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Kharitonov's theorem and the second method of Lyapunov

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Abstract: In this paper Kharitonov's theorem for the robust stability of interval polynomials is proved using the second method of Lyapunov. The Hermite matrix is taken as the matrix of the quadratic form which is used as a Lyapunov function to prove Hurwitz stability. It is shown that if the four Hermite matrices corresponding to the four Kharitonov extreme polynomials are positive definite, the Hermite matrix of any polynomial of the polynomial family remains positive definite.

Keywords: Lyapunov theory; Kharitonov theorem; interval polynomials.

1. Introduction

Consider an interval polynomial

$$f(s)=\sum_{i=0}^{n} a_i s^i \tag{1}$$

where

$$a_{\mathrm{L}i} \leqslant a_i \leqslant a_{\mathrm{U}i} \tag{2}$$

Kharitonov [1] has shown that the polynomial family defined by (1) and (2) is Hurwitz stable if and only if four extreme polynomials, namely,

$$f_1(s)=a_{\mathrm{U}0}+a_{\mathrm{L}1}s+a_{\mathrm{L}2}s^2+a_{\mathrm{U}3}s^3+a_{\mathrm{U}4}s^4+a_{\mathrm{L}5}s^5+a_{\mathrm{L}6}s^6+\cdots, \tag{3}$$

$$f_2(s)=a_{\mathrm{U}0}+a_{\mathrm{U}1}s+a_{\mathrm{L}2}s^2+a_{\mathrm{L}3}s^3+a_{\mathrm{U}4}s^4+a_{\mathrm{U}5}s^5+a_{\mathrm{L}6}s^6+\cdots, \tag{4}$$

$$f_3(s)=a_{\mathrm{L}0}+a_{\mathrm{U}1}s+a_{\mathrm{U}2}s^2+a_{\mathrm{L}3}s^3+a_{\mathrm{L}4}s^4+a_{\mathrm{U}5}s^5+a_{\mathrm{U}6}s^6+\cdots, \tag{5}$$

$$f_4(s)=a_{\mathrm{L}0}+a_{\mathrm{L}1}s+a_{\mathrm{U}2}s^2+a_{\mathrm{U}3}s^3+a_{\mathrm{L}4}s^4+a_{\mathrm{L}5}s^5+a_{\mathrm{U}6}s^6+\cdots, \tag{6}$$

are Hurwitz stable.

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In [2] it was shown that for 3rd degree, 4th degree and 5th degree polynomials $f(s)$, the Hurwitz stability of $f_1(s)$; $f_1(s)$ and $f_2(s)$; $f_1(s)$ and $f_2(s)$ and $f_3(s)$ respectively is necessary and sufficient for the Hurwitz stability of the interval polynomial.

Kharitonov's theorem has different proofs with the simplest one perhaps being given in [3], using the value set concept and the Cremer–Leonard–Michailov criterion.

The Hurwitz stability criterion was proved in [4], using the second method of Lyapunov with the Hermite matrix as the matrix of the quadratic form Lyapunov function. In [5] a reduced Hermite criterion is derived which makes use of a symmetric matrix of half the dimension of the original Hermite matrix and which we shall use in the sequel. The idea of the proof of Kharitonov's theorem using Lyapunov's second method is based on the following: If we assume the positive definiteness of the reduced Hermite matrix of the four Kharitonov polynomials we have to prove that the reduced Hermite matrix remains positive definite for any value of the parameters in the Kharitonov box. It is clearly sufficient to establish simply that its determinant does not go to zero. We divide the problem in two parts: We prove the above for fixed even coefficients and then for fixed odd coefficients.

The determinant of the reduced Hermite matrix is determined as a function of the even or the odd parts of the polynomial, following the line of thought in [6]. The change of the even part or the odd part of the polynomial inside the box is determined following a similar argument to that used in the proof of Markov–Chebychev theorem in [7]. In the following section we bring two lemmas which are used in Section 3 to prove Kharitonov's theorem.

