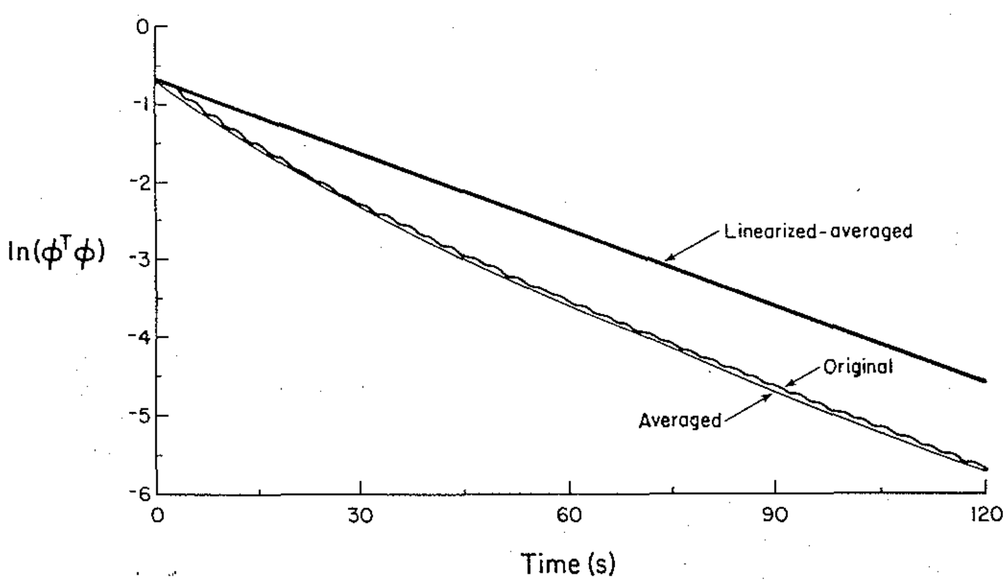

Fig. 1. Logarithm of the Lyapunov function.

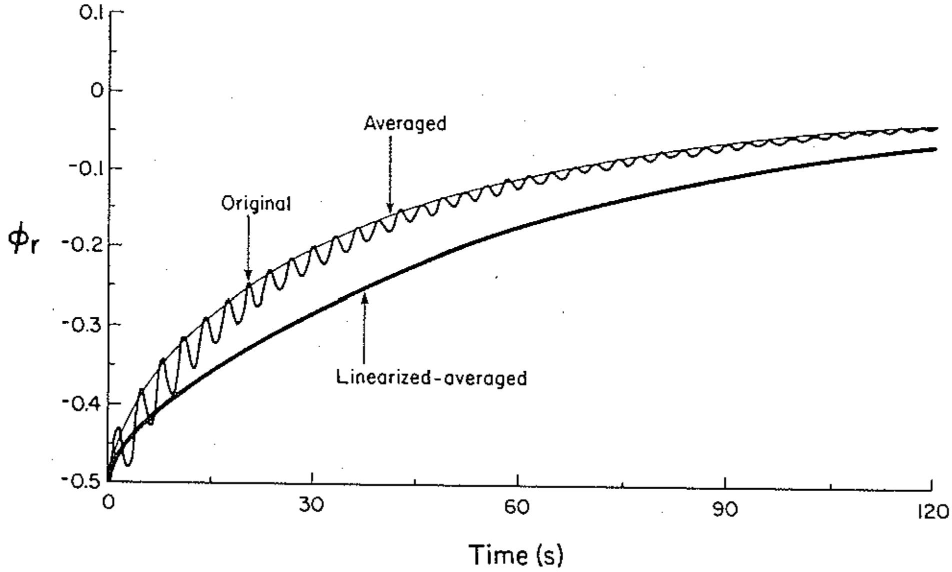

Fig. 2. Parameter error ϕ_r.

