By using (3.9) and Corollary 3.15, part i), we contract $\tilde{K}(t)$ to

$$K(t)=\begin{bmatrix} K_{11}(t) & 0 \\ K_{21}(t) & K_{32}(t) \\ 0 & K_{42}(t) \end{bmatrix} \tag{4.8}$$

so that the estimator E (aggregation of $\tilde{E}$) becomes

$$E:\begin{bmatrix} \hat{x}_1(t+1) \\ \hat{x}_2(t+1) \\ \hat{x}_3(t+1) \end{bmatrix} = \begin{bmatrix} A_{11}(t) & A_{12}(t) & A_{13}(t) \\ A_{21}(t) & A_{22}(t) & A_{23}(t) \\ A_{31}(t) & A_{32}(t) & A_{33}(t) \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \\ \hat{x}_3(t) \end{bmatrix} + \begin{bmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \\ B_{31}(t) & B_{32}(t) \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + \begin{bmatrix} K_{11}(t) & 0 \\ K_{21}(t) & K_{32}(t) \\ 0 & K_{42}(t) \end{bmatrix} \left(\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} - \begin{bmatrix} C_{11}(t) & C_{12}(t) & C_{13}(t) \\ C_{21}(t) & C_{22}(t) & C_{23}(t) \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \\ \hat{x}_3(t) \end{bmatrix} \right) \tag{4.9}$$

for the original system S (aggregation of $\tilde{S}$).

The suboptimality aspects of the decentralized estimator design are discussed in [5].

B. Restriction and Control

In the restriction case we have $\tilde{x}(t) = Vx(t)$, where

$$V=\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2/2 & 0 \\ 0 & I_2/2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}, \quad U=\begin{bmatrix} I_1 & 0 & 0 & 0 \\ 0 & I_2/2 & I_2/2 & 0 \\ 0 & 0 & 0 & I_3 \end{bmatrix} \tag{4.10}$$

and

$$M_A(t)=\begin{bmatrix} 0 & A_{12}(t)/2 & -A_{12}(t)/2 & 0 \\ 0 & A_{22}(t)/2 & -A_{22}(t)/2 & 0 \\ 0 & -A_{22}(t)/2 & A_{22}(t)/2 & 0 \\ 0 & -A_{32}(t)/2 & A_{32}(t)/2 & 0 \end{bmatrix}, \quad N_B(t)=0, \quad N_\Gamma(t)=0,$$

$$N_C(t)=\begin{bmatrix} 0 & C_{12}(t)/2 & -C_{12}(t)/2 & 0 \\ 0 & -C_{22}(t)/2 & C_{22}(t)/2 & 0 \end{bmatrix} \tag{4.11}$$

and they satisfy restriction conditions (2.9), (2.10). The expansion $\tilde{S}$ has the form

$$\tilde{S}:\begin{bmatrix} \tilde{x}_1(t+1) \\ \tilde{x}_2(t+1) \end{bmatrix} = \left[\begin{array}{cc:cc} A_{11}(t) & A_{12}(t) & 0 & A_{13}(t) \\ A_{21}(t) & A_{22}(t) & 0 & A_{23}(t) \\ \hdashline A_{21}(t) & 0 & A_{22}(t) & A_{23}(t) \\ A_{31}(t) & 0 & A_{32}(t) & A_{33}(t) \end{array}\right] \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix} + \left[\begin{array}{c:c} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \\ \hdashline B_{21}(t) & B_{22}(t) \\ B_{31}(t) & B_{32}(t) \end{array}\right] \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + \left[\begin{array}{c:c} \Gamma_{11}(t) & \Gamma_{12}(t) \\ \Gamma_{21}(t) & \Gamma_{22}(t) \\ \hdashline \Gamma_{21}(t) & \Gamma_{22}(t) \\ \Gamma_{31}(t) & \Gamma_{32}(t) \end{array}\right] \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \left[\begin{array}{cc:cc} C_{11}(t) & C_{12}(t) & 0 & C_{13}(t) \\ \hdashline C_{21}(t) & 0 & C_{22}(t) & C_{23}(t) \end{array}\right] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} v_2(t) \\ v_2(t) \end{bmatrix} \tag{4.12}$$

and the subsystems appear as disjoint.