2. Mathematical background

The polynomial $f(s)$ in (1) is partitioned into its even and odd parts as follows:

$$f(s) = \sum_{i=0}^{n} a_i s^i = h(s^2) + sg(s^2). \tag{7}$$

The four Kharitonov polynomials are given (in obvious notation) by:

$$f_1 = (h_U, g_L), \quad f_2 = (h_U, g_U), \quad f_3 = (h_L, g_U), \quad f_4 = (h_L, g_L). \tag{8}$$

The $n \times n$ Hermite matrix P associated with $f(s)$ is given by:

$$p_{ij} = \begin{cases} \sum_{k=1}^{i} (-1)^{k+1} a_{n-k+1} a_{n-i-j+k} & \text{if } j \geqslant i,\ j+i \text{ even}, \\ p_{ji} & \text{if } j < i,\ j+i \text{ even}, \\ 0 & \text{if } j+i \text{ odd}. \end{cases} \tag{9}$$

For $n = 6$ for example,

$$P = \begin{bmatrix} a_6a_5 & 0 & a_6a_3 & 0 & a_6a_1 & 0 \\ 0 & -a_6a_3 + a_5a_4 & 0 & -a_6a_1 + a_5a_2 & 0 & a_5a_0 \\ a_6a_3 & 0 & a_6a_1 - a_5a_2 + a_4a_3 & 0 & -a_5a_0 + a_3a_1 & 0 \\ 0 & -a_6a_1 + a_5a_2 & 0 & a_5a_0 - a_4a_1 + a_3a_2 & 0 & a_3a_0 \\ a_6a_1 & 0 & -a_5a_0 + a_4a_1 & 0 & -a_3a_0 + a_2a_1 & 0 \\ 0 & a_5a_0 & 0 & a_3a_0 & 0 & a_1a_0 \end{bmatrix}. \tag{10}$$

For simplicity and without real loss of generality we shall assume that $a_n = 1$.

The Hermite criterion states that $f(s)$ is Hurwitz stable if and only if P is positive definite. Consider the matrices

$$A=\begin{bmatrix} 0 & 1 & & \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix}, \qquad J=\begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & & 0 \\ 0 & & & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}, \qquad P^* = JPJ. \tag{11}$$

Then we get the Lyapunov equation

$$A^{\mathrm{T}}P^* + P^*A = -2b_{\mathrm{L}}b_{\mathrm{L}}^{\mathrm{T}} \tag{12}$$

where $b_{\mathrm{L}}^{\mathrm{T}} = [\cdots \ 0 \ a_{n-3} \ 0 \ a_{n-1}]$ and $x^{\mathrm{T}}P^*x_{\mathrm{L}}$ is the Lyapunov function which proves the Hurwitz stability of $f(s)$.

The reduced Hermite matrices C and D derived in [5] are submatrices of the Hermite matrix P obtainable by deleting even-numbered rows and columns from P or deleting odd-numbered rows and columns respectively.

For $n = 6$ for example,

$$C=\begin{bmatrix} a_6a_5 & a_6a_3 & a_6a_1 \\ a_6a_3 & a_6a_1 - a_5a_2 + a_4a_3 & -a_5a_0 + a_4a_1 \\ a_6a_1 & -a_5a_0 + a_4a_1 & -a_3a_0 + a_2a_1 \end{bmatrix}, \tag{13}$$

$$D=\begin{bmatrix} -a_6a_3 + a_5a_4 & -a_6a_1 + a_5a_2 & a_5a_0 \\ -a_6a_1 + a_5a_2 & a_5a_0 - a_4a_1 + a_3a_2 & a_3a_0 \\ a_5a_0 & a_3a_0 & a_1a_0 \end{bmatrix}. \tag{14}$$

A form of the result in [5] can be stated as follows:

If $a_i > 0$, $i = 0, 1, \ldots, n$, then

$$P > 0 \ \Leftrightarrow \ C > 0 \ \Leftrightarrow \ D > 0. \tag{15}$$

If (with $a_n = 1$)

$$A_1=\begin{bmatrix} 0 & 1 & & & \\ \vdots & 0 & 1 & & \\ \vdots & \vdots & 0 & \ddots & \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & & 1 \\ -a_0 & -a_2 & -a_4 & & -a_{n-2} \end{bmatrix} \tag{16}$$

then there holds

$$(JCJ)A_1 = A_1^{\mathrm{T}}(JCJ) = -JDJ \tag{17}$$

(where J is as above except for its dimension).