using this result, or using (3.13)–(3.14), we find that for $v = \phi^{\mathrm{T}}\phi$,

$$-\dot{v} = 2\varepsilon \frac{b_p}{b_m}\left(\frac{\omega_0^2 + a_m^2}{\omega_0^2 + \left(a_m - b_p\phi_y\right)^2}\right)\frac{r_0^2}{2}\phi^{\mathrm{T}}\begin{pmatrix} \dfrac{a_m b_m}{a_m^2 + \omega_0^2} & \dfrac{b_m^2\left(a_m^2 - \omega_0^2\right)}{\left(a_m^2 + \omega_0^2\right)^2} \\ \dfrac{b_m^2}{a_m^2 + \omega_0^2} & \dfrac{a_m b_m^3}{\left(a_m^2 + \omega_0^2\right)^2} \end{pmatrix}\phi. \tag{3.20}$$

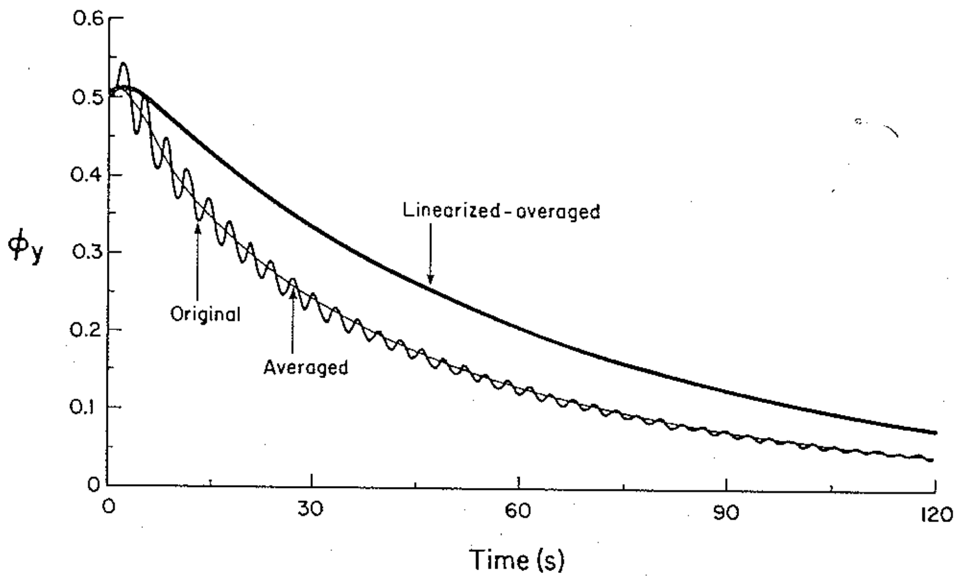

Fig. 3. Parameter error ϕ_y.

