Stability and the Matrix Lyapunov Equation for Discrete 2-Dimensional Systems

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Abstract—The stability of two-dimensional, linear, discrete systems is examined using the 2-D matrix Lyapunov equation. While the existence of a positive definite solution pair to the 2-D Lyapunov equation is sufficient for stability, the paper proves that such existence is not necessary for stability, disproving a long-standing conjecture.

I. Introduction

The introduction of state-space models for 2-D discrete systems allows the investigation of stability using the state-space approach. The question then arises as to how one of the standard 1-D linear stability tools, the Lyapunov matrix equation, could be extended to the 2-D case. One would hope that the extension of the Lyapunov equation to the 2-D case would give necessary and sufficient conditions for the characteristic polynomial of a linear discrete 2-D system to be void of zeros in the unit bidisc in terms of properties of the solutions to the 2-D Lyapunov equation.

There are essentially two different approaches to this problem. The first consists of developing a 2-D Lyapunov equation with constant coefficients [1], [2], while the second approach is considering a 1-D Lyapunov equation with coefficients which are functions in a complex variable [3], [4]. In [1], the formulation of the Lyapunov equation for n-dimensional continuous systems was first considered. This formulation is extended to discrete systems in [2] using the multi-dimensional bilinear transformation. Furthermore, it has been asserted that the existence of positive definite matrices satisfying the 2-D Lyapunov equation is necessary and sufficient for the 2-D characteristic polynomial to be void of zeros in the unit bidisc. Although the sufficiency part of this assertion can be easily proven, the necessity part has not been satisfactorily established yet.

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The aim of this paper is to establish a necessary and sufficient condition for the existence of positive definite solutions of the 2-D Lyapunov equation. This condition is developed based on properties of strictly bounded real matrices. Furthermore, it is shown that the developed condition is stronger than the condition that the characteristic polynomial has no zeros in the unit bidisc. This is demonstrated with an example of a stable 2-D system for which no positive definite solutions of the 2-D Lyapunov equation can be found.

The paper is organized as follows: In Section II the 2-D discrete state-space model is introduced and some properties of strictly bounded real matrices are briefly discussed. In Section III the necessary and sufficient condition for the existence of positive definite solutions of the 2-D Lyapunov equation is developed, and compared with the condition that the characteristic polynomial has no zeros in the unit bidisc. Finally, in Section IV the result of Section III is applied to two special classes of 2-D systems.

Notation: $\overline{U}^2$ denotes the closed unit bidisc.

$$\overline{U}^2 \colon \{(z_1 z_2) \mid |z_1| \leq 1, |z_2| \leq 1\}$$

I_n the $n \times n$ unity matrix and $\oplus$ the direct sum of matrices.

II. Preliminaries

Linear shift invariant 2-D discrete systems can be represented by the following state space model [5]:

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^{h(i,j)} \\ x^{v(i,j)} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(i,j) \quad (1)$$

$$y(i,j) = [C_1, C_2] \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} \quad (2)$$

where $x^h \in \mathbb{R}^n$, $x^v \in \mathbb{R}^m$ represent the horizontal and vertical states, respectively, u is the input and y the output. The system matrix A is given by

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad (3)$$

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with the submatrices A_{ij}, $i, j=1,2$ of appropriate dimensions. In the investigation of stability of a 2-D discrete system realized with a model of the above type, it is required to establish that the zeros of the characteristic polynomial are outside $\overline{U}^2$, i.e.,

$$C(z_1, z_2) = \det\begin{bmatrix} I_n - z_1 A_{11} & -z_1 A_{12} \\ -z_2 A_{21} & I_m - z_2 A_{22} \end{bmatrix} \neq 0 \quad \text{in } \overline{U}^2 \tag{4}$$

Although it is possible to test the stability condition (4) in the frequency domain using one of the known methods for testing stability of 2-D polynomials [6], the 2-D Lyapunov equation could be an important tool in the stability analysis and in the design of 2-D systems in the state-space.