By way of digression we may recall [5,8] that

$$\begin{aligned} &\text{for } n \text{ even:} \quad \det C = \det H_{n-1}, \quad \det D = a_0 \det H_{n-1}, \\ &\text{for } n \text{ odd:} \quad \det C = a_0 \det H_{n-1}, \quad \det D = \det H_{n-1} \end{aligned} \tag{18}$$

where det H_{n-1} is the $n-1$ Hurwitz-determinant. H_n is given by

$$H_n = \begin{bmatrix} a_1 & a_0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ a_3 & a_2 & a_1 & a_0 & \cdots & \cdots & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & & & & & \ddots & \vdots \\ \vdots & \vdots & & & & & \ddots & a_{n-2} \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & a_n \end{bmatrix}. \tag{19}$$

It is clear that if C or $D > 0$ then det H_{n-1} is positive. The quantity det H_{n-1} determines the critical stability condition. If coefficients of an originally stable system are perturbed, stability will be lost if det C (or det D) becomes zero.

Without loss of generality we shall consider in the sequel even polynomials by setting $n = 2m$. A similar treatment can be used for odd n.

Lemma 1. *With notation as above,*

$$\det C = (-1)^{m(m-1)/2} \prod g(\alpha_i) = (-1)^{m(m-1)/2} a_{n-1} \prod h(\beta_j) \tag{20}$$

where α_i and β_j are the roots of $h(\lambda)$ and $g(\lambda)$ respectively.

The second key result which we need is as follows:

Lemma 2. *With quantities as defined above,*

$$\lambda_L^T C^* \lambda_L = g(\lambda) \frac{dh(\lambda)}{d\lambda} - h(\lambda) \frac{dg(\lambda)}{d\lambda} \quad \textit{where} \quad \lambda_L^T = [1 \quad \lambda \quad \cdots \quad \lambda^{m-1}]. \tag{21}$$

The proofs of Lemma 1 and Lemma 2 are given in the Appendix.

3. Proof of Kharitonov's theorem

We need the following additional lemma:

Lemma 3. *Consider the family of polynomials defined by*

$$f(s) = \sum_{i=0}^{n} a_i s^i \quad \textit{and} \quad a_{L2j+1} \leqslant a_{2j+1} \leqslant a_{U2j+1} \quad \textit{for } j = 0, 1, \ldots, m-1,$$

while a_{2j} is fixed at the values of $\tilde{h}$ where $\tilde{h} = \{h_L, h_U\}$. Suppose that C is positive definite for the four Kharitonov polynomials. Then det C *is nonzero for all polynomials in the family.*

Proof. Let $a_i = (-1)^{(i-1)/2} \phi_i(t)$ for $i = 1, 3, 5, \ldots, n-1$, where $\phi_i(t)$ are monotonically increasing functions and differentiable when t varies from $t = t_1$ to $t = t_2$. Taking $t = t_1$ gives g_L and taking $t = t_2$ gives g_U. Then

$$\frac{dg}{dt} = \sum (-1)^{(i-1)/2} \frac{\partial g}{\partial a_i} \frac{d\phi_i}{dt} = \frac{d\phi_1}{dt} - \lambda \frac{d\phi_3}{dt} + \lambda^2 \frac{d\phi_5}{dt} - \cdots. \tag{22}$$