For the decentralized control purposes, we set all the off-(block) diagonal terms in the matrices $\tilde{A}(t)$ and $\tilde{B}(t)$ of (4.12) to zero so the subsystems become decoupled. Then we build decentralized control laws for the decoupled subsystems, and we make sure that the restriction conditions are satisfied [2]. The resulting controller gain matrix $\tilde{L}(t)$ = diag $[L_1(t), L_2(t)]$ can then be used as a suboptimal decentralized control for the expansion $\tilde{S}$ of (4.12). In order to get the gain matrix $L(t)$ for the original system S of (4.1), we represent $\tilde{L}(t)$ as

$$\tilde{L}(t)=\begin{bmatrix} L_{11}(t) & L_{12}(t) & 0 & 0 \\ 0 & 0 & L_{23}(t) & L_{24}(t) \end{bmatrix} \tag{4.13}$$

to conform with the representation (4.12). Then we contract $\tilde{L}(t)$ [2] by using $L(t) = \tilde{L}(t)V$ and get

$$L(t)=\begin{bmatrix} L_{11}(t) & L_{12}(t) & 0 \\ 0 & L_{23}(t) & L_{24}(t) \end{bmatrix} \tag{4.14}$$

which can be implemented in the original system S.

For obvious reasons, the proposed control design is not optimal in general. The stability and suboptimality aspects of the continuous version are discussed in [5] and [2].

V. Conclusion

A solution to the decentralized estimation and control problems with overlapping information structure constraints has been presented. The described method is based on the expansion-contraction framework of the stochastic inclusion principle which is extended here to cover inclusion of Gauss–Markov models and associated estimators. It has been shown how a separation principle can be used to formulate independently the estimation and control laws for systems composed of interconnected overlapping subsystems.

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A Local Stability Analysis for a Class of Adaptive Systems

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***Abstract*—An analysis of adaptive systems is presented where a local L_∞-stability is ensured under a persistent excitation condition.**

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Introduction

In this note we combine some earlier results [1]-[4] to provide a framework for stability analysis of adaptive systems. We consider here the continuous-time adaptive control of a scalar plant[1] with input u and output y, described by

$$\text{Plant: } y = d + Pu \tag{1a}$$

$$\text{Control: } u = -\hat{\theta}'z \tag{1b}$$

$$\text{Adaptation: } \dot{\hat{\theta}} = \gamma ze, \ \hat{\theta}(0) \in R^p \tag{1c}$$

where P is linear with strictly proper transfer function $P(s)$, d is an external disturbance, $\hat{\theta}$ is the adjustable parameter vector, $\gamma > 0$ is the constant adaptive gain, z is the regressor (information) vector consisting of filtered measurable signals, e.g., u, y, and references, and e is an error signal which drives the adaptation. System (1) can also be described in an error system form (e.g., [7], [8]) by proceeding as follows.

Define the parameter error by

$$\tilde{\theta} := \hat{\theta} - \theta_* \tag{2}$$

where $\theta_* \in R^p$ is a constant vector of tuned parameters, i.e., the parameters that would be selected if the plant P were known. Using (2) we can rewrite (1b) as

$$u = -\theta_*'z - v$$

where v is the *adaptive control error*. An equivalent representation of (1) is given by the adaptive error system depicted in Fig. 1 and described by

$$e = e_* - H_{ev}v \tag{4a}$$

$$z = z_* - H_{zv}v \tag{4b}$$

$$v = z'\tilde{\theta} \tag{4c}$$

$$\dot{\tilde{\theta}} = \gamma ze, \ \tilde{\theta}(0) = \hat{\theta}(0) - \theta_* \tag{4d}$$

where (e_*, z_*) are the outputs of the tuned system which is defined as system (1) with $\hat{\theta}(t) = \theta_*$. The operators H_{ev} and H_{zv} are linear with strictly proper transfer functions $H_{ev}(s)$ and $H_{zv}(s)$, respectively, which are dependent on the tuned parameter θ_*. From the definition of the tuned system [3], [4], it follows that $H_{ev}(s)$ and $H_{zv}(s)$ are exponentially stable. By the same reasoning the tuned signals $e_*(t)$ and $z_*(t)$ are bounded.