The n-D Lyapunov equation was first presented in [1] as a condition for the multivariable characteristic polynomial of an n-D continuous system to be strictly Hurwitzian (i.e., to have no zeros in the region $\text{Re}(s_i) \geqslant 0$, $i=1,\cdots,n$). This was extended to the discrete case using the double bilinear transformation in [2]. The resulting discrete 2-D Lyapunov condition is as follows:

2-D Lyapunov condition: There exist matrices $W_1(n\times n)$, $W_2(m\times m)$, $W=W_1\oplus W_2$ and Q symmetric and positive definite such that

$$A^TWA - W = -Q \tag{5}$$

is satisfied.

It has been asserted [2], [1] that the above 2-D Lyapunov condition is necessary and sufficient for the stability condition (4). In the next section it will be shown that the 2-D Lyapunov condition is in general only sufficient for the stability condition (4) and not necessary. To show this the following definitions and preliminary results for strictly bounded real matrices will be needed.

Definition 1: Let $S(z^{-1})$ be a square matrix of real rational functions in the complex variable z^{-1}. Then $S(z^{-1})$ is called strictly bounded real if

i) all poles of $S(z^{-1})$ lie in $|z^{-1}|<1$;

ii) $I - S^T(e^{-j\omega})S(e^{j\omega})$ is positive definite for all $\omega \in [0,2\pi]$.

This definition is given in [7] with the only difference that z is used as the complex variable instead of z^{-1}. z^{-1} is used here in order to conform with the fact that in 2-D stability theory the stability region is defined as the area outside the unit bidisc. Conditions i) and ii) on $S(z^{-1})$ can be reduced to conditions on the matrices of a minimal state-space realization of $S(z^{-1})$ using the Bounded Real Lemma [7]. Suppose that $S(z^{-1})$ has a minimal realization given by the quadruple $\{F, G, H, J\}$ such that

$$S(z^{-1}) = J + H^T(z^{-1}I - F)^{-1}G \tag{6}$$

then the strictly Bounded Real Lemma can be formulated:

Lemma 1: Let $S(z^{-1})$ be a square matrix of real rational functions in z^{-1} and $\{F, G, H, J\}$ a minimal realization of $S(z^{-1})$. Then $S(z^{-1})$ is strictly bounded real if, and only if, there exists a symmetric positive definite matrix P such that the matrix Q_1 given by

$$Q_1 = \begin{bmatrix} I - J^TJ - G^TPG & -(F^TPG + HJ)^T \\ -F^TPG - HJ & P - F^TPF - HH^T \end{bmatrix} \tag{7}$$

is positive definite.

This result was essentially established by Anderson and Vongpanitlerd ([7, lemma 11.2.1]) except for two minor differences. The first is that in [7] the lemma is stated using a particular minimal realization of $S(z^{-1})$ with a coordinate basis in which $P=I$. Matrix Q_1, (7), results after changing the coordinate basis used in [7, lemma 11.2.1] by applying a transformation matrix T such that $T^TT=P$. The second difference is that in [7], the case of $S(z^{-1})$ being only bounded real and not strictly bounded real is considered. As a result of this the condition on Q_1 in [7] is weaker than the one in Lemma 1. In particular, the condition for $S(z^{-1})$ to be strictly bounded real is that Q_1, (7), is positive definite, while in [7] the condition for $S(z^{-1})$ to be bounded real is that Q_1 is nonegative definite.

Remark: If $\{F, G, H, J\}$ is not a minimal realization of $S(z^{-1})$ and there exists a positive definite P such that matrix Q_1 in (7) is positive definite, then $S(z^{-1})$ is still strictly bounded real. However, the converse cannot be established.