At the roots α_i of $\tilde{h}(\lambda)$, which are negative and independent of t, we get

$$\frac{\mathrm{d}g(\alpha_i)}{\mathrm{d}t} > 0. \tag{23}$$

Now from Lemma 2 and the stability of the four Kharitonov polynomials

$$g_{\mathrm{U}}(\alpha_i)\frac{\mathrm{d}\tilde{h}(\alpha_i)}{\mathrm{d}\alpha_i} > 0 \quad \text{and} \quad g_{\mathrm{L}}(\alpha_i)\frac{\mathrm{d}\tilde{h}(\alpha_i)}{\mathrm{d}\alpha_i} > 0 \tag{24}$$

which means that $g(\alpha_i)$ does not change sign as t varies from t_1 to t_2. Then from Lemma 1, det C does not go to zero. Hence C remains positive definite for all values of $a_1, a_3, \ldots, a_{n-1}$ in the given range. □

Now let us return to the family defined by (1) and (2), with the assumption that the four Kharitonov polynomials are Hurwitz stable. By Lemma 3, if we choose any $g(\lambda)$ consistent with (2), the matrix C associated with $g(\lambda)$ and either $h_{\mathrm{U}}(\lambda)$ or $h_{\mathrm{L}}(\lambda)$ is positive definite. Now let $a_i = (-1)^{i/2}\phi_i(t)$, $i = 0, 2, 4, \ldots, n-2$, where $\phi_i(t)$ are monotonically increasing and differentiable when t varies from $t = t_1$ to $t = t_2$. Taking $t = t_1$ gives h_{L} and taking $t = t_2$ gives h_{U}. Then

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \sum(-1)^{i/2}\frac{\partial h}{\partial a_i}\frac{\mathrm{d}\phi_i}{\mathrm{d}t} = \frac{\mathrm{d}\phi_0}{\mathrm{d}t} - \lambda\frac{\mathrm{d}\phi_2}{\mathrm{d}t} + \lambda^2\frac{\mathrm{d}\phi_4}{\mathrm{d}t} - \cdots \tag{25}$$

and at all roots β_j of $g(\lambda)$ which are negative we get

$$\frac{\mathrm{d}h(\beta_j)}{\mathrm{d}t} > 0. \tag{26}$$

Then from Lemma 2

$$-h_{\mathrm{U}}(\beta_j)\frac{\mathrm{d}g(\beta_j)}{\mathrm{d}\beta_j} > 0 \quad \text{and} \quad -h_{\mathrm{L}}(\beta_j)\frac{\mathrm{d}g(\beta_j)}{\mathrm{d}\beta_j} > 0, \tag{27}$$

which means that $h(\beta_j)$ does not change sign as t varies from t_1 to t_2. Then from Lemma 1, det C does not go to zero. Hence C remains positive definite for all values of a_i in the given range for $i = 0, 1, \ldots, n$. Therefore, if C is positive definite for the four Kharitonov polynomials it remains positive definite for the whole box.

4. Conclusions

It was shown that he positive definiteness of the Hermite matrix for the four Kharitonov polynomials guarantees its positive definiteness for all other values of the coefficients inside the Kharitonov box.

As the Hermite matrix can be used to construct a Lyapunov function to prove Hurwitz stability, then the above result can be considered as a Lyapunov–Kharitonov link.

Nowhere in our argument have we used the concept of a value set, nor monotony of the argument of a Hurwitz stable polynomial evaluated on the imaginary axis. This monotony can actually be easily derived by using a property like that of Lemma 2, but applied to the full size Hermite matrix P, rather than C.

Appendix

Proof of Lemma 1. The proof follows the same lines as [6]. Consider the transfer function

$$w_1(\lambda) = \frac{g(\lambda)}{h(\lambda)} = \frac{a_1 + a_3\lambda + \cdots + a_{n-1}\lambda^{m-1}}{a_0 + a_2\lambda + \cdots + \lambda^m}. \tag{A.1}$$

A realisation of (A.1) is given by

$$A_1 = \begin{bmatrix} 0 & 1 & & \\ 0 & & \ddots & \\ & & & 1 \\ -a_0 & -a_2 & \cdots & -a_{n-2} \end{bmatrix}, \quad b_{L1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad c_{L1}^{T} = [a_1 \quad a_3 \quad \cdots \quad a_{n-1}]. \tag{A.2}$$

