INT. J. CONTROL, 1966, VOL. 3, NO. 6, 535–540

Stability of Distributed-parameter Dynamical Systems with Multiple Non-linearities†

By B. D. O. ANDERSON

Stanford Electronics Laboratories, Stanford University,
Stanford, California

[Received June 28, 1966]

ABSTRACT

The Popov criterion for the stability of a linear finite-dimensional system containing a single non-linearity is extended to infinite-dimensional systems with multiple non-linearities. Rather than examining stability of the states under zero input conditions the paper examines stability of the output.

§ 1. INTRODUCTION

ONE of the few results of general applicability in the study of the stability of non-linear control systems is the Popov criterion (Popov 1961, 1963). The criterion is concerned with setting down conditions under which a linear time-invariant dynamical system with a non-linear memoryless feedback law will be zero-input stable, or stable in the sense of Lyapunov.

Popov's (1961) original study is concerned with single-input, single-output systems which are finite-dimensional, i.e. the system transfer function or the Laplace transform of the system impulse response is a ratio of two polynomials in a complex variable s. The comparatively *ad hoc* procedures used to derive the result can be replaced by the more motivated procedures of Kalman (1963), where a Lyapunov function is found as a result of a system theoretic characterization of positive real functions.

Two possible directions for generalizing the theory are possible. One is to allow a *multivariable* linear time-invariant part of the system, corresponding to the presence of multiple non-linearities, perhaps separated by a linear part of the overall plant. Such a generalization may be found in Anderson (1966). The exact statement of the criterion requires the concept of *positive real matrices*, and the proof of the criterion depends on a system theoretic characterization of the positive real property. A second possibility for generalization is to relax the condition of finite dimensionality on the linear part of the plant, permitting distributed systems to be considered. Such a generalization can be found in Desoer (1965).

† Communicated by the Author.

In this paper we carry out a double generalization to consider multivariable systems which are not finite dimensional. The treatment given is more like that of Desoer (1965) than Anderson (1966), but provides a perhaps neater proof than that of Desoer (1965) for the single-variable case, by using a simple property of positive real matrices in a direct fashion.

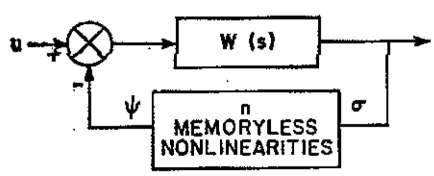
In § 2 of the paper we present a precise statement of the problem, including the various restrictions necessary on both the non-linear and linear parts of the system. Some of the latter are automatically satisfied for the finite-dimensional case (corresponding to a transfer function matrix with elements that are rational functions of s). Section 3 of the paper states and proves the main results.

§ 2. Description of the System

The prototype system is shown in the figure; we shall denote it by Ⓢ. The matrix $W(s)$ is an $n \times n$ matrix of transfer functions; we denote by $w(\cdot)$ impulse response of which $W(s)$ is the Laplace transform, and by Ⓦ the sub-system which has $w(\cdot)$ as impulse response. Without knowledge of the internal structure of Ⓦ, it is a problem to define its state explicity and to discuss the stability of the state. Accordingly we shall examine rather *the stability of the output of* Ⓢ, *under the conditions of zero input*; thus true Lyapunov stability is not considered. It is also helpful for the analysis to consider another modification of the Lyapunov asymptotic stability definition: we shall actually prove that under certain conditions the output σ is bounded and square integrable, i.e.

$$\int_0^\infty \sigma'(t)\,\sigma(t)\,dt < \infty. \tag{1}$$

Fig. 1



The system Ⓢ.

Here the superscript prime denotes matrix transposition.

The (non-external) input to the Ⓦ part of Ⓢ is $-\psi$, which is determined by the non-linearities as a vector function of the output vector σ. The notation $\psi(t)$ will be used as a loose way of writing $\psi(\sigma(t))$. We shall assume:

(A 1) With ψ_i and σ_i the ith components of ψ and σ respectively:

$$\sigma_i\psi_i \geqslant (k_{1i}+\epsilon)\psi_i^2 \quad (i=1,2,\dots n) \quad (\epsilon>0),\ (k_{1i}>0) \tag{2a}$$

and

$$\sigma_i\psi_i \geqslant (k_{2i}+\epsilon)\sigma_i^2 \quad (i=1,2,\dots n) \quad (\epsilon>0),\ (k_{2i}>0). \tag{2b}$$

For future use, we define the matrices K_1 and K_2 by:

$$K_1 = \operatorname{diag}\{k_{11}, k_{12}, \dots k_{1n}\} \tag{3a}$$

and

$$K_2 = \operatorname{diag}\{k_{21}, k_{22}, \dots k_{2n}\}. \tag{3b}$$

It is actually possible to consider a more general form of assumption than (2), as discussed by Anderson (1966). Note also that eqns. (2) are essentially the vector equivalent of the familiar condition used in the single non-linearity case, see Kalman (1963) or Desoer (1965).