One of the very useful features of this error system is that the nonlinear effect of the adaptive algorithm can be analyzed separately from the analysis of the tuned system. The tuned system represents an ideal which would be achieved with the given structure of the adaptive control. Hence, the algebraic design procedure is separated from the nonlinear stability analysis. It is convenient, therefore, to view e_*, z_*, and $\tilde{\theta}_O$ as "inputs" to the error system. The assumption, naturally, is that e_* and z_* are well behaved with e_* small. Note that $\tilde{\theta}_O$ need not be small. In the ideal case, assuming perfect model following and no disturbances in the tuned system, $e_*(t) = 0$. If the disturbances are of a special kind then $e_*(t) \to 0$, i.e., the tuned system exhibits servo action. The more realistic case, however, is when $e_* \in L_\infty$ due to bounded disturbances which cannot be asymptotically rejected.

Global Stability Conditions

By global stability of (2) we mean that all bounded inputs e_*, z_*, and $\tilde{\theta}_O$ produce bounded outputs e, $\tilde{\theta}$, and z. In general, no restrictions are placed on the initial parameter error $\tilde{\theta}_O$ other than boundedness. Sufficient conditions for global stability can be obtained for (2) using passivity theory (e.g., [5, p. 182]). A detailed analysis can be found in [3], [4]. One of the conditions is that $H_{ev}(s)$ is *strictly positive real* (SPR), i.e., $H_{ev}(s)$ is strictly proper[2], exponentially stable and there exist a positive constant p such that

$$\text{Re } H_{ev}(j\omega) \geq \rho |H_{ev}(j\omega)|^2, \quad \forall \omega \in R. \tag{5}$$

Unfortunately, $H_{ev}(s) \in$ SPR is not robust with respect to even mild modeling error, particularly high-frequency unmodeled dynamics [6]. For example, $H_{ev}(s) \in$ SPR implies that the relative degree of $H_{ev}(s)$ cannot exceed one, from which it follows that applying this restriction to (1) imposes the same relative degree restriction on $P(s)$ as well. This is unrealistic, even in this simple example.

Local Stability Conditions

Conditions for local stability require not only that the inputs e_*, z_*, and $\tilde{\theta}_O$ are bounded, but that these bounds are not arbitrary. The local analysis is facilitated by transforming the error system (4) to the variational form

$$x = x_L - Gf(x) \tag{6a}$$

where x, x_L, G, and $f(x)$ are defined by

$$x := \begin{pmatrix} \tilde{e} \\ \tilde{z} \\ \tilde{\theta} \end{pmatrix} := \begin{pmatrix} e - e_* \\ z - z_* \\ \hat{\theta} - \theta_* \end{pmatrix}, \ f(x) := \begin{pmatrix} \tilde{z}'\tilde{\theta} \\ \tilde{z}\tilde{e} \end{pmatrix} \tag{6b}$$

$$x_L := \begin{pmatrix} \tilde{e}_L \\ \tilde{z}_L \\ \tilde{\theta}_L \end{pmatrix} := \begin{pmatrix} -H_{ev}z_*'\tilde{\theta}_L \\ -H_{zv}z_*'\tilde{\theta}_L \\ (I+LM)^{-1}\tilde{\theta}_o + Kz_*e_* \end{pmatrix} \tag{6c}$$

$$G := \begin{bmatrix} H_{ev}(1 - z_*'KN) & H_{ev}z_*'K \\ H_{zv}(1 - z_*'KN) & H_{zv}z_*'K \\ KN & -K \end{bmatrix} \tag{6d}$$