II. The 2-D Lyapunov Equation

In this section the necessary and sufficient condition for the existence of positive definite matrices $W=W_1\oplus W_2$, Q satisfying the 2-D Lyapunov (5) will be presented. This result is based on the Bounded Real Lemma outlined in the previous section. The condition presented in Theorem 1 will be subsequently compared with the stability condition (4) and it will be shown by means of an example that the stability condition (4) is not sufficient for the existence of positive definite matrices W and Q satisfying the 2-D Lyapunov equation.

Theorem 1: Consider a 2-D discrete system described by (1). If for some nonsingular T, $S(z_2^{-1})$ given by

$$S(z_2^{-1}) = T\left[A_{11} + A_{12}(z_2^{-1}I - A_{22})^{-1}A_{21}\right]T^{-1} \tag{8}$$

is strictly bounded real and if (A_{22}, A_{21}) and (A_{22}, A_{12}^T) are, respectively, completely reachable and completely observable, then there exists some positive definite matrices W_1, W_2, and Q such that $W=W_1\oplus W_2$ and Q satisfy the 2-D Lyapunov equation:

$$A^TWA - W = -Q. \tag{9}$$

Conversely, if (9) holds for positive definite $W=W_1\oplus W_2$ and Q, then there exists a nonsingular matrix T such that $S(z_2^{-1})$ is strictly bounded real.

Proof: Sufficiency: T nonsingular and (A_{22}, A_{21}), (A_{22}, A_{12}^T) completely reachable and observable imply that the quadruple $\{A_{22}, A_{21}T^{-1}, A_{12}^TT^T, TA_{11}T^{-1}\}$ is a minimal realization of $S(z_2^{-1})$. From lemma 1, it follows that there exists a positive definite matrix P such that the matrix:

$$Q_1=\begin{bmatrix} I-T^{-T}A_{11}^TT^TTA_{11}T^{-1}-T^{-T}A_{21}^TPA_{21}T^{-1} & -\left(A_{22}^TPA_{21}T^{-1}+A_{12}T^TTA_{11}T^{-1}\right)^T \\ -A_{22}^TPA_{21}T^{-1}-A_{12}^TT^TTA_{11}T^{-1} & P-A_{22}^TPA_{22}-A_{12}^TT^TTA_{12} \end{bmatrix} \tag{10}$$

is positive definite. Equation (10) gives after premultiplication with $(T^T\oplus I_m)$ and postmultiplication with $(T\oplus I_m)$:

$$\begin{bmatrix} T^T & 0 \\ 0 & I_m \end{bmatrix} Q_1 \begin{bmatrix} T & 0 \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} T^TT & 0 \\ 0 & P \end{bmatrix} - \begin{bmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{bmatrix} \begin{bmatrix} T^TT & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{11}$$

which is the 2-D Lyapunov equation (9) with $W=T^TT\oplus P$ and $Q=(T^T\oplus I_m)$.

Necessity: Suppose (9) is satisfied. Set $T=W_1^{1/2}$, $F=A_{22}$, $G=A_{21}T^{-1}$, $H^T=TA_{12}$ and $J=TA_{11}T^{-1}$. Then premultiply and postmultiply (9) by $(T^{-1}\oplus I_m)$ and (7) results with $P=W_2$ and $Q_1=(T^{-1}\oplus I_m)Q(T^{-1}\oplus I_m)$. The strictly bounded property of $S(z_2^{-1})$ follows then directly from Lemma 1.

The stability condition (4) will now be compared with the condition for the existence of positive definite matrices satisfying the 2-D Lyapunov equation given in Theorem 1. It can be assumed without loss of generality that the systems considered have completely reachable and observable pairs (A_{22}, A_{21}) and (A_{22}, A_{12}^T), respectively. For in the next section it will be shown that if the reachability and/or observability condition is not satisfied, then this system can be decomposed into two subsystems. The first subsystem is a 1-D system and the second is a 2-D one satisfying the observability and reachability conditions. The stability condition (4) can be rewritten as

$$\det\begin{bmatrix} I_n-z_1S_1(z_2^{-1}) & 0 \\ -z_2A_{21} & I_m-z_2A_{22} \end{bmatrix}\neq 0, \quad \text{for all } (z_1, z_2)\in\overline{U}^2 \tag{12}$$

where

$$S_1(z_2^{-1})=A_{11}+A_{12}\left(z_2^{-1}I_m-A_{22}\right)^{-1}A_{21}. \tag{13}$$