Our second assumption concerns the system Ⓦ. Without knowledge of the internal structure of Ⓦ, it is a problem to define its state explicitly. A useful related concept however is the collection of zero-input responses of Ⓦ. These are the outputs appearing in the interval $[0, \infty)$ as a result of Ⓦ being in some initial state at $t=0$, and these responses are observable. We assume

(A 2) For any initial state, the zero-input response $z(\cdot)$ of Ⓦ is bounded on $[0, \infty)$ and is such that z is square integrable on $[0, \infty)$

Secondly,

(A 3) The corresponding time-derivative $\dot{z}(\cdot)$ exists, and $\dot{z}$ is bounded and square integrable on $[0, \infty)$.

Finally, we shall require:

$$\text{(A 4)} \quad |\lim_{s\to\infty} W(s)| < \infty. \tag{4}$$

In the finite-dimensional case, (A 2), (A 3), and (A 4) are implied by the requirement that the poles of $W(s)$ lie in the left half plane, and that $W(\infty)=0$.

In the stability analysis to follow, we shall need the concept of a positive real matrix. A matrix $V(s)$ of functions of a complex variable s is positive real if (Newcomb 1966):

(1) $V(s)+V'(s^*)$ is a non-negative definite hermetian matrix in the half plane $\operatorname{Re} s>0$;

(2) $V^*(s)=V(s^*)$ for $\operatorname{Re} s>0$;

(3) The elements of $V(s)$ are all analytic in $\operatorname{Re} s>0$.

For our purposes, condition (1) is more usefully expressed in time domain notation as (Newcomb 1966):

$$\int_{t_0}^{t_0+t} x'(\lambda)\int_{t_0}^{\lambda} v(\lambda-\tau)x(\tau)\,d\tau\,d\lambda \geqslant 0, \tag{5}$$

where $v(\cdot)$ is the impulse response matrix corresponding to $V(s)$ (and may contain a delta function or even a derivative of a delta function), $x(\cdot)$ is an arbitrary vector function defined over $[t_0, t_0+t]$ for which the integral exists, and t is arbitrary and positive.

Finally, it should be noted that the location of the non-linearities that has been assumed in the figure is not as restrictive as it may at first sight appear. Anderson (1966) pointed out how many different types of system can be redrawn in the form of the figure, thus permitting application of the theory.

§ 3. Statement and Proof of the Main Theorem

Theorem. Consider the multivariable time-invariant feedback system Ⓢ shown in the figure, where the system Ⓦ meets assumptions (A 2) to (A 4) of § 2, and the non-linearities meet assumption (A 1) of § 2. If there exists a non-negative scalar β such that

$$K_1 + (1+\beta s)\, W(s) = V(s) \tag{6}$$

is a positive real matrix, then for any initial state the zero-input response σ of Ⓢ is bounded and square-integrable on $[0, \infty)$.

Proof. We first establish square integrability. Let the initial state be such that the zero-input response of Ⓦ without the non-linear feedback is z. The system Ⓢ is described by:

$$\sigma(t) = z(t) - \int_0^t w(t-\tau)\psi(\tau)\, d\tau. \tag{7}$$

Then

$$\dot\sigma(t) = \dot z(t) - \frac{d}{dt}\int_0^t w(t-\tau)\psi(\tau)\, d\tau \tag{8}$$

and, using (6);

$$K_1\psi(t) - \sigma(t) - \beta\dot\sigma(t) = -z(t) - \beta\dot z(t) + \int_0^t v(t-\tau)\psi(\tau)\, d\tau, \tag{9}$$

from which:

$$\int_0^t [\psi'(\lambda)K_1\psi(\lambda) - \psi'(\lambda)\sigma(\lambda)]\, d\lambda - \int_0^t \beta\psi'(\lambda)\dot\sigma(\lambda)\, d\lambda$$
$$+\int_0^t \psi'(\lambda)[z(\lambda)+\beta\dot z(\lambda)]\, d\lambda = \int_0^t \psi'(\lambda)\, d\lambda \int_0^\lambda v(\lambda-\tau)\psi(\tau)\, d\tau. \tag{10}$$

Note that eqns. (4) and (6) establish that v may contain a delta function, but not the derivative of a delta function. The integral on the right of (10) is accordingly well defined for all 'reasonably behaved' ψ.