Define also

$$C^* = JCJ, \qquad D^* = JDJ. \tag{A.3}$$

The observability matrix is given by

$$S_1 = \begin{bmatrix} c_{L1}^{T} \\ c_{L1}^{T} A_1 \\ \vdots \\ c_{L1}^{T} A_1^{m-1} \end{bmatrix} = S_1(A_1, c_{L1}^{T}). \tag{A.4}$$

Let $e_{L1}, e_{L2}, \ldots, e_{Lm}$ be the unit vectors; then it is easily verified that

$$S_1(A_1, e_{L1}^{T}) = I, \quad S_1(A_1, e_{L2}^{T}) = A_1, \quad \ldots, \quad S_1(A_1, e_{Lm}^{T}) = A_1^{m-1}.$$

Therefore,

$$S_1(A_1, c_{L}^{T}) = a_1 I + a_3 A_1 + \cdots + a_{n-1} A_1^{m-1} = g(A_1). \tag{A.5}$$

But

$$C^* e_{Lm} = \begin{bmatrix} a_1 \\ a_3 \\ \vdots \\ a_{n-3} \\ a_{n-1} \end{bmatrix} = c_{L1}$$

and by (17),

$$C^* A_1^k = (A_1^{T})^k C^*, \quad k = 1, 2, 3, \ldots .$$

Then it can be seen by inspection that

$$C^* [e_{Lm} \quad A_1 e_{Lm} \quad \cdots \quad A_1^{m-1} e_{Lm}] = S_1^{T}. \tag{A.6}$$

But $[e_{Lm} \quad A_1 e_{Lm} \quad \cdots \quad A_1^{m-1} e_{Lm}]$ is clearly a lower triangular matrix with second diagonal elements equal to one. Therefore, $\det C^* = (-1)^{m(m-1)/2} \det S_1^{T}$ and from (A.5),

$$\det C^* = (-1)^{m(m-1)/2} \det g(A_1). \tag{A.7}$$

From a well known result in matrix theory $\det g(A_1) = \prod g(\alpha_i)$ where α_i are the eigenvalues of A_1. Therefore,

$$\det C = (-1)^{m(m-1)/2} \prod g(\alpha_i) \tag{A.8}$$

where α_i are the roots of $h(\lambda)$.

For the proof of the second equality of Lemma 1, consider the transfer function

$$w_2(\lambda)=\frac{h(\lambda)}{\lambda^2 g(\lambda)/a_{n-1}}=\frac{a_0+a_2\lambda+\cdots+\lambda^m}{(a_1/a_{n-1})\lambda^2+(a_3/a_{n-1})\lambda^3+\cdots+\lambda^{m+1}}. \tag{A.9}$$

A realisation of (A.9) is given by

$$A_2=\begin{bmatrix}0 & 1\\ 0 & B\end{bmatrix}\quad\text{where}\quad B=\begin{bmatrix}0 & 1 & & \\ \vdots & & \ddots & \\ \vdots & & & 1\\ 0 & -\frac{a_1}{a_{n-1}} & \cdots & -\frac{a_{n-3}}{a_{n-1}}\end{bmatrix},$$

$$b_{L2}=\begin{bmatrix}0\\ \vdots\\ 0\\ 1\end{bmatrix},\qquad c_{L2}^{T}=[a_0\quad a_2\quad\cdots\quad 1]. \tag{A.10}$$

The observability matrix S_2 is given by

$$S_2=\begin{bmatrix}c_{L2}^T\\ c_{L2}^T A_2\\ \vdots\\ c_{L2}^T A_2^m\end{bmatrix}=\begin{bmatrix}a_0 & a_2 & \cdots & 1\\ 0 & a_0 & \cdots & \cdots\\ \vdots & & \ddots & \\ 0 & 0 & 0 & a_{0+\cdots}\end{bmatrix}=S_2(A_2,c_{L2}^T). \tag{A.11}$$

$h(B)$(established below)

Now $S_2(A_2, e_{L1}^T)=I$, $S_2(A_2, e_{L2}^T)=A_2,\dots,S_2(A_2, e_{Lm+1}^T)=A_2^m$, so that

$$S_2(A_2,c_{L2}^T)=a_0 I+a_2A_2+\cdots+A_2^m=h(A_2). \tag{A.12}$$

If we denote the $m\times m$ submatrix of A_2 obtained by deleting the first row and column by B then the corresponding submatrix of S_2 is evidently $h(B)$.