Using Huang's condition [9], [4], it follows that condition (12) is equivalent to the following two conditions:

$$\det(I-z_2A_{22})\neq 0, \qquad \text{for all } |z_2|\leqslant 1 \tag{14}$$

$$\det\left(I-z_1S_1(z_2-1)\right)\neq 0, \qquad \text{for all } |z_1|\leqslant 1 \text{ and } |z_2|=1. \tag{15}$$

These two conditions require that the eigenvalues of A_{22} and $S_1(e^{-j\omega})$, denoted by $\lambda_i\{A_{22}\}$ and $\lambda_i\{S_1(e^{-j\omega})\}$, satisfy

$$|\lambda_i\{A_{22}\}|<1, \qquad i=1,\cdots,m \tag{16}$$

and

$$|\lambda_i\{S_1(e^{j\omega})\}|<1, \qquad i=1,\cdots,n \text{ for all } \omega\in[0,2\pi] \tag{17}$$

and for any nonsingular T (17) is equivalent to

$$|\lambda_i\{S(e^{j\omega})\}|=|\lambda_i\{TS_1(e^{j\omega})T^{-1}\}|<1, \qquad i=1,\cdots,n \text{ for all } \omega\in[0,2\pi] \tag{18}$$

The strict bounded real property of $S(z_2^{-1})$, required in Theorem 1, can now be compared to conditions (16) and (18), which are equivalent to the stability condition (4).

In the following lemma it is shown that the strict bounded real property of $S(z_2^{-1})$ implies conditions (16) and (18). However, as it will be seen later, the converse is not true.

Lemma 2: Consider $S(z_2^{-1})$, given by (8), with the pairs (A_{22}, A_{21}) and (A_{22}, A_{12}^T) being completely reachable and observable respectively. Then (16) and (18) hold if $S(z_2^{-1})$ is strictly bounded real.

Proof: If $S(z_2^{-1})$ is strictly bounded real, it follows for the eigenvalues at A_{22}, the system matrix in the minimal realization of $S(z_2^{-1})$, that

$$|\lambda_i\{A_{22}\}|<1 \qquad \text{for } i=1,\cdots,m.$$

From the strict bounded real property of $S(z_2^{-1})$, it follows that

$$I-S^T(e^{-j\omega})S(e^{j\omega})>0 \qquad \text{for all } \omega\in[0,2\pi] \tag{19}$$

which implies

$$\lambda_i\{S(e^{j\omega})\}<1 \qquad \text{for all } \omega\in[0,2\pi] \text{ and } i=1,\cdots,n.$$

The converse of the above lemma is not true. In other words, the stability condition (4) does not imply that $S(z_2^{-1})$ is strict bounded real, and hence, there exist systems with a stable characteristic polynomial and a $S(z_2^{-1})$ which is not strictly bounded real. From Theorem 1 it follows that for such a system positive definite matrices $W=W_1\oplus W_2$ and Q satisfying the 2-D Lyapunov equation do not exist. In example 1 such a system is discussed.

Example: Consider the 2-D discrete system described by the following system matrix:

$$A=\left[\begin{array}{c|c|cccccc} 0.5 & 0.0070.0 & 0.012 & -0.008 & 0.028 & 0 & 0 & 0 \\ -0.007 & 0.5 & 0 & 0 & 0 & 0.012 & 0.008 & 0.012 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0.845 & -2.657 & 2.81 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & -0.845 & -2.657 & -2.81 \end{array}\right] \tag{20}$$

where $x^h \in R^2$ and $x^v \in R^6$.

Claim 1: The 2-D characteristic polynomial of matrix A, (20), has no zeros in $\bar{U}^2$.