Equation (10) may now be used to show that σ is square integrable, or, what is completely equivalent by (2 *a*) and (2 *b*) that ψ is square integrable.

For convenience denote the left side of the above equation by $I_1+I_2+I_3$, the right side by I_4, where each I_j corresponds to an integral multiplied by the associated constant coefficient.

From (2 *a*) it is evident that

$$I_1<0. \tag{11}$$

From (2 *b*) it follows that

$$I_2 \leqslant -\frac{\beta}{2}\,[\sigma'(t)K_2^{-1}\sigma(t)-\sigma'(0)K_2^{-1}\sigma(0)], \tag{12}$$

the equality holding if $\beta=0$.

For I_3, we have the inequality:

$$|I_3|<\left(\int_0^t \psi'(\lambda)\psi(\lambda)\,d\lambda\right)^{1/2}\left(\int_0^t [\alpha z'(\lambda)+\beta\dot{z}'(\lambda)]\ [\alpha z(\lambda)+\beta\dot{z}(\lambda)]\,d\lambda\right)^{1/2}$$

$$<\left(\int_0^t \psi'(\lambda)\psi(\lambda)\,d\lambda\right)^{1/2}M, \tag{13}$$

where M is a constant independent of t existing by (A 2) and (A 3).

Since I_1 is evidently

$$0\left[\int_0^t \psi'(\lambda)\psi(\lambda)\,d\lambda\right]$$

it follows by (11) that if ψ is not square integrable, I_1 diverges to $-\infty$. Moreover, if I_2 diverges, it must diverge also to $-\infty$ since $\beta>0$. Finally, from (13) it is clear that I_3 cannot diverge as fast as I_1. Hence for sufficiently large t, if ψ is not square integrable:

$$I_1+I_2+I_3<0. \tag{14}$$

The positive real character of $V(s)$ however implies (see eqn. (5)) that

$$I_4>0, \tag{15}$$

which is a contradiction of (10). Consequently $\psi(\cdot)$ is square integrable.

Turning now to the boundedness of σ, we note that in (7), w is square integrable on $[0, \infty)$, having elements identical with the elements of some z (Desoer 1965), which are square integrable by (A 2); second, ψ is square integrable by the material just proved. This implies that there exists a constant N such that

$$\int_0^t w(t-\tau)\psi(\tau)\,d\tau<N<\infty \tag{16}$$

for all t.

Since z is bounded by (A 2), it follows from (7) that σ is bounded. This proves the theorem.

§ 4. Conclusions

One of the few general results on the stability of non-linear control systems has been extended to a more general situation than has hitherto been available. Because of our comparative lack of ability to deal with the state of an arbitrary distributed system, the generality of the Lyapunov stability has been sacrificed to discuss stability of the output under zero input conditions. Certainly though there is practical significance in doing this, and we are not the first to do so (Desoer 1965). An interesting feature of the proof of the principal result is the simple way in which the positive real property is employed.

Acknowledgment

This work was carried out under the sponsorship of Joint Services Contract Nonr 225(83).

References

Anderson, B. D. O., 1966, Report No. SEL-66-012 (TR No. 6560-2), Stanford Electronics Laboratories, Stanford, California, February.

Desoer, C. A., 1965, *IEEE Transactions on Automatic Control*, **AC-10**, 182.

Kalman, R. E., 1963, *Proc. nat. Acad. Sci., Wash.*, **49**, 201.

Newcomb, R. W., 1966, *Linear Multiport Synthesis* (New York : McGraw-Hill).

Popov, V. M., 1961, *Automat. Telemech., Moscow*, **22**, 961 ; 1962, *Automation and Remote Control*, **22**, 857.

Popov, V. M., and Halanay, A., 1963, *Automation and Remote Control*, **23**, 783.