We have

$$\det h(A)=a_0\det h(B) \tag{A.13}$$

Now it can be easily established by direct calculation that

$$D^*B=B^TD^* \tag{A.14}$$

and

$$D^*e_{Lm}=a_{n-1}\begin{bmatrix}a_0\\ a_2\\ \vdots\\ \vdots\\ a_{n-2}\end{bmatrix}-\begin{bmatrix}0\\ a_1\\ a_3\\ \vdots\\ a_{n-3}\end{bmatrix}=f_L \tag{A.15}$$

Also

$$c_{L2}^TA_2=\frac{1}{a_{n-1}}[0\quad f_L^T]. \tag{A.16}$$

Recalling the construction of S_2 above and its submatrix $h(B)$, we see that

$$[f_L \quad Bf_L \quad \cdots \quad B^{m-1}f_L] = a_{n-1}[h(B)]^T \tag{A.17}$$

and accordingly,

$$D^*[e_{Lm} \quad \cdots \quad B^{m-1}e_{Lm}] = \left[D^*e_{Lm} \quad B^TD^*e_{Lm} \quad \cdots \quad (B^T)^{m-1}D^*e_{Lm}\right] = a_{n-1}[h(B)]^T. \tag{A.18}$$

Now $[e_{Lm} \quad \cdots \quad B^{m-1}e_{Lm}]$ is a lower triangular matrix with unity elements in the second diagonal. Therefore,

$$\begin{aligned}\det D^* &= (-1)^{m(m-1)/2}a_{n-1} \det h(B) = (-1)^{m(m-1)/2}a_{n-1}h(0)\Pi h(\beta_j)\\ &= (-1)^{m(m-1)/2}a_0a_{n-1}\Pi h(\beta_j)\end{aligned} \tag{A.19}$$

where β_j are the roots of $g(\lambda)$. Hence

$$\det C = \frac{1}{a_0} \det D^* = (-1)^{m(m-1)/2}a_{n-1}\Pi h(\beta_j). \qquad \square$$

Proof of Lemma 2. It is easily verified that

$$\frac{h(\mu)g(\lambda) - h(\lambda)g(\mu)}{\mu - \lambda} = [1 \quad \mu \quad \cdots \quad \mu^{m-1}]C^*\begin{bmatrix}1\\ \lambda\\ \vdots\\ \lambda^{m-1}\end{bmatrix}. \tag{A.20}$$

It is straightforward to show that if we substitute $\mu = \lambda + \varepsilon$ and let ε tend to zero, then

$$\frac{h(\lambda+\varepsilon)g(\lambda) - h(\lambda)g(\lambda+\varepsilon)}{\varepsilon} = \frac{[h(\lambda+\varepsilon) - h(\lambda)]g(\lambda) - [g(\lambda+\varepsilon) - g(\lambda)]h(\lambda)}{\varepsilon}$$

so that

$$g(\lambda)\frac{\mathrm{d}h(\lambda)}{\mathrm{d}\lambda} - h(\lambda)\frac{\mathrm{d}g(\lambda)}{\mathrm{d}\lambda}.$$

Therefore,

$$\lambda_L^T C^* \lambda_L = g(\lambda)\frac{\mathrm{d}h(\lambda)}{\mathrm{d}\lambda} - h(\lambda)\frac{\mathrm{d}g(\lambda)}{\mathrm{d}\lambda}. \qquad \square \tag{A.21}$$

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