with

$$N := z_*H_{ev} + e_*H_{zv} \tag{6e}$$

$$M := Nz_*' \tag{6f}$$

$$K := (I+LM)^{-1}L \tag{6g}$$

and where L has the transfer function,

$$L(s) = \frac{1}{s}\gamma \tag{6h}$$

with γ from the adaptive algorithm (1c). This error system (6) is arrived at by separating the nonlinear cross-product terms in $f(x)$ from the linear terms in x_L. We shall refer to x_L as the response of the *linearized system*. This is almost identical to the linearized system studied by Rohrs *et al.* [6], which was arrived at by a "final approach analysis." Note that in this case the linearized system is the input to the nonlinear system. The operators K and G are linear and time-varying due to their dependence on the tuned signals. If the linearized response x_L in (6c) is small, and if the nonlinear term $f(x)$ is suitably restricted, then intuitively, x would be attracted to some neighborhood of x_L. The following theorem makes this

[1] Extension to MIMO plant is straightforward, e.g., [3].

[2] When $H_{ev}(s)$ is proper but not strictly proper, then SPR is defined as Re $H_{ev}(j\omega) \geq \epsilon > 0$, $\forall \omega \epsilon R$.

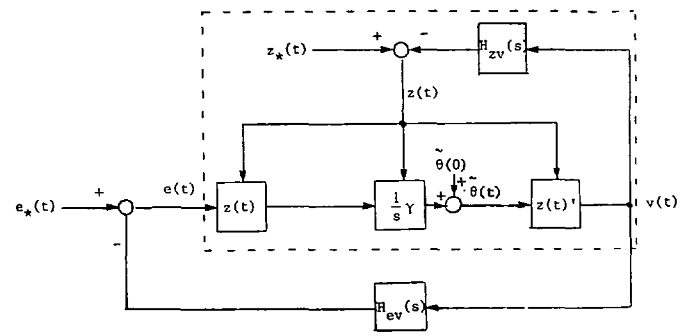

Fig. 1. Adaptive error system.

notion precise. We use the notation $\gamma_\infty(\cdot)$ and $\|\ \|_\infty$ to denote L_∞-gain and L_∞-norm, respectively.

Theorem 1: Suppose there exist finite positive constants g, ϵ, and $\delta(\epsilon)$ such that

$$\gamma_\infty(G) \le g < 1/\epsilon \tag{7a}$$

$$|x| < \delta(\epsilon) \Rightarrow |f(x)| < \epsilon |x|. \tag{7b}$$

Then

$$\|x_L\|_\infty \le (1 - g\epsilon)\delta(\epsilon) \tag{7c}$$

implies

$$\|x\|_\infty \le \delta(\epsilon). \tag{7d}$$

Theorem 1 follows directly from the linearization theorem of [5, p. 131]. Theorem 1 asserts that the error outputs x of the adaptive error system are L_∞-bounded in an ϵ-neighborhood of the linearized response, provided that the linearized response is small enough and that $G \in L_\infty$-stable. Condition (7d) shows that the actual response can be arbitrarily close to the linearized response. Since Theorem 1 provides sufficient conditions, instability does *not* follow if $x_L \in L_\infty^n$ but exceeds the magnitude constraint of (7c).

The function $\delta(\epsilon)$ in (7b) can be determined from the definition of $f(x)$ in (6b) and the norm selected. For example, if the norm on R^n is defined as $|x| = \text{Max}_i\ |x_i|$ and $\|x\|_\infty = \sup_t |x(t)|$, then