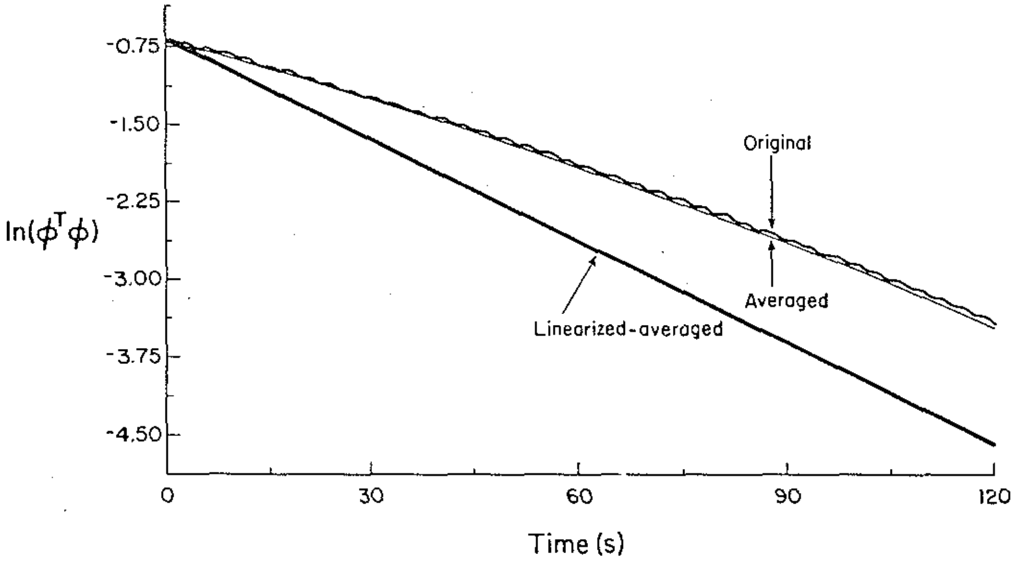

Fig. 4. Logarithm of the Lyapunov function.

It can be checked that when the first term in brackets is equal to 1 (i.e. with ϕ_y replaced by zero), the result is the same as the result obtained by first linearizing the system, then averaging it. Also, given any prescribed B_h (but such that $\hat{D}_\phi(s)$ is Hurwitz), (3.20) can be used to obtain estimates of the rates of convergence of the *nonlinear* system.

We reproduce here simulations for the following values of the parameters: $a_m = 3$, $b_m = 3$, $a_p = 1$, $b_p = 2$, $r_0 = 1$, $\omega_0 = 1$, $\varepsilon = 1$. The first set of figures is a simulation for initial conditions $\phi_r(0) = -0.5$, and $\phi_y(0) = 0.5$. Fig. 1 represents the time variation of the function $\ln(v = \phi^T\phi)$ for the original, averaged, and linearized-averaged systems (the minimum slope of the curve gives the rate of convergence). It shows the close approximation of the original system by the averaged system. The slope for the linearized-averaged system is asymptotically identical to that of the averaged system, since parameters eventually get arbitrarily close to their nominal values. Fig. 2 and Fig. 3 show the approximation of the trajectories of ϕ_r, and ϕ_y.

Fig. 4 represents the logarithm of the Lyapunov function for a simulation with identical parameters, but initial conditions $\phi_r(0) = 0.5$, $\phi_y(0) = -0.5$. Due to the change of sign in $\phi_y(0)$, the rate of convergence is less here than the rate of the linearized system, while it was larger in the previous case. These simulations demonstrate the close approximation by the averaged system, and it should be noted that this is achieved despite an adaptation gain ε equal to 1. This shows that the averaging method is useful for values of ε which are not necessarily infinitesimal (i.e. not necessarily for very slow adaptation), but for values which are often practical ones.

4. Conclusions

Averaging methods have been extended to include differential systems representing adaptive control schemes. Previous results on the frequency-domain analysis of an adaptive control scheme were generalized to the full nonlinear equations, and some estimates of the parameter convergence rates were obtained. One general restriction is that the parameter estimates along trajectories have the property that the associated frozen closed-loop system is stable. This application is only one of many possible uses of averaging, but it demonstrates its usefulness as a method of analysis of adaptive systems.

Appendix

Generalized Bellman–Gronwall Lemma (cf. e.g. [12], p. 169). *If* $x(t)$, $a(t)$, $u(t)$ *are positive functions satisfying*

$$x(t) \leqslant \int_0^t a(\tau)x(\tau)\,\mathrm{d}\tau + u(t) \tag{A1}$$

for all $t \in [0, T]$, *and* $u(t)$ *is differentiable*, ***then***

$$x(t) \leqslant u(0)\exp\left\{\int_0^t a(\sigma)\,\mathrm{d}\sigma\right\} + \int_0^t \dot{u}(\tau)\exp\left\{\int_\tau^t a(\sigma)\,\mathrm{d}\sigma\right\}\mathrm{d}\tau \tag{A2}$$

for all $t \in [0, T]$.

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