Proof: The characteristic polynomial of A can be given by

$$\det\begin{bmatrix} I_n - z_1 S_1(z_2^{-1}) & 0 \\ -z_2 A_{21} & I_m - z_2 A_{22} \end{bmatrix} \tag{21}$$

where

$$S_1(z_2^{-1}) = A_{11} + A_{12}(z_2^{-1} I - A_{22})^{-1} A_{21} \tag{22}$$

which gives

$$S_1(z_2^{-1}) = \begin{bmatrix} 0.5 & S_a(z_2^{-1}) \\ S_b(z_2^{-1}) & 0.5 \end{bmatrix} \tag{23}$$

with

$$S_a(z_2^{-1}) = \frac{0.028 z_2^{-2} - 0.008 z_2^{-1} + 0.012}{z_2^{-3} - 2.81 z_2^{-2} + 2.657_2^{-1} - 0.845} + 0.007 \tag{24}$$

and

$$S_b(z_2^{-1}) = \frac{0.028 z_2^{-2} + 0.008 z_2^{-1} + 0.012}{z_2^{-3} + 2.81 z_2^{-2} + 2.657 z_2^{-1} + 0.845} - 0.007. \tag{25}$$

In order to prove that the characteristic polynomial has no zeros in $\bar{U}^2$, we have to show that conditions (16)

$$|\lambda_i\{A_{22}\}| < 1, \qquad \text{for } i = 1, \cdots, 6$$

and (17)

$$|\lambda_i\{S_1(e^{j\omega})\}| < 1, \qquad \text{for } i = 1,2 \text{ and all } \omega \in [0, 2\pi]$$

are satisfied.

$$P - S_1^T(1) P S_1(1) = \begin{bmatrix} 0.75 + \alpha p_{12} - \alpha^2 p_{22} & -0.5\beta - (0.25 - \alpha\beta) p_{12} + 0.5\alpha p_{22} \\ -0.5\beta - (0.25 - \alpha\beta) p_{12} + 0.5\alpha p_{22} & 0.75 p_{22} - \beta p_{12} - \beta^2 \end{bmatrix}. \tag{35}$$

The first condition can be easily proven by computing the eigenvalues of the A_{22} (or the poles of $S_a(z_2^{-1})$ and $S_b(z_2^{-1})$). For the second condition consider

$$\det[I_n - z_1 S_1(e^{j\omega})] = z_1^2\left[(z_1^{-1} - 0.5)^2 - S_a(e^{j\omega}) \cdot S_b(e^{j\omega})\right]. \tag{26}$$

It can easily be verified by evaluating $|S_a(e(^{j\omega}) \cdot S_b(e^{j\omega})|$ for all $\omega \in [0, 2\pi]$ that

$$|S_a(e^{j\omega}) \cdot S_b(e^{j\omega})| < 0.02 \tag{27}$$

which implies that the values of z_1^{-1} for which (26) is zero, are all close to 0.5. These values are precisely the eigenvalues of S_1, so that

$$|\lambda_i\{S_1(e^{j\omega})\}| < 1 \qquad \text{for } \omega \in [0, 2\pi], \qquad i = 1, 2$$

and consequently the characteristic polynomial of A has no zeros in $\bar{U}^2$.

Claim 2: There is no nonsingular matrix T for which $S(z_2^{-1}) = T S_1(z_2^{-1}) T^{-1}$ satisfies

$$I - S^T(e^{-j\omega}) \cdot S(e^{j\omega}) > 0, \qquad \text{for all } \omega \in [0, 2\pi] \tag{28}$$

Proof: Let us assume that there exists a nonsingular T such that (28) is satisfied and we will show that this results in a contradiction. Let

$$P = T^T T \tag{29}$$

then, (28) can be rewritten as

$$P - S_1^T(e^{-j\omega}) P S_1(e^{j\omega}) > 0, \qquad \text{for all } \omega \in [0, 2\pi].$$

It will be shown that there exists no positive definite P such that

$$P - S_1^T(1) P S_1(1) > 0 \tag{30}$$

and

$$P - S_1^T(-1) P S_1(-1) > 0 \tag{31}$$

are satisfied. From (23) to (25) we obtain

$$S_1(1) = \begin{bmatrix} 0.5 & \beta \\ -\alpha & 0.5 \end{bmatrix} \tag{32}$$

and

$$S_1(-1) = \begin{bmatrix} 0.5 & \alpha \\ -\beta & 0.5 \end{bmatrix} \tag{33}$$

where $\alpha = 0.000435$ and $\beta = 16.007$.

P is positive definite and without loss of generality we can assume that $p_{11} = 1$, which implies the following condition:

$$|p_{12}| < \sqrt{p_{22}}. \tag{34}$$

Consider (30)

The positive definiteness condition requires

$$0.75 p_{22} - \beta p_{12} - \beta^2 > 0 \tag{36}$$

or

$$0.75 p_{22} > \beta^2 - \beta p_{12}$$

which gives using (34)

$$0.75_{22} > \beta^2 - |\beta| \sqrt{p_{22}}$$

or

$$\left(\sqrt{p_{22}} + \tfrac{2}{3}|\beta|\right)^2 - \tfrac{16}{9}\beta^2 > 0$$

which implies

$$\sqrt{p_{22}} > \tfrac{2}{3}|\beta| \approx 10. \tag{37}$$

Consider next (31)

$$P - S_1^T(-1) P S_1(-1) = \begin{bmatrix} 0.75 + \beta p_{12} - \beta^2 p_{22} & -0.5\alpha - (0.25 - \alpha\beta) p_{12} + 0.5\beta p_{22} \\ -0.5\alpha - (0.25 - \alpha\beta) p_{12} + 0.5\beta p_{22} & 0.75 p_{22} - \alpha p_{12} - \alpha^2 \end{bmatrix}. \tag{38}$$

The condition for positive definiteness requires

$$0.75+\beta p_{12}-\beta^2 p_{22}>0 \tag{39}$$

which, after similar calculations as in the previous case, yields the condition

$$\sqrt{p_{22}}<\frac{3}{2|\beta|}\approx 0.1. \tag{40}$$

Condition (40) contradicts condition (37), and, therefore, there is no nonsingular matrix T such that (28) is satisfied.

The implication of Claim 2 and Theorem 1 is that for this 2-D system there exist no positive definite matrices $W=W_1\oplus W_2$ and Q such that the 2-D Lyapunov equation is satisfied. This same system has a characteristic polynomial which has no zeros in $\overline{U}^2$ as shown in Claim 1. Consequently, the assertion that the stability of the characteristic polynomial implies the existence of positive definite matrices $W=W_1\oplus W_2$ and Q satisfying the 2-D Lyapunov equation is not correct. In other words, the necessity part of the 2-D Lyapunov condition is different than in the 1-D case. This establishes another case where 1-D results cannot be extended to the higher-dimensional case. Furthermore, it should be noted that these results can easily be extended to the continuous case.

IV. Special Cases

In this section Theorem 1 is applied to the following two classes of 2-D discrete systems.

i) 2-D realizations with a nonreachable pair (A_{22}, A_{21});

ii) 2-D systems with a characteristic polynomial which is first order in at least one variable.

Consider the 2-D discrete system described with (1) where the pair (A_{22}, A_{21}) is not reachable. This implies that the quadruple $\{A_{22}, A_{21}, A_{12}^{\mathrm{T}}, A_{11}\}$ is not a minimal realization of $S_1(z_2^{-1})$ and, therefore, theorem 1 cannot be applied. However, it is well known that a minimal realization of $S(z_2^{-1})$ can be obtained by separating out the reachable part of $\{A_{22}, A_{21}, A_{12}^T, A_{11}\}$. There exists a nonsingular matrix R such that

$$R^{-1}A_{22}R=\left[\begin{array}{c|c} A_{221} & A_{222}\\ \hline 0 & A_{223}\end{array}\right]\begin{matrix} q\\ m-q\end{matrix} \tag{41}$$

$$R^{-1}A_{21}=\left[\begin{array}{c} A_{211}\\ \hline 0\end{array}\right]\begin{matrix} q\\ m-q\end{matrix} \tag{42}$$

and

$$A_{12}R=\overset{\quad q \qquad\quad m-q}{[A_{121} \;\vert\; A]_{122}}\,. \tag{43}$$

Further, the pair (A_{221}, A_{211}) is completely reachable. Applying the coordinate transformation $(I_n\oplus R)$ to the 2-D system described by (1), the system yields

$$\hat{A}=\begin{bmatrix} A_{11} & A_{121} & A_{122}\\ A_{211} & A_{221} & A_{222}\\ 0 & 0 & A_{223}\end{bmatrix}$$

with the following characteristic polynomial,

$$C(z_1,z_2)=\det\begin{bmatrix} I_n-z_1A_{11} & -z_2A_{121}\\ -z_1A_{211} & I_q-z_2A_{221}\end{bmatrix}\cdot\det\left[I_{m-q}-z_2A_{223}\right]. \tag{44}$$

The structure of the system matrix A in the new coordinate basis allows the decomposition of the original 2-D system into two subsystems. The first is one with a 1-D characteristic polynomial, and the second has a 2-D characteristic polynomial, and in addition $\{A_{221}, A_{211}, A_{121}, A_{11}\}$ is a minimal realization of $S(z_2^{-1})$. Theorem 1 can now be applied to the reduced 2-D system and if $S(z_2^{-1})$ is strictly bounded real, there exist positive definite solutions of the 2-D Lyapunov equation for the reduced 2-D system. It should be noticed, however, that even if positive definite solutions of the Lyapunov equations for both the 1-D and the 2-D subsystems exist, it is not necessary that positive definite solutions to the 2-D Lyapunov equation for the original 2-D system exist. Obviously, similar observations can be made in the case where the pair (A_{22}, A_{12}^T) is not observable.

Consider now the second special class of 2-D systems, namely those with a characteristic polynomial which is first order in at least one of the variables z_1, z_2. For these systems Corollary 1 can be given.

Corollary 1: Consider a 2-D discrete system described by (1) with $n=1$. If the characteristic polynomial of this system has no zeros in $\overline{U}^2$ and if the pairs (A_{11}, A_{21}) and (A_{11}, A_{12}^T) are completely reachable and observable, respectively, then there exist some positive definite matrices $W=W_1\oplus W_2$ and Q such that the 2-D Lyapunov equation

$$A^TWA-W=-Q \tag{45}$$

is satisfied. Conversely, if (45) is satisfied, then the characteristic polynomial of the system has no zeros in $\overline{U}^2$.

Proof: The proof of this theorem follows from theorem 1 and the fact that for $n=1$, $S(z_2^{-1})$ is scalar function in z_2^{-1}. Consequently, the two scalar conditions

$$|\lambda_1\{S(e^{j\omega})\}|<1, \qquad \text{for all } \omega\in[0,2\pi]$$

and

$$1-S(e^{-j\omega})S(e^{j\omega})>0, \qquad \text{for all } \omega\in[0,2\pi]$$

are equivalent.

A similar corollary can be given for the case $m=1$. In the case $m=n=1$ A_{ij} $(i,j=1,2)$ become scalar and, therefore, the reachability and observability conditions are not needed, which yields the same condition given in Theorem 2 in [6].

V. Conclusions

The necessary and sufficient condition for the existence of positive definite matrices satisfying the 2-D Lyapunov equation has been developed. It has been shown that the existence of such solution pairs implies that the 2-D characteristic polynomial has no zeros in the unit bidisc but the converse is not true. A stable 2-D system is given for which no positive definite matrices satisfying the 2-D Lyapunov equation exist.

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