$$\delta(\epsilon) = \epsilon \tag{8a}$$

and using the corresponding induced matrix norm, we obtain

$$g = \max\ \{g_1, g_2\},$$

$$g_1 = g_0(1 + \|z_*\|_\infty k(1+n)) \tag{8b}$$

$$g_2 = k(1+n)$$

where

$$g_0 \ge \max\ \{\gamma_\infty(H_{ev}), \gamma_\infty(H_{zv})\}$$

$$n \ge \gamma_\infty(N),\ k \ge \gamma_\infty(K). \tag{8c}$$

Although Theorem 1 provides conditions for local L_∞-stability, these do not immediately provide a *region of attraction*, i.e., bounds on e_*, z_*, and $\tilde\theta_o$. These bounds in turn are determined from the set of allowable reference commands, plant initial conditions, and disturbances. Since e_* and z_* are bounded by predetermined performance goals of the tuned system, it follows that $\tilde\theta_o$ is the unknown driving factor governing the size of $\|x_L\|_\infty$. That the initial parameter error vector occupies this position of villainy should come as no surprise. One way to offset large initial parameter errors is to keep the adaptation gain γ small. This has the effect of reducing large system transients, however, this may be less than prudent if the system is initially unstable or lightly damped.

No claims are made in Theorem 1 about the mechanism that provides $x_L \in L_\infty^n$ and $G \in L_\infty$-stable. However, it follows from the definition of the tuned system that $e_* \in L_\infty$, $z_* \in L_\infty^n$ and $H_{ev}, H_{zv} \in L_\infty$-stable, thus, M in (6f) is L_∞-stable. Hence, a term by term inspection of G (6d) and x_L (6c) reveals that $x_L \in L_\infty^n$ and $G \in L_\infty$-stable, *if and only if* $\tilde\theta_L \in L_\infty^p$. Looking at (6c) we can also describe $\tilde\theta_L(t)$ as the solution to the differential equation

$$\dot\xi(t) = -\gamma(M\xi)(t) + \gamma w(t) \tag{9}$$

with $w = z_* e_*$ and $\epsilon(0) = \tilde\theta_o$. Referring to (6) and (9), the operator K is equivalent to the mapping from w into ξ. Hence, the stability analysis of (9) is of fundamental importance.

Persistent Excitation and Exponential Stability

Equations similar to (9) have been studied by invoking a persistent excitation condition on $z_*(t)$. The following definition and lemma from [1] provides the basic result.

Definition: A regulated function $f(\cdot): R_+ \to R^n$ is *persistently exciting*, denoted $f \in PE$, if there exist positive constants α_1, α_2, and α_3 such that

$$\alpha_1 I_n \le \int_s^{s+\alpha_3} f(t)f(t)'\ dt \le \alpha_2 I_n, \qquad \forall\ s \in R_+. \tag{10}$$

Lemma 1: Consider the differential equation:

$$\dot\xi(t) = -\gamma f(t)(Hf'\xi)(t) + \gamma w(t), \qquad t \ge 0. \tag{11a}$$

If $f \in PE$ and $H(s) \in$ SPR then the map $(\xi(0), w)| \to \xi$ is exponentially stable, i.e., there exist positive constants m and λ such that,

$$|\xi(t)| \le me^{-\lambda t}|\xi(0)| + \int_0^t me^{-\lambda(t-\tau)}|w(\tau)|\ d\tau. \tag{11b}$$

The usefulness of applying Lemma 1 to determine stability conditions of (9) is made apparent by writing H_{ev} as,

$$H_{ev} = \bar H_{ev} + \tilde H_{ev} \tag{12}$$

where $\bar H_{ev}$ is the nominal representation of H_{ev} and $\tilde H_{ev}$ is the deviation induced, for example, by modeling error. Combining (12) with (9), and using the definitions in (6) gives

$$\dot\xi = -\gamma z_* \bar H_{ev} z'_* \xi + \gamma Q\xi + \gamma w \tag{13a}$$

where

$$Q := M - z_* \bar H_{ev} z'_* = z_* \tilde H_{ev} z'_* + e_* H_{zv} z'_*. \tag{13b}$$

If $\bar{H}_{ev}(s) \in$ SPR and $z_* \in PE$, then using Lemma 1 gives,

$$|\xi(t)| \leq me^{-\lambda t}|\xi(0)| + \int_0^t \gamma me^{-\lambda(t-\tau)}|(Q\xi)(\tau) + w(\tau)| \; d\tau. \tag{14}$$

Hence, k from (8) is,

$$k = \frac{m}{\lambda} \geq \gamma_\infty(K) \tag{15a}$$

and from (14) with ξ replaced by $\tilde{\theta}_L$ we get,

$$\|\tilde{\theta}_L\|_\infty \leq (1-\gamma mq)^{-1}[|\tilde{\theta}_o| + \gamma m\|z_* e_*\|_\infty/\lambda] \tag{15b}$$

provided $\gamma mq < 1$ where

$$q = \|z_*\|_\infty^2 \gamma_\infty(\bar{H}_{ev}) + g_O\|z_*\|_\infty\|e_*\|_\infty \geq \gamma_\infty(Q). \tag{15c}$$

Combining (8), (15), and Theorem 1 gives the following result.

Lemma 2: The adaptive system (1) or (2) is locally L_∞-stable if for some $\epsilon < 1/g$,

$$\sigma = 1 - \frac{|\tilde{\theta}_O| + \gamma m\|z_* e_*\|_\infty/\lambda}{(1-g\epsilon)\epsilon} > 0 \tag{16a}$$

and

$$\gamma mq \leq \sigma. \tag{16b}$$

Lemma 2 together with (15) and (8) provides an *explicit upper bound* on $\|\tilde{\theta}_L\|_\infty$, $|\theta_o|$, and the amount by which H_{ev} can deviate from a nominal $\bar{H}_{ev}$ which is SPR. If the bounds are satisfied then Theorem 1 asserts that the signals in the adaptive system (1) are all bounded.

Unlike the global stability case where the bound on the deviation $\tilde{H}_{ev}$ is severely restricted, the bound here can be large.

Concluding Remarks

The stability analysis provided here involves establishing the exponential stability of a differential equation (9) which arises in the study of most adaptive systems. Although the connection between exponential stability of (a) and persistent excitation is known [1], it is important here to obtain specific formulas for the rates and gains involved, e.g., (8), (15), (16). Other methods to obtain these values can be found in [9] and [10]. Note also that Theorem 1 only requires L_∞-stability which is certainly provided when (9) is exponentially stability. However, L_∞-stability can be obtained by using a nonlinear adaptation gain in (1c), i.e., $\dot{\theta} = \gamma h(z, e)$. For example, $h(z, e)$ can arise from using a dead-zone, leakage, or normalization [11]. Such schemes can be incorporated in the general framework presented here but require further analysis in order to obtain explicit signal bounds.

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Sufficient Sequences and State-Space Models for Random Processes

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***Abstract*—Let $(U_1, U_2, \cdots)$ be a sequence of observed random variables whose probability distributions are described by a parameterized family of density functions $\{p_k(u_1, \cdots, u_k; \theta)\}$. If there exists a sequence of sufficient statistics for $\theta(T_1(U_1), T_2(U_1, U_2), \cdots)$, and if a realizability assumption holds, then there is a finite-dimensional state-space model whose output process agrees with $(U_1, U_2, \cdots)$ in distribution.**

I. Introduction

The purpose of this note is to develop some stochastic system theory related to parameter estimation problems arising in applications such as linear predictive modeling of signals. We will focus attention on problems which admit a nontrivial sufficient statistic. In previous work [1], sufficient statistics were introduced as a means of classifying the structure of finite observation records of discrete-time, stationary Gaussian random processes. In the study of problems where the observation record increases, it is natural for the system theorist to view a sequence of sufficient statistics as defining the input/output map of a dynamical system. In [2], some realization theory was developed and applied to a study of sufficient sequences from the perspective of nonlinear filtering theory.

An example derived in [2] shows that there are sufficient sequences admitting no smooth finite-dimensional realization. In particular, for observations from a stationary Gaussian process with unknown mean and known, nonrational power spectral density function, there is a one-dimensional sufficient sequence defining a linear input/output map that has no smooth finite-dimensional realization, linear or nonlinear. Viewed from the perspective of the observed process, this result is perhaps not unexpected, since there is no smooth finite-dimensional model which generates such observations [